MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

SVD-BASED MODEL ORDER REDUCTION

Peter Benner

October 28, 2020

Methods of Model Order Reduction
Shanghai University

Outline

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Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)), \quad x(t_0) = x_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.

Model Reduction for Dynamical Systems

Original System

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Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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Goal:

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Secondary goal: reconstruct approximation of x from $\hat{x}$.

Linear Systems in Frequency Domain

Linear, Time-Invariant (LTI) Systems

$$\Sigma : \begin{cases} \dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \\ y &= Cx + Du, \qquad C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times m}. \end{cases}$$

Assumptions: $t_0 = 0$, $x_0 = x(0) = 0$.

Laplace Transform / Frequency Domain

Application of Laplace transform

$$\mathcal{L} : x(t) \mapsto x(s) = \int_0^\infty e^{-st} x(t)\, dt \quad (\Rightarrow \ \dot{x}(t) \mapsto sx(s))$$

with $s \in \mathbb{C}$ leads to linear system of equations:

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s).$$

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Laplace Transform / Frequency Domain

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s)$$

yields I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)}\Big)u(s) = G(s)u(s).$$

G is the transfer function of Σ, $G : \mathcal{L}_2^m \to \mathcal{L}_2^p \quad (\mathcal{L}_2 := \mathcal{L}(L_2(-\infty, \infty)))$.

Model Reduction as Approximation Problem

Model Order Reduction Problem

Approximate the dynamical system

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by reduced-order system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \qquad \hat{A} \in \mathbb{R}^{r\times r}, \quad \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, \qquad \hat{C} \in \mathbb{R}^{p\times r}, \quad \hat{D} \in \mathbb{R}^{p\times m}. \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \left\| Gu - \hat{G}u \right\| \le \left\| G - \hat{G} \right\| \|u\| \le \text{tolerance} \cdot \|u\| .$$

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$\Longrightarrow$ Approximation problem:

$$\min_{\text{order}\,(\hat{G}) \leq r} \|G - \hat{G}\| \quad \text{for } \mathcal{H}_2/\mathcal{H}_\infty \text{ norm.}$$

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
- Memory: $4 \cdot n_x \cdot n_y$ bytes.

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Theorem: (Schmidt-Mirsky/Eckart-Young)

Best rank-r approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$\widehat{X} = \sum\nolimits_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U \Sigma V^T$ is the singular value decomposition (SVD) of X.
The approximation error is $\|X - \widehat{X}\|_2 = \sigma_{r+1}$.

Motivation for SVD-based Methods

An Inspiration: Image Compression by Truncated SVD

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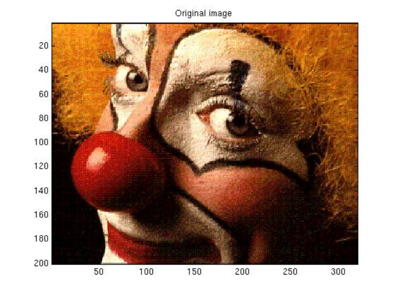
Idea for dimension reduction

Instead of X save $u_1, \ldots, u_r$, $\sigma_1 v_1, \ldots, \sigma_r v_r$.
$\rightsquigarrow$ memory $= 4r \times (n_x + n_y)$ bytes.

Example: Clown



320×200 pixel

$\rightsquigarrow \quad \approx 256$ kb

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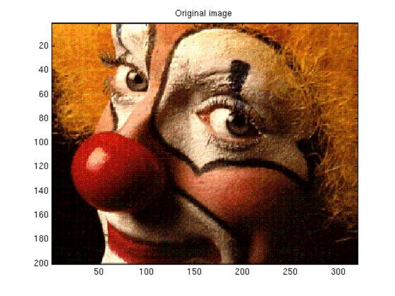

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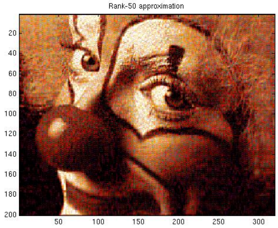
- rank $r = 50$, ≈ 104 kb



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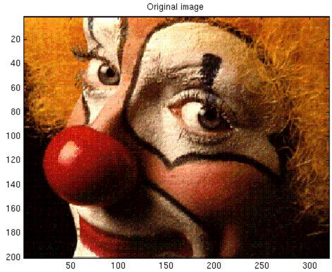

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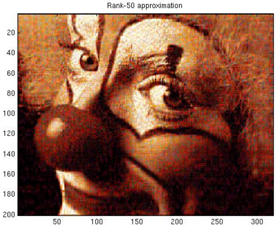
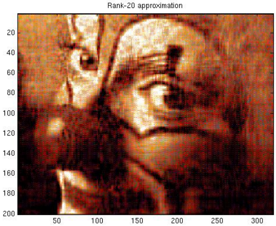
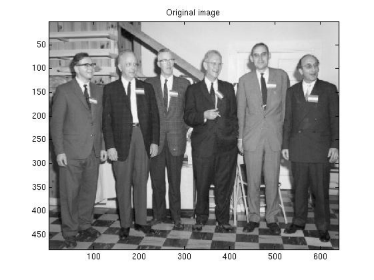


- rank $r = 20$, ≈ 42 kb



Example: Gatlinburg

Organizing committee
Gatlinburg/Householder Meeting 1964:
James H. Wilkinson, Wallace Givens, George Forsythe, Alston Householder, Peter Henrici, Fritz L. Bauer.



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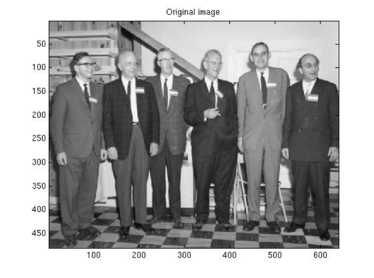
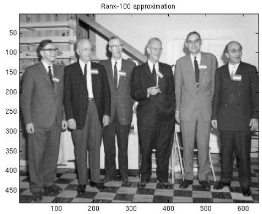


640×480 pixel, ≈ 1229 kb

- rank $r = 100$, ≈ 448 kb



- rank $r = 50$, ≈ 224 kb

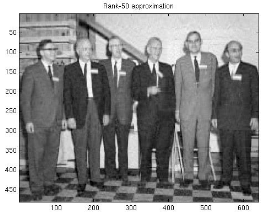
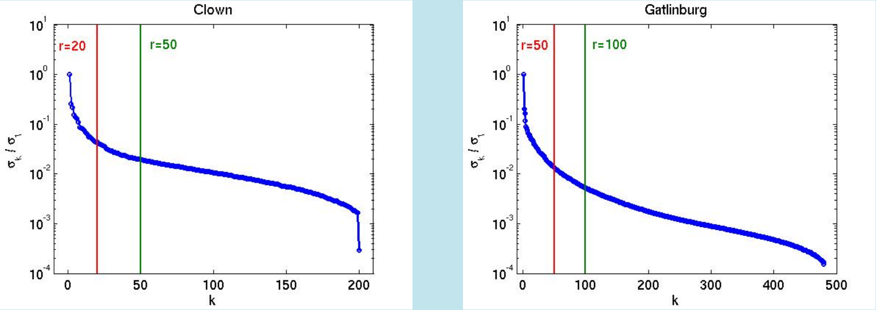


Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices



Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x} &= f(t,x,u) &= Ax + Bu, \qquad & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= g(t,x,u) &= Cx + Du, \qquad & C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{aligned}$$

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State-Space Description for I/O-Relation

Variation-of-constants $\Longrightarrow$

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t\in\mathbb{R}.$$

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- Problem: in general, $\mathcal{S}$ does not have a discrete SVD and can therefore not be approximated as in the matrix case!

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Alternative to State-Space Operator: Hankel operator

Instead of

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use Hankel operator

$$\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\, d\tau \quad \text{for all } t > 0.$$

Linear, Time-Invariant (LTI) Systems

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$\rightsquigarrow$ *Hankel singular values* $\{\sigma_j\}_{j=1}^{\infty} : \ \sigma_1 \geq \ldots \geq \sigma_n \geq \sigma_{n+1} = 0 = \ldots = 0.$

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$\Longrightarrow$ SVD-type approximation of $\mathcal{H}$ possible!

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$\Downarrow$

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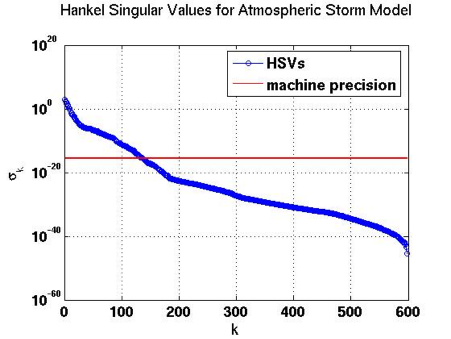
$\Downarrow$

Hankel singular values



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$\Rightarrow$ solution: Adamjan-Arov-Krein (AAK Theory, 1971/78).

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But: computationally challenging for large-scale systems.

Recent progress in [B./Werner 2020].

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- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

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$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

("system does not generate energy").

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- If $\mathcal{V}$ is an A-invariant subspace corresponding to a subset of A's spectrum, then we call P a spectral projector.
- Let $\mathcal{W} \subset \mathbb{R}^n$, $\dim \mathcal{W} = r$, with basis matrix $W = [\, w_1, \ldots, w_r \,]$, then $P = V(W^T V)^{-1} W^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.

Methods:

1. Modal Truncation
2. Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods)
3. Balanced Truncation
4. many more...

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$$\mathrm{range}(V) = \mathcal{V}, \quad \mathrm{range}(W) = \mathcal{W}, \quad W^T V = I_r.$$

Then, with $\hat{x} = W^T x$, we obtain $x \approx V\hat{x}$ so that

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|,$$

and the reduced-order model is

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

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- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

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Base enrichment

Static modes are defined by setting $\dot{x} = 0$ and assuming unit loads, i.e., $u(t) \equiv e_j$, $j = 1, \ldots, m$:

$$0 = Ax(t) + Be_j \quad \Longrightarrow \quad x(t) \equiv -A^{-1}b_j.$$

Projection subspace $\mathcal{V}$ is then augmented by $A^{-1}[\, b_1, \ldots, b_m \,] = A^{-1}B$.

Interpolation-projection framework $\Longrightarrow G(0) = \hat{G}(0)$!

If two-sided projection is used, complimentary subspace can be augmented by $A^{-T}C^T \Longrightarrow G'(0) = \hat{G}'(0)$!

Note: if $m \neq q$, add random vectors or delete some of the columns in $A^{-T}C^T$.

Guyan reduction (static condensation)

Partition states in masters $x_1 \in \mathbb{R}^r$ and slaves $x_2 \in \mathbb{R}^{n-r}$ (FEM terminology)
Assume stationarity, i.e., $\dot{x} = 0$ and solve for x_2 in

$$\begin{aligned} 0 &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ \Rightarrow \quad x_2 &= -A_{22}^{-1} A_{21} x_1 - A_{22}^{-1} B_2 u. \end{aligned}$$

Inserting this into the first part of the dynamic system

$$\dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_1 u, \quad y = C_1 x_1 + C_2 x_2$$

then yields the reduced-order model

$$\begin{aligned} \dot{x}_1 &= (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_1 + (B_1 - A_{12} A_{22}^{-1} B_2) u \\ y &= (C_1 - C_2 A_{22}^{-1} A_{21}) x_1 - C_2 A_{22}^{-1} B_2 u. \end{aligned}$$

Definition

A realization (A, B, C, D) of a linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\} \quad \text{(w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1\text{)}.$$

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When does a balanced realization exist?
Assume A to be Hurwitz, i.e. $\Lambda(A) \subset \mathbb{C}^-$. Then:

Theorem

Given a stable minimal linear system $\Sigma : (A, B, C, D)$, a balanced realization is obtained by the state-space transformation with

$$T_b := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$, $Q = R^T R$ (e.g., Cholesky decompositions) and $SR^T = U \Sigma V^T$ is the SVD of SR^T.

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Proof. Exercise!

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$$(\hat{A}, \hat{B}, \hat{C}, D) = (TAT^{-1}, TB, CT^{-1}, D)$$

be any transformed realization with associated controllability Lyapunov equation

$$0 = \hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T = TAT^{-1}\hat{P} + \hat{P}T^{-T}A^TT^T + TBB^TT^T.$$

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The uniqueness of the solution of the Lyapunov equation implies that $\hat{P} = TPT^T$ and, analogously, $\hat{Q} = T^{-T}QT^{-1}$. Therefore,

$$\hat{P}\hat{Q} = TPQT^{-1},$$

showing that $\Lambda(\hat{P}\hat{Q}) = \Lambda(PQ) = \{\sigma_1^2, \ldots, \sigma_n^2\}$.

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Remark

For non-minimal systems, the Gramians can also be transformed into diagonal matrices with the leading $\hat{n} \times \hat{n}$ submatrices equal to $\mathrm{diag}(\sigma_1, \ldots, \sigma_{\hat{n}})$, and

$$\hat{P}\hat{Q} = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_{\hat{n}}^2, 0, \ldots, 0).$$

see [Laub/Heath/Paige/Ward 1987, Tombs/Postlethwaite 1987].

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Basic principle:

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Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC \; = \; 0.$
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  Proof: Recall Hankel operator

$$y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau = Ce^{At}z.$$

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$$\Longleftrightarrow \qquad PQz = \sigma^2 z. \quad \square$$

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- An LTI system Σ, realized by (A, B, C, D), is called balanced, if the Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right) \end{aligned}$$

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$.

Motivation:

HSVs are system invariants: they are preserved under
$\mathcal{T} : (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D)$:

in transformed coordinates, the Gramians satisfy

$$\begin{aligned}(TAT^{-1})(TPT^T) + (TPT^T)(TAT^{-1})^T + (TB)(TB)^T &= 0,\\ (TAT^{-1})^T(T^{-T}QT^{-1}) + (T^{-T}QT^{-1})(TAT^{-1}) + (CT^{-1})^T(CT^{-1}) &= 0\end{aligned}$$

$$\Rightarrow \ (TPT^T)(T^{-T}QT^{-1}) = TPQT^{-1},$$

hence $\Lambda(PQ) = \Lambda((TPT^T)(T^{-T}QT^{-1}))$.

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HSVs determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

In balanced coordinates ... energy transfer from u_- to y_+:

$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int_0^\infty y(t)^T y(t)\, dt}{\int_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2$$

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$\Longrightarrow$ Truncate states corresponding to “small” HSVs

$\Longrightarrow$ complete analogy to best approximation via SVD!

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1. Compute (Cholesky) factors of the Gramians, $P = S^T S$, $Q = R^T R$.

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$\Longrightarrow VW^T$ is an oblique projector, hence balanced truncation is a Petrov-Galerkin projection method.

Properties:

- Reduced-order model is minimal (controllable and observable) and stable with HSVs $\sigma_1, \ldots, \sigma_r$.

Balanced Truncation

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- Reduced-order model is minimal (controllable and observable) and stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right) \|u\|_2 .$$

Properties:

General misconception: complexity $\mathcal{O}(n^3)$ – true for several implementations! (e.g., MATLAB, SLICOT).

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Use low-rank techniques ideas from numerical linear algebra:

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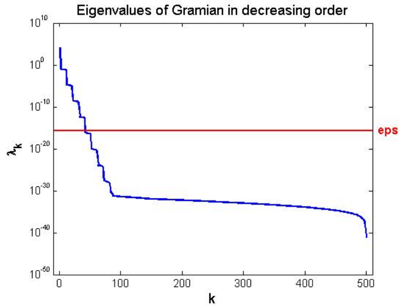
Use low-rank techniques ideas from numerical linear algebra:

- Instead of Gramians P, Q compute $S, R \in \mathbb{R}^{n \times k}$, $k \ll n$, such that

$$P \approx SS^T, \quad Q \approx RR^T.$$

- Compute S, R with problem-specific Lyapunov solvers of "low" complexity directly.



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Use low-rank techniques ideas from numerical linear algebra:

Sparse Balanced Truncation:

- Implementation using sparse Lyapunov solver ($\rightarrow$ ADI+sparse LU).
- Complexity $\mathcal{O}(n(k^2 + r^2))$.
- Software:
  + MATLAB toolbox LyaPack (Penzl 1999),
  + Software library M.E.S.S.[a] in C/MATLAB [B./Saak/Köhler/uvm.],
  + pyMOR.

[a] Matrix Equation Sparse Solvers

Recall Peaceman-Rachford ADI:

Consider $Au = s$ where $A \in \mathbb{R}^{n\times n}$ spd, $s \in \mathbb{R}^n$.

ADI iteration idea: decompose $A = H + V$ with $H, V \in \mathbb{R}^{n\times n}$ such that

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\begin{aligned}
(H + pI)v &= r\\
(V + pI)w &= t
\end{aligned}
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can be solved easily/efficiently.

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ADI Iteration

If H, V spd $\Rightarrow \exists p_k,\ k = 1, 2, \ldots,$ such that

$$
\begin{array}{rcl}
u_0 & = & 0 \\
(H + p_k I)u_{k-\frac{1}{2}} & = & (p_k I - V)u_{k-1} + s \\
(V + p_k I)u_k & = & (p_k I - H)u_{k-\frac{1}{2}} + s
\end{array}
$$

converges to $u \in \mathbb{R}^n$ solving $Au = s$.

The (linear) Lyapunov operator

$$\mathcal{L}: \quad X \quad \mapsto \quad AX + XA^T$$

can be decomposed into the linear operators

$$\mathcal{L}_H : X \mapsto AX, \qquad \mathcal{L}_V : X \mapsto XA^T.$$

In analogy to the standard ADI method we find the

ADI iteration for the Lyapunov equation **[Wachspress 1988]**

$$\begin{aligned} X_0 &= 0, \\ (A + p_k I)X_{k-\frac{1}{2}} &= -W - X_{k-1}(A^T - p_k I), \\ (A + p_k I)X_k^T &= -W - X_{k-\frac{1}{2}}^T(A^T - p_k I). \end{aligned}$$

Consider $AX + XA^T = -BB^T$ for stable A, $B \in \mathbb{R}^{n \times m}$ with $m \ll n$.

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For $k = 1, \ldots, k_{\max}$

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Rewrite as one step iteration and factorize $X_k = Z_k Z_k^T$, $k = 0, \ldots, k_{\max}$

$$\begin{array}{rcl} Z_0 Z_0^T & = & 0 \\ Z_k Z_k^T & = & -2p_k (A + p_k I)^{-1} BB^T (A + p_k I)^{-T} \\ & & +(A + p_k I)^{-1}(A - p_k I) Z_{k-1} Z_{k-1}^T (A - p_k I)^T (A + p_k I)^{-T} \end{array}$$

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\end{aligned}
$$

$\ldots \rightsquigarrow$ low-rank Cholesky factor ADI [Penzl 1997/2000, Li/White 1999/2002, B./Li/Penzl 1999/2008, Gugercin/Sorensen/Antoulas 2003]

$$Z_k = [\sqrt{-2p_k}(A + p_k I)^{-1}B,\ (A + p_k I)^{-1}(A - p_k I)Z_{k-1}]$$ [Penzl 2000]

$Z_k = [\sqrt{-2p_k}(A + p_k I)^{-1}B,\ (A + p_k I)^{-1}(A - p_k I)Z_{k-1}]$ [Penzl 2000]

Observing that $(A - p_i I)$, $(A + p_k I)^{-1}$ commute, we rewrite $Z_{k_{\max}}$ as

$Z_{k_{\max}} = [z_{k_{\max}},\ P_{k_{\max}-1}z_{k_{\max}},\ P_{k_{\max}-2}(P_{k_{\max}-1}z_{k_{\max}}),\ \ldots,\ P_1(P_2 \cdots P_{k_{\max}-1}z_{k_{\max}})]$,

where

$$z_{k_{\max}} = \sqrt{-2p_{k_{\max}}}(A + p_{k_{\max}} I)^{-1}B$$

and

$$P_i := \frac{\sqrt{-2p_i}}{\sqrt{-2p_{i+1}}}\left[I - (p_i + p_{i+1})(A + p_i I)^{-1}\right].$$

[Li/White 2002]

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[Li/White 2002]

$\rightsquigarrow$ Need to solve only one (sparse) linear system with m right-hand sides per iteration!

Algorithm [Penzl 1997/2000, Li/White 1999/2002, B. 2004, B./Li/Penzl 1999/2008]

$$V_1 \leftarrow \sqrt{-2\,\mathrm{re}\,p_1}(A + p_1 I)^{-1}B, \qquad Z_1 \leftarrow V_1$$

FOR $k = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{re}\,p_k}{\mathrm{re}\,p_{k-1}}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1}V_{k-1}\right)$$

$$Z_k \leftarrow \left[\begin{array}{cc} Z_{k-1} & V_k \end{array}\right]$$

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At convergence, $Z_{k_{\max}} Z_{k_{\max}}^T \approx X$, where (without column compression)

$$Z_{k_{\max}} = \begin{bmatrix} V_1 & \ldots & V_{k_{\max}} \end{bmatrix}, \qquad V_k = \boxed{} \in \mathbb{C}^{n \times m}.$$

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Note: Implementation in real arithmetic is possible: combine two steps [B./Li/Penzl 1999/2008] or employ the relations of consecutive complex factors [B./Kürschner/Saak 2011].

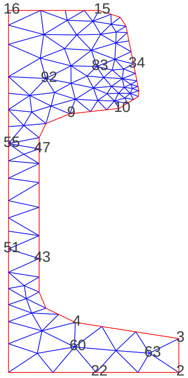
Current implementations (pyMOR, M.E.S.S.) employ low-rank property of residual, update residual in each step, and compute new shifts on the fly!

- Mathematical model: boundary control for linearized 2D heat equation.

$$
\begin{aligned}
c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, && \xi \in \Omega \\
\lambda \frac{\partial}{\partial n} x &= \kappa (u_k - x), && \xi \in \Gamma_k,\ 1 \leq k \leq 7, \\
\frac{\partial}{\partial n} x &= 0, && \xi \in \Gamma_7.
\end{aligned}
$$

$$\Longrightarrow m = 7, p = 6.$$

- FEM Discretization, different models for initial mesh ($n = 371$), 1, 2, 3, 4 steps of mesh refinement $\Rightarrow$ $n = 1357, 5177, 20209, 79841$.

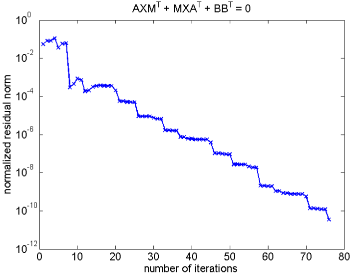


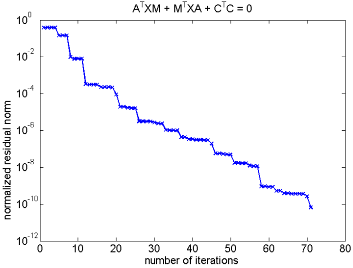
Source: Physical model: courtesy of Mannesmann/Demag.
Math. model: Tröltzsch/Unger 1999/2001, Penzl 1999, Saak 2003.

- Solve dual Lyapunov equations needed for balanced truncation, i.e.,

$$APM^T + MPA^T + BB^T = 0, \quad A^TQM + M^TQA + C^TC = 0,$$

for $n = 79,841$.
- 25 shifts chosen by Penzl heuristic from 50/25 Ritz values of A of largest/smallest magnitude, no column compression performed.
- M.E.S.S. requires no factorization of mass matrix.





Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis range(Z), $Z \in \mathbb{R}^{n\times r}$, for subspace $\mathcal{Z} \subset \mathbb{R}^n$, $\dim \mathcal{Z} = r$.
2. Set $\hat{A} := Z^T AZ$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A}\hat{X} + \hat{X}\hat{A}^T + \hat{B}\hat{B}^T = 0$.
4. Use $X \approx Z\hat{X}Z^T$.

Examples:

- Krylov subspace methods, i.e., for $m = 1$:

$$\mathcal{Z} = \mathcal{K}(A, B, r) = \mathrm{span}\{B, AB, A^2B, \ldots, A^{r-1}B\}$$

[Saad 1990, Jaimoukha/Kasenally 1994, Jbilou 2002–08].

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- Extended (and rational) Krylov method (EKSM, RKSM) [Simoncini 2007, Druskin/Knizhnerman/Simoncini 2011],

$$\mathcal{Z} = \mathcal{K}(A, B, r) \cup \mathcal{K}(A^{-1}, B, r).$$

Projection-based methods for Lyapunov equations with $A + A^T < 0$:

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4. Use $X \approx Z\hat{X}Z^T$.

Examples:

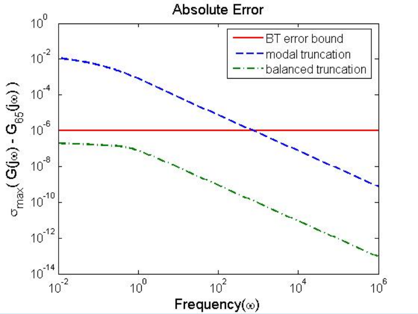
- ADI subspace [B./R.-C. Li/Truhar 2008]:

$$\mathcal{Z} = \mathrm{colspan}\left[\begin{array}{ccc} V_1, & \ldots, & V_r \end{array}\right].$$

  Note:

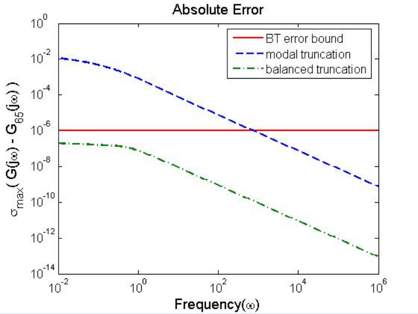
  1. ADI subspace is rational Krylov subspace [J.-R. Li/White 2002].
  2. Similar approach: ADI-preconditioned global Arnoldi method [Jbilou 2008].

$n = 1357$, **Absolute Error**



- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.

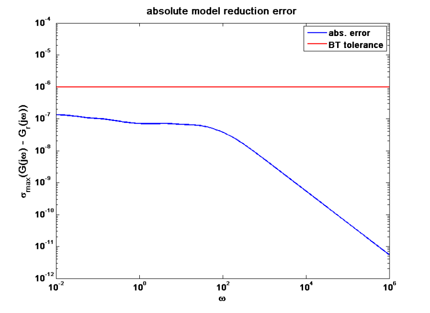
$n = 1357$, Absolute Error



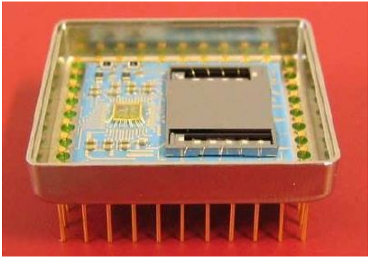
– BT model computed with sign function method,

– MT w/o static condensation, same order as BT model.

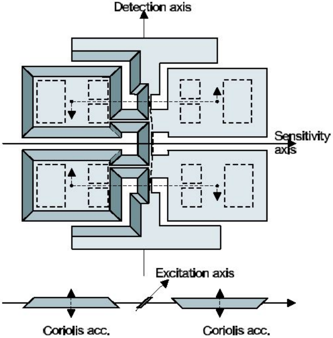
$n = 79841$, Absolute Error



– BT model computed using M-M.E.S.S. in MATLAB,



- Vibrating micro-mechanical gyroscope for inertial navigation.
- Rotational position sensor.



- By applying AC voltage to electrodes, wings are forced to vibrate in anti-phase in wafer plane.
- Coriolis forces induce motion of wings out of wafer plane yielding sensor data.

Source: http://modelreduction.org/index.php/Modified_Gyroscope

- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  $\rightsquigarrow$ $n = 34,722$, $m = 1$, $p = 12$.
- Reduced model computed using ADI-based balanced truncation, $r = 30$.

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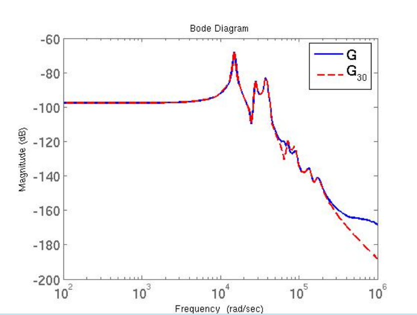
Frequency Repsonse Analysis



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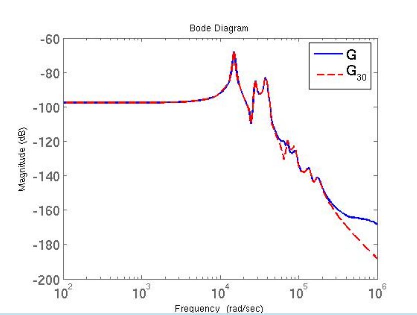
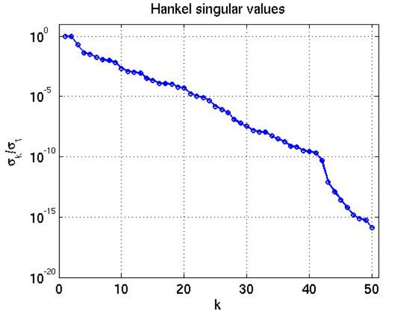


Hankel Singular Values



Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

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Classical Balanced Truncation (BT) [Mullis/Roberts 1976, Moore 1981]

- $P =$ controllability Gramian of system given by (A, B, C, D).
- $Q =$ observability Gramian of system given by (A, B, C, D).
- P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0.$$

Balancing-Related Model Reduction

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Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

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LQG Balanced Truncation (LQGBT)

[Jonckheere/Silverman 1983]

- $P/Q =$ controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= AP + PA^T - PC^T CP + B^T B, \\ 0 &= A^T Q + QA - QBB^T Q + C^T C. \end{aligned}$$

Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Balanced Stochastic Truncation (BST)

[Desai/Pal 1984, Green 1988]

- $P =$ controllability Gramian of system given by (A, B, C, D), i.e., solution of Lyapunov equation $AP + PA^T + BB^T = 0$.
- $Q =$ observability Gramian of right spectral factor of power spectrum of system given by (A, B, C, D), i.e., solution of ARE

$$\hat{A}^T Q + Q\hat{A} + QB_W(DD^T)^{-1}B_W^T Q + C^T(DD^T)^{-1}C = 0,$$

where $\hat{A} := A - B_W(DD^T)^{-1}C$, $B_W := BD^T + PC^T$.

Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Positive-Real Balanced Truncation (PRBT) [Green 1988]

- Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.
- P, Q solve dual AREs

$$\begin{aligned} 0 &= \bar{A}P + P\bar{A}^T + PC^T\bar{R}^{-1}CP + B\bar{R}^{-1}B^T, \\ 0 &= \bar{A}^TQ + Q\bar{A} + QB\bar{R}^{-1}B^TQ + C^T\bar{R}^{-1}C, \end{aligned}$$

where $\bar{R} = D + D^T$, $\bar{A} = A - B\bar{R}^{-1}C$.

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and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [Opdenacker/Jonckheere 1988];
- H_∞ balanced truncation (HinfBT) – closed-loop balancing based on H_∞ compensator [Mustafa/Glover 1991].

Both approaches require solution of dual AREs.

- Frequency-weighted versions of the above approaches.

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  - stability (all),

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- Computable error bounds, e.g.,

$$\text{BT:} \quad \|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_j^{BT},$$

$$\text{LQGBT:} \quad \|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}}$$

$$\text{BST:} \quad \|G - G_r\|_\infty \leq \left(\prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \|G\|_\infty,$$

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$$

- Can be combined with singular perturbation approximation (= Guyan reduction applied to balanced realization!) for improved steady-state performance.

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\end{aligned}$$

- Can be combined with singular perturbation approximation (= Guyan reduction applied to balanced realization!) for improved steady-state performance.
- Computations can be modularized $\rightsquigarrow$ software packages M-M.E.S.S., MORLAB, see http://www.mpi-magdeburg.mpg.de/823508/software.

Outline

Current Research Topics

- Special methods for second-order (mechanical), switched and delay systems.
- Time- and frequency-limited variants.
- Empirical variants using snapshots and integral representation of Gramians.
- Extensions to bilinear, quadratic-bilinear, polynomial, and stochastic systems.
- MOR methods for discrete-time systems.
- Extensions to descriptor systems $E\dot{x} = Ax + Bu$, E singular.
- Parametric model reduction:

$$\dot{x} = A(p)x + B(p)u, \quad y = C(p)x,$$

where $p \in \mathbb{R}^d$ is a free parameter vector; parameters should be preserved in the reduced-order model.

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