SVD-BASED MODEL ORDER REDUCTION

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Methods of Model Order Reduction
Shanghai University
1. Introduction to SVD-based Model Order Reduction

2. Model Reduction by Projection

3. Balanced Truncation

4. Final Remarks
1. Introduction to SVD-based Model Order Reduction
   - Model Reduction for Dynamical Systems
   - Motivation for SVD-based Methods
   - SVD-based MOR for LTI Systems

2. Model Reduction by Projection

3. Balanced Truncation

4. Final Remarks
Model Reduction for Dynamical Systems

Dynamical Systems

\[
\Sigma : \begin{cases} 
\dot{x}(t) &= f(t, x(t), u(t)), \quad x(t_0) = x_0, \\
y(t) &= g(t, x(t), u(t))
\end{cases}
\]

with

- **states** \(x(t) \in \mathbb{R}^n\),
- **inputs** \(u(t) \in \mathbb{R}^m\),
- **outputs** \(y(t) \in \mathbb{R}^p\).
Model Reduction for Dynamical Systems

Original System

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y(t) = g(t, x(t), u(t)). \end{cases} \]

- states \( x(t) \in \mathbb{R}^n \),
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Goal:

\[ \|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \text{ for all admissible input signals.} \]
Model Reduction for Dynamical Systems

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**Reduced-Order System**

\[ \hat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \hat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \hat{g}(t, \hat{x}(t), u(t)) \end{cases} \]

- states \( \hat{x}(t) \in \mathbb{R}^r, r \ll n \),
- inputs \( u(t) \in \mathbb{R}^m \),
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**Goal:**

\[ \| y - \hat{y} \| < \text{tolerance} \cdot \| u \| \text{ for all admissible input signals.} \]
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- States \( \hat{x}(t) \in \mathbb{R}^r, r \ll n \)
- Inputs \( u(t) \in \mathbb{R}^m \)
- Outputs \( \hat{y}(t) \in \mathbb{R}^p \).

Goal:
\[ \|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \text{ for all admissible input signals.} \]

Secondary goal: reconstruct approximation of \( x \) from \( \hat{x} \).
Linear, Time-Invariant (LTI) Systems

\[ \Sigma : \begin{cases} 
    \dot{x} &= Ax + Bu, \\
    y &= Cx + Du,
\end{cases} \quad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \quad C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times m}. \]

Assumptions: \( t_0 = 0, \ x_0 = x(0) = 0. \)

Laplace Transform / Frequency Domain

Application of Laplace transform

\[ \mathcal{L} : x(t) \mapsto x(s) = \int_0^\infty e^{-st} x(t) \, dt \quad (\Rightarrow \ \dot{x}(t) \mapsto sx(s)) \]

with \( s \in \mathbb{C} \) leads to linear system of equations:

\[ sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s). \]
Linear Systems in Frequency Domain

Linear, Time-Invariant (LTI) Systems

\[ \Sigma : \begin{cases} \dot{x} &= Ax + Bu, & A \in \mathbb{R}^{n \times n}, & B \in \mathbb{R}^{n \times m}, \\ y &= Cx + Du, & C \in \mathbb{R}^{p \times n}, & D \in \mathbb{R}^{p \times m}. \end{cases} \]

Assumptions: \( t_0 = 0, \ x_0 = x(0) = 0. \)

Laplace Transform / Frequency Domain

\[ sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s) \]

yields I/O-relation in frequency domain:

\[ y(s) = \left( C(sI_n - A)^{-1}B + D \right)u(s) = G(s)u(s). \]

\( G \) is the transfer function of \( \Sigma \), \( G : \mathcal{L}_2^m \to \mathcal{L}_2^p \) \((\mathcal{L}_2 := \mathcal{L}(L_2(-\infty, \infty)))\).
Model Order Reduction Problem

Approximate the dynamical system

\[ \dot{x} = Ax + Bu, \quad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \]
\[ y = Cx + Du, \quad C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times m}. \]

by reduced-order system

\[ \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u, \quad \hat{A} \in \mathbb{R}^{r \times r}, \quad \hat{B} \in \mathbb{R}^{r \times m}, \]
\[ \hat{y} = \hat{C}\hat{x} + \hat{D}u, \quad \hat{C} \in \mathbb{R}^{p \times r}, \quad \hat{D} \in \mathbb{R}^{p \times m}. \]

of order \( r \ll n \), such that

\[ \|y - \hat{y}\| = \|Gu - \hat{Gu}\| \leq \|G - \hat{G}\| \|u\| \leq \text{tolerance} \cdot \|u\|. \]
Model Order Reduction Problem

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\dot{x} = Ax + Bu, \quad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \\
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of order \( r \ll n \), such that

\[
\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \|u\| \leq \text{tolerance} \cdot \|u\|.
\]

\[\implies\] Approximation problem:

\[
\min_{\text{order } (\hat{G}) \leq r} \|G - \hat{G}\| \quad \text{for } \mathcal{H}_2/\mathcal{H}_\infty \text{ norm.}
\]
A digital image with \( n_x \times n_y \) pixels can be represented as matrix 
\( X \in \mathbb{R}^{n_x \times n_y} \), where \( x_{ij} \) contains color information of pixel \((i,j)\).

Memory: \( 4 \cdot n_x \cdot n_y \) bytes.
A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where $x_{ij}$ contains color information of pixel $(i, j)$.

Memory: $4 \cdot n_x \cdot n_y$ bytes.

**Theorem: (Schmidt-Mirsky/Eckart-Young)**

Best rank-$r$ approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$\hat{X} = \sum_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U \Sigma V^T$ is the singular value decomposition (SVD) of $X$. The approximation error is $\|X - \hat{X}\|_2 = \sigma_{r+1}$. 
A digital image with \( n_x \times n_y \) pixels can be represented as matrix \( X \in \mathbb{R}^{n_x \times n_y} \), where \( x_{ij} \) contains color information of pixel \((i,j)\).

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The approximation error is \( \|X - \hat{X}\|_2 = \sigma_{r+1} \).

**Idea for dimension reduction**

Instead of \( X \) save \( u_1, \ldots, u_r, \sigma_1 v_1, \ldots, \sigma_r v_r \).

\(~\rightarrow\) memory = \( 4r \times (n_x + n_y) \) bytes.
Motivation for SVD-based Methods
Example: Image Compression by Truncated SVD

Example: Clown

Original image

320 × 200 pixel
⇽ ≈ 256 kb
Motivation for SVD-based Methods
Example: Image Compression by Truncated SVD

Example: Clown

320 × 200 pixel
⇒ ≈ 256 kb

rank $r = 50$, ≈ 104 kb
Motivation for SVD-based Methods

Example: Image Compression by Truncated SVD

Example: Clown

- rank $r = 50$, $\approx 104$ kb
- rank $r = 20$, $\approx 42$ kb

$320 \times 200$ pixel
$\approx \approx 256$ kb
Example: Gatlinburg

Organizing committee

640 × 480 pixel, ≈ 1229 kb
Motivation for SVD-based Methods
Dimension Reduction via SVD

Example: Gatlinburg

Organizing committee
Gatlinburg/Householder Meeting 1964:
James H. Wilkinson, Wallace Givens,
George Forsythe, Alston Householder,
Peter Henrici, Fritz L. Bauer.

- rank $r = 100$, $\approx 448$ kb
- rank $r = 50$, $\approx 224$ kb

640 $\times$ 480 pixel, $\approx 1229$ kb
Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices
Linear, Time-Invariant (LTI) Systems

\[
\begin{align*}
\dot{x} &= f(t, x, u) = Ax + Bu, & A &\in \mathbb{R}^{n \times n}, & B &\in \mathbb{R}^{n \times m}, \\
y &= g(t, x, u) = Cx + Du, & C &\in \mathbb{R}^{p \times n}, & D &\in \mathbb{R}^{p \times m}.
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Assumptions (for now): \( t_0 = 0, \ x_0 = x(0) = 0, \ D = 0. \)
### SVD-based MOR for LTI Systems

**Linear, Time-Invariant (LTI) Systems**

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\]

**State-Space Description for I/O-Relation**

Variation-of-constants \( \implies \)

\[
S : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)} Bu(\tau) \, d\tau \quad \text{for all } t \in \mathbb{R}.
\]
Linear, Time-Invariant (LTI) Systems

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\( S : \mathcal{U} \rightarrow \mathcal{Y} \) is a linear operator between (function) spaces.
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- Recall: matrix in \(\mathbb{R}^{n \times m}\) is a linear operator, mapping \(\mathbb{R}^{m} \to \mathbb{R}^{n}\)!
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- Basic Idea: use SVD approximation as for matrix \(A\)!
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- Recall: matrix in \(\mathbb{R}^{n \times m}\) is a linear operator, mapping \(\mathbb{R}^m \rightarrow \mathbb{R}^n\)!
- Basic Idea: use SVD approximation as for matrix \(A\)!
- Problem: in general, \(S\) does not have a discrete SVD and can therefore not be approximated as in the matrix case!
Linear, Time-Invariant (LTI) Systems

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Alternative to State-Space Operator: Hankel operator

Instead of

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S : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)} Bu(\tau) \, d\tau \quad \text{for all } t \in \mathbb{R}.
\]

use Hankel operator

\[
\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau) \, d\tau \quad \text{for all } t > 0.
\]
## Linear, Time-Invariant (LTI) Systems

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\(\mathcal{H}\) compact, finite-dimensional \(\Rightarrow\) \(\mathcal{H}\) has discrete SVD

\(\sim\) Hankel singular values \(\{\sigma_j\}_{j=1}^{\infty} : \sigma_1 \geq \ldots \geq \sigma_n \geq \sigma_{n+1} = 0 = \ldots = 0.\)
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\(~\sim Hankel singular values\quad \{\sigma_j\}_{j=1}^{\infty} : \sigma_1 \geq \ldots \geq \sigma_n \geq \sigma_{n+1} = 0 = \ldots = 0.\)

\(\Rightarrow\) SVD-type approximation of \(\mathcal{H}\) possible!
Linear, Time-Invariant (LTI) Systems

\[ \dot{x} = Ax + Bu, \quad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \]
\[ y = Cx, \quad C \in \mathbb{R}^{p \times n}. \]

Alternative to State-Space Operator: Hankel operator

- \( \mathcal{H} \) compact
- \( \mathcal{H} \) has discrete SVD
- Hankel singular values

Hankel Singular Values for Atmospheric Storm Model

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SVD-based MOR
Linear, Time-Invariant (LTI) Systems

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y = Cx, \quad C \in \mathbb{R}^{p \times n}.
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\( \mathcal{H} \) compact \( \Rightarrow \) \( \mathcal{H} \) has discrete SVD

\( \Rightarrow \) Best approximation problem w.r.t. 2-induced operator norm well-posed
Linear, Time-Invariant (LTI) Systems

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\[ \mathcal{H} \text{ compact} \Rightarrow \mathcal{H} \text{ has discrete SVD} \]

⇒ Best approximation problem w.r.t. 2-induced operator norm well-posed
### Linear, Time-Invariant (LTI) Systems

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### Alternative to State-Space Operator: Hankel operator

The Hankel operator \( \mathcal{H} \) can be defined as

\[
\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} C e^{A(t-\tau)} B u(\tau) \, d\tau \quad \text{for all } t > 0.
\]

If \( \mathcal{H} \) is compact, then \( \mathcal{H} \) has a discrete SVD. This makes the best approximation problem with respect to the 2-induced operator norm well-posed.

Furthermore, the solution to this problem is given by the Adamjan-Arov-Krein (AAK Theory, 1971/78).

However, this can be computationally challenging for large-scale systems.

Recent progress in [B./Werner 2020].
1. Introduction to SVD-based Model Order Reduction

2. Model Reduction by Projection
   - Linear Algebra Basics
   - Projection Basics
   - Extensions

3. Balanced Truncation

4. Final Remarks
Goals

- Automatic generation of compact models.

\[ \| y - \hat{y} \| < \text{tolerance} \cdot \| u \| \quad \forall u \in L^2(\mathbb{R}, \mathbb{R}^m) \]

Need computable error bound/estimate!

Preserve physical properties:
- stability (poles of \( G \) in \( \mathbb{C}^- \)),
- minimum phase (zeroes of \( G \) in \( \mathbb{C}^- \)),
- passivity

\[ \int_{-\infty}^t u(\tau) T y(\tau) \, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \forall u \in L^2(\mathbb{R}, \mathbb{R}^m) \]

"system does not generate energy".
- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want
  \[ \|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m). \]

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Preserve physical properties:
Model Reduction by Projection
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Model Reduction by Projection

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- Preserve physical properties:
  - stability (poles of \( G \) in \( \mathbb{C}^- \)),
  - minimum phase (zeroes of \( G \) in \( \mathbb{C}^- \)),
  - passivity

  \[ \int_{-\infty}^{t} u(\tau)^T y(\tau) d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m). \]

  ("system does not generate energy").
A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. 

Let $V = \text{range}(P)$, then $P$ is a projector onto $V$. 

If \{ $v_1, ..., v_r$ \} is a basis of $V$ and $V = [v_1, ..., v_r]$, then $P = V (V^T V)^{-1} V^T$ is a projector onto $V$. 

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- If \( \mathcal{V} \) is an \( A \)-invariant subspace corresponding to a subset of \( A \)'s spectrum, then we call \( P \) a spectral projector.
- Let \( \mathcal{W} \subset \mathbb{R}^n \), \( \dim \mathcal{W} = r \), with basis matrix \( W = [w_1, \ldots, w_r] \), then \( P = V(W^TV)^{-1}W^T \) is an oblique projector onto \( \mathcal{V} \) along \( \mathcal{W} \).
Methods:

1. Modal Truncation
2. Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods)
3. Balanced Truncation
4. many more...

Joint feature of these methods: computation of reduced-order model (ROM) by projection!
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Assume trajectory \( x(t; u) \) is contained in low-dimensional subspace \( \mathcal{V} \). Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto \( \mathcal{V} \) along complementary subspace \( \mathcal{W} \): \( x \approx \mathcal{V} \mathcal{W}^T x =: \hat{x} \), where

\[
\text{range}(\mathcal{V}) = \mathcal{V}, \quad \text{range}(\mathcal{W}) = \mathcal{W}, \quad \mathcal{W}^T \mathcal{V} = I_r.
\]

Then, with \( \hat{x} = \mathcal{W}^T x \), we obtain \( x \approx \mathcal{V} \hat{x} \) so that

\[
\|x - \hat{x}\| = \|x - \mathcal{V} \hat{x}\|,
\]

and the reduced-order model is

\[
\hat{A} := \mathcal{W}^T \mathcal{A} \mathcal{V}, \quad \hat{B} := \mathcal{W}^T \mathcal{B}, \quad \hat{C} := \mathcal{C} \mathcal{V}, \quad (\hat{D} := D).
\]
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Important observation:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$W^T (\dot{\tilde{x}} - A\tilde{x} - Bu) = W^T (VW^T \dot{x} - AVW^T x - Bu)$$
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\]
Base enrichment

**Static modes** are defined by setting $\dot{x} = 0$ and assuming unit loads, i.e., $u(t) \equiv e_j$, $j = 1, \ldots, m$:

$$0 = Ax(t) + Be_j \implies x(t) \equiv -A^{-1}b_j.$$  

Projection subspace $\mathcal{V}$ is then augmented by $A^{-1}[b_1, \ldots, b_m] = A^{-1}B$.

**Interpolation-projection framework** $\implies G(0) = \hat{G}(0)$!

If two-sided projection is used, complimentary subspace can be augmented by $A^{-T}C^T \implies G'(0) = \hat{G}'(0)$!

Note: if $m \neq q$, add random vectors or delete some of the columns in $A^{-T}C^T$. 

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SVD-based MOR
Guyan reduction (static condensation)

Partition states in masters \( x_1 \in \mathbb{R}^r \) and slaves \( x_2 \in \mathbb{R}^{n-r} \) (FEM terminology)

Assume stationarity, i.e., \( \dot{x} = 0 \) and solve for \( x_2 \) in

\[
\begin{bmatrix}
0 \\
\end{bmatrix} = 
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22} \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix} + 
\begin{bmatrix}
B_1 \\
B_2 \\
\end{bmatrix} u
\]

\[
\Rightarrow x_2 = -A^{-1}_{22}A_{21}x_1 - A^{-1}_{22}B_2 u.
\]

Inserting this into the first part of the dynamic system

\[
\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u,
\]

\[
y = C_1x_1 + C_2x_2
\]

then yields the reduced-order model

\[
\dot{x}_1 = (A_{11} - A_{12}A^{-1}_{22}A_{21})x_1 + (B_1 - A_{12}A^{-1}_{22}B_2) u
\]

\[
y = (C_1 - C_2A^{-1}_{22}A_{21})x_1 - C_2A^{-1}_{22}B_2 u.
\]
1. Introduction to SVD-based Model Order Reduction

2. Model Reduction by Projection

3. Balanced Truncation
   Balanced Realizations
   The basic method
   ADI Methods for Lyapunov Equations
   Balancing-Related Model Reduction

4. Final Remarks
Definition

A realization \((A, B, C, D)\) of a linear system \(\Sigma\) is balanced if its infinite controllability/observability Gramians \(P/Q\) satisfy

\[
P = Q = \text{diag}\{\sigma_1, \ldots, \sigma_n\} \quad (\text{w.l.o.g. } \sigma_j \geq \sigma_{j+1}, \ j = 1, \ldots, n-1).
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When does a balanced realization exist?

Assume \(A\) to be Hurwitz, i.e. \(\Lambda(A) \subset \mathbb{C}^-\). Then:

Theorem

Given a stable minimal linear system \(\Sigma : (A, B, C, D)\), a balanced realization is obtained by the state-space transformation with

\[
T_b := \Sigma^{-\frac{1}{2}} V^T R,
\]

where \(P = S^T S, \ Q = R^T R\) (e.g., Cholesky decompositions) and \(SR^T = U\Sigma V^T\) is the SVD of \(SR^T\).
### Definition

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\(\sigma_1, \ldots, \sigma_n\) are the **Hankel singular values** of \(\Sigma\).

**Note:** \(\sigma_1, \ldots, \sigma_n \geq 0\) as \(P, Q \geq 0\) by definition, and \(\sigma_1, \ldots, \sigma_n > 0\) in case of minimality!
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Proof. Exercise!
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Theorem

The Hankel singular values (HSV$s$) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

Proof. In balanced coordinates, the HSV$s$ are $\Lambda (PQ)^{1/2}$. Now let

$$(\hat{A}, \hat{B}, \hat{C}, D) = (TAT^{-1}, TB, CT^{-1}, D)$$

be any transformed realization with associated controllability Lyapunov equation

$$0 = \hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T = TAT^{-1}\hat{P} + \hat{P}T^{-T}A^TT^T + TBB^TT^T.$$
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\]

This is equivalent to

\[
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\]

The uniqueness of the solution of the Lyapunov equation implies that \( \hat{P} = TPT^T \) and, analogously, \( \hat{Q} = T^{-T}QT^{-1} \). Therefore,

\[
\hat{P}\hat{Q} = TPQT^{-1},
\]

showing that \( \Lambda (\hat{P}\hat{Q}) = \Lambda (PQ) = \{\sigma_1^2, \ldots, \sigma_n^2\} \).
Definition

A realization \((A, B, C, D)\) of a stable linear system \(\Sigma\) is balanced if its infinite controllability/observability Gramians \(P/Q\) satisfy

\[ P = Q = \text{diag}\{\sigma_1, \ldots, \sigma_n\} \quad (\text{w.l.o.g. } \sigma_j \geq \sigma_{j+1}, \, j = 1, \ldots, n - 1). \]

\(\sigma_1, \ldots, \sigma_n\) are the Hankel singular values of \(\Sigma\).

Note: \(\sigma_1, \ldots, \sigma_n \geq 0\) as \(P, Q \geq 0\) by definition, and \(\sigma_1, \ldots, \sigma_n > 0\) in case of minimality!

Remark

For non-minimal systems, the Gramians can also be transformed into diagonal matrices with the leading \(\hat{n} \times \hat{n}\) submatrices equal to \(\text{diag}(\sigma_1, \ldots, \sigma_{\hat{n}})\), and

\[ \hat{P} \hat{Q} = \text{diag}(\sigma_1^2, \ldots, \sigma_{\hat{n}}^2, 0, \ldots, 0). \]

see [Laub/Heath/Paige/Ward 1987, Tombs/Postlethwaite 1987].
Balanced Truncation

Basic principle:

An LTI system $\Sigma$, realized by $(A, B, C, D)$, is called balanced, if the Gramians, i.e., solutions $P, Q$ of the Lyapunov equations

\[
AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC = 0,
\]

satisfy: $P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$. 
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- $\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the **Hankel singular values (HSV)s** of $\Sigma$. 
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**Proof:** Recall Hankel operator

\[
y(t) = H u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau) \, d\tau
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\]

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SVD-based MOR
Balanced Truncation

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Hankel singular values = square roots of eigenvalues of \( \mathcal{H}^*\mathcal{H} \),

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y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau) \, d\tau = Ce^{At} z.
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Hankel singular values = square roots of eigenvalues of \( \mathcal{H}^*\mathcal{H} \),

\[
\mathcal{H}^*y(t) = B^T e^{-A^T t} \int_{0}^{\infty} e^{A^T \tau} C^T y(\tau) \, d\tau.
\]

Hence,

\[
\mathcal{H}^*\mathcal{H}u(t) = B^T e^{-A^T t} \int_{0}^{\infty} e^{A^T \tau} C^T Ce^{A\tau} z \, d\tau
\]
Balanced Truncation

Basic principle:

- Lyapunov eqns.: \( AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0. \)
- \( \Lambda (PQ)^{1/2} = \{\sigma_1, \ldots, \sigma_n\} \) are the Hankel singular values (HSV) of \( \Sigma \).

Proof: Recall Hankel operator

\[
y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau) \, d\tau = Ce^{At} z.
\]

Hankel singular values = square roots of eigenvalues of \( \mathcal{H}^*\mathcal{H} \),

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\]

Hence,

\[
\mathcal{H}^* \mathcal{H} u(t) = B^T e^{-A^T t} \int_{0}^{\infty} e^{A^T \tau} C^T Ce^{A^T} z \, d\tau
\]

\[
= B^T e^{-A^T t} \left[ \int_{0}^{\infty} e^{A^T \tau} C^T Ce^{A^T} \, d\tau \right] z \\
\equiv Q
\]
Balanced Truncation

**Basic principle:**

- Lyapunov eqns.: \[ AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0. \]
- \( \Lambda (PQ)^{1/2} = \{\sigma_1, \ldots, \sigma_n\} \) are the Hankel singular values (HSV) of \( \Sigma \).

**Proof:** Recall Hankel operator

\[ y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau) d\tau = Ce^{At} z. \]

Hankel singular values = square roots of eigenvalues of \( \mathcal{H}^* \mathcal{H} \),

\[ \mathcal{H}^* y(t) = B^T e^{-A^T t} \int_{0}^{\infty} e^{A^T \tau} C^T y(\tau) d\tau. \]

Hence,

\[ \mathcal{H}^* \mathcal{H} u(t) = B^T e^{-A^T t} \int_{0}^{\infty} e^{A^T \tau} C^T Ce^{A\tau} z \ d\tau \]

\[ = B^T e^{-A^T t} Qz \]
### Balanced Truncation

**Basic principle:**

- Lyapunov eqns.: \( AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0. \)
- \( \Lambda \left( PQ \right)^{1/2} = \{ \sigma_1, \ldots, \sigma_n \} \) are the Hankel singular values (HSV) of \( \Sigma \).

**Proof:** Recall Hankel operator

\[
y(t) = \mathcal{H} u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau) \, d\tau = Ce^{At} z.
\]

Hence, Hankel singular values = square roots of eigenvalues of \( \mathcal{H}^* \mathcal{H} \),

\[
\mathcal{H}^* y(t) = B^T e^{-A^T t} \int_0^\infty e^{A^T \tau} C^T y(\tau) \, d\tau.
\]

Hence,

\[
\mathcal{H}^* \mathcal{H} u(t) = B^T e^{-A^T t} Qz.
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Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0$, $A^T Q + QA + C^T C = 0$.
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Proof: Recall Hankel operator

$$y(t) = \mathcal{H} u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau) \, d\tau = Ce^{At} z.$$  

Hankel singular values = square roots of eigenvalues of $\mathcal{H}^* \mathcal{H}$,

$$\mathcal{H}^* y(t) = B^T e^{-At} \int_{0}^{\infty} e^{At} C^T y(\tau) \, d\tau.$$  

Hence,

$$\mathcal{H}^* \mathcal{H} u(t) = B^T e^{-At} Qz \cong \sigma^2 u(t).$$
Balanced Truncation

Basic principle:

- Lyapunov eqns.: \( AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC = 0. \)
- \( \Lambda(PQ)^{1/2} = \{\sigma_1, \ldots, \sigma_n\} \) are the Hankel singular values (HSVs) of \( \Sigma \).

Proof: Hankel singular values = square roots of eigenvalues of \( \mathcal{H}^*\mathcal{H} \),

\[
\mathcal{H}^*\mathcal{H}u(t) = B^Te^{-A^Tt}Qz = \sigma^2 u(t).
\]

\[ \implies u(t) = \frac{1}{\sigma^2} B^Te^{-A^Tt}Qz \]
Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0$, $A^T Q + QA + C^T C = 0$.
- $\Lambda (PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSV) of $\Sigma$.

Proof: Hankel singular values $= \sqrt{\text{eigenvalues of } H^* H}$,

\[
H^* H u(t) = B^T e^{-A^T t} Q z = \sigma^2 u(t).
\]

$\implies u(t) = \frac{1}{\sigma^2} B^T e^{-A^T t} Q z \implies (\text{recalling } z = \int_{-\infty}^{0} e^{-A\tau} B u(\tau) d\tau)$
Basic principle:

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\[
z = \int_{-\infty}^0 e^{-A^T \tau} B \frac{1}{\sigma^2} B^T e^{-A^T \tau} Qz d\tau
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**Basic principle:**

- Lyapunov eqns.: $AP + PA^T + BB^T = 0$, $A^TQ + QA + C^TC = 0$.
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**Proof:** Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$
\mathcal{H}^*\mathcal{H} u(t) = B^T e^{-A^Tt} Qz \doteq \sigma^2 u(t).
$$

$$
\implies u(t) = \frac{1}{\sigma^2} B^T e^{-A^Tt} Qz 
\implies (\text{recalling } z = \int_{-\infty}^{0} e^{-A\tau} Bu(\tau) d\tau)
$$

$$
z = \int_{-\infty}^{0} e^{-A\tau} B \frac{1}{\sigma^2} B^T e^{-A^T\tau} Qz d\tau
$$

$$
= \frac{1}{\sigma^2} \int_{-\infty}^{0} e^{-A\tau} BB^T e^{-A^T\tau} d\tau Qz
$$
Balanced Truncation

Basic principle:

- Lyapunov eqns.: \( AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0 \).
- \( \Lambda (PQ)^{1/2} = \{ \sigma_1, \ldots, \sigma_n \} \) are the Hankel singular values (HSVs) of \( \Sigma \).

Proof: Hankel singular values = square roots of eigenvalues of \( \mathcal{H}^* \mathcal{H} \),

\[
\mathcal{H}^* \mathcal{H} u(t) = B^T e^{-A^T t} Qz \equiv \sigma^2 u(t).
\]

\( \implies \) \( u(t) = \frac{1}{\sigma^2} B^T e^{-A^T t} Qz \implies \) (recalling \( z = \int_{-\infty}^{0} e^{-A \tau} B u(\tau) \, d\tau \))

\[
z = \int_{-\infty}^{0} e^{-A \tau} B \frac{1}{\sigma^2} B^T e^{-A^T \tau} Qz \, d\tau
\]

\[
= \frac{1}{\sigma^2} \int_{-\infty}^{0} e^{-A \tau} B B^T e^{-A^T \tau} \, d\tau \, Qz
\]

\[
= \frac{1}{\sigma^2} \int_{0}^{\infty} e^{A t} B B^T e^{A^T t} \, dt \, Qz
\]

\( \equiv P \)
### Basic principle:

- **Lyapunov eqns.**:
  
  \[ AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0. \]

- \( \Lambda (PQ)^{1/2} = \{ \sigma_1, \ldots, \sigma_n \} \) are the Hankel singular values (HSV) of \( \Sigma \).

**Proof**: Hankel singular values \( = \) square roots of eigenvalues of \( \mathcal{H}^* \mathcal{H} \),

\[
\mathcal{H}^* \mathcal{H} u(t) = B^T e^{-A^T t} Qz = \sigma^2 u(t).
\]

\( \implies u(t) = \frac{1}{\sigma^2} B^T e^{-A^T t} Qz \implies (\text{recalling } z = \int_{-\infty}^{0} e^{-A \tau} B u(\tau) d\tau) \)

\[
z = \int_{-\infty}^{0} e^{-A \tau} B \frac{1}{\sigma^2} B^T e^{-A^T \tau} Qz d\tau
\]

\[
= \frac{1}{\sigma^2} \int_{0}^{\infty} e^{At} BB^T e^{A^T t} dt \quad Qz \quad \equiv P
\]

\[= \frac{1}{\sigma^2} PQz \]
Balanced Truncation

Basic principle:

- Lyapunov eqns.: \( AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0 \).
- \( \Lambda (PQ)^{\frac{1}{2}} = \{ \sigma_1, \ldots, \sigma_n \} \) are the Hankel singular values (HSV)s of \( \Sigma \).

**Proof:** Hankel singular values = square roots of eigenvalues of \( H^*H \),

\[
H^*Hu(t) = B^T e^{-A^Tt}Qz = \sigma^2 u(t).
\]

\[u(t) = \frac{1}{\sigma^2} B^T e^{-A^Tt}Qz \implies (\text{recalling } z = \int_{-\infty}^{0} e^{-A\tau} B u(\tau) d\tau)\]

\[
z = \int_{-\infty}^{0} e^{-A\tau} B \frac{1}{\sigma^2} B^T e^{-A^T\tau}Qz d\tau
\]

\[
= \frac{1}{\sigma^2} \int_{0}^{\infty} e^{At} BB^T e^{A^Tt} dt \cdot Qz
\]

\[
= \frac{1}{\sigma^2} PQz \tag{\ref{eq:svd-based}}
\]

\[\iff PQz = \sigma^2 z. \square\]
Balanced Truncation

Basic principle:

- An LTI system $\Sigma$, realized by $(A, B, C, D)$, is called balanced, if the Gramians, i.e., solutions $P, Q$ of the Lyapunov equations
  \[
  AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0,
  \]
  satisfy: $P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

- $\Lambda(PQ)^{1/2} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSV) of $\Sigma$.

- Compute balanced realization of the system via state-space transformation

  \[
  \mathcal{T} : (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D)
  \]

  \[
  = \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right)
  \]
## Balanced Truncation

### Basic principle:
- An LTI system $\Sigma$, realized by $(A, B, C, D)$, is called balanced, if the Gramians, i.e., solutions $P, Q$ of the Lyapunov equations
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- $\Lambda(PQ)^{1/2} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSV$s)$ of $\Sigma$.
- Compute balanced realization of the system via state-space transformation
  \[
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  = \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right)
  \]
- Truncation $\leadsto (\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$. 
Motivation:

HSVs are system invariants: they are preserved under
\( \mathcal{T} : (A, B, C, D) \mapsto (\mathcal{T}A\mathcal{T}^{-1}, \mathcal{T}B, \mathcal{T}C\mathcal{T}^{-1}, D) \):

in transformed coordinates, the Gramians satisfy

\[
(TA\mathcal{T}^{-1})(\mathcal{T}P\mathcal{T}^T) + (\mathcal{T}P\mathcal{T}^T)(TA\mathcal{T}^{-1})^T + (TB)(TB)^T = 0,
\]
\[
(TA\mathcal{T}^{-1})^T(T^{-T}Q\mathcal{T}^{-1}) + (T^{-T}Q\mathcal{T}^{-1})(TA\mathcal{T}^{-1}) + (CT^{-1})^T(CT^{-1}) = 0
\]

\[
\Rightarrow (\mathcal{T}P\mathcal{T}^T)(T^{-T}Q\mathcal{T}^{-1}) = \mathcal{T}PQ\mathcal{T}^{-1},
\]

hence \( \Lambda(PQ) = \Lambda((\mathcal{T}P\mathcal{T}^T)(T^{-T}Q\mathcal{T}^{-1})) \).
Motivation:

HSV\textsc{s} are system invariants: they are preserved under 
\( \mathcal{T} : (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D) \).

HSV\textsc{s} determine the energy transfer given by the Hankel map

\[ \mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+ . \]

In balanced coordinates \ldots energy transfer from \( u_- \) to \( y_+ \):

\[ E := \sup_{u \in L_2(-\infty, 0]} \frac{\int_0^\infty y(t)^T y(t) \, dt}{\int_{-\infty}^0 u(t)^T u(t) \, dt} = \frac{1}{\| x_0 \|_2} \sum_{j=1}^n \sigma_j^2 x_0^2, j \]
Motivation:

HSVIs are **system invariants**: they are preserved under 
\[ \mathcal{T} : (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D). \]

HSVIs determine the energy transfer given by the Hankel map

\[ \mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_. \]

In balanced coordinates . . . energy transfer from \( u_- \) to \( y_+ \):

\[
E := \sup_{u \in L_2(-\infty, 0], x(0) = x_0} \int_{-\infty}^{\infty} y(t)^T y(t) \, dt \quad = \quad \frac{1}{\|x_0\|_2^2} \sum_{j=1}^{n} \sigma_j^2 x_0^2, 
\]

\[ \implies \text{Truncate states corresponding to “small” HSVIs} \]

\[ \implies \text{complete analogy to best approximation via SVD!} \]
Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S$, $Q = R^T R$. 
Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, \( P = S^T S, \ Q = R^T R \).

2. Compute SVD \( S R^T = [U_1, U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \).
## Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S$, $Q = R^T R$.

2. Compute SVD $SR^T = [U_1, U_2] \begin{bmatrix} \Sigma_1 & \quad \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$.

3. ROM is $(W^T A V, W^T B, C V, D)$, where

\[
W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \quad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.
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Implementation: SR Method

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Note:

\[
V^T W = (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}})
\]
Balanced Truncation

Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, \( P = S^T S, \ Q = R^T R \).

2. Compute SVD \( SRT = [U_1, U_2] \begin{bmatrix} \Sigma_1 & \ \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \).

3. ROM is \( (W^T AV, W^T B, CV, D) \), where

\[
W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \quad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.
\]

Note:

\[
V^T W = (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}}) = \Sigma_1^{-\frac{1}{2}} U_1^T U \Sigma V^T V_1 \Sigma_1^{-\frac{1}{2}}
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Balanced Truncation

Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, \( P = S^T S, \ Q = R^T R \).

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3. ROM is \((W^T A V, W^T B, C V, D)\), where

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W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \quad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.
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Note:

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V^T W = (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}}) = \Sigma_1^{-\frac{1}{2}} U_1^T U \Sigma V^T V_1 \Sigma_1^{-\frac{1}{2}}
\]

\[
= \Sigma_1^{-\frac{1}{2}} [I_r, 0] \begin{bmatrix} \Sigma_1 & \ \\ \ & \Sigma_2 \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \Sigma_1^{-\frac{1}{2}}
\]
## Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S$, $Q = R^T R$.

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   $$W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \quad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.$$ 

**Note:**

$$V^T W = (\Sigma_1^{-\frac{1}{2}} U_1^T S) (R^T V_1 \Sigma_1^{-\frac{1}{2}}) = \Sigma_1^{-\frac{1}{2}} U_1^T U \Sigma V^T V_1 \Sigma_1^{-\frac{1}{2}}$$

$$= \Sigma_1^{-\frac{1}{2}} [I_r, 0] \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \Sigma_1^{-\frac{1}{2}} = \Sigma_1^{-\frac{1}{2}} \Sigma_1 \Sigma_1^{-\frac{1}{2}} = I_r$$

$\implies V W^T$ is an oblique projector, hence balanced truncation is a Petrov-Galerkin projection method.
Balanced Truncation

Properties:

- Reduced-order model is minimal (controllable and observable) and stable with HSVs $\sigma_1, \ldots, \sigma_r$. 
Properties:

- Reduced-order model is minimal (controllable and observable) and stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- **Adaptive choice of $r$** via computable error bound:

$$
\| y - \hat{y} \|_2 \leq \left( 2 \sum_{k=r+1}^{n} \sigma_k \right) \| u \|_2.
$$
### Properties:

**General misconception:** complexity $O(n^3)$ – true for several implementations! (e.g., MATLAB, SLICOT).
Properties:

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Use low-rank techniques ideas from numerical linear algebra:
Balanced Truncation

Properties:

General misconception: complexity $O(n^3)$ – true for several implementations! (e.g., MATLAB, SLICOT).

Use low-rank techniques ideas from numerical linear algebra:

- Instead of Gramians $P, Q$ compute $S, R \in \mathbb{R}^{n \times k}, k \ll n$, such that
  \[ P \approx SS^T, \quad Q \approx RR^T. \]

- Compute $S, R$ with problem-specific Lyapunov solvers of “low” complexity directly.

Eigenvalues of Gramian in decreasing order

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SVD-based MOR
Balanced Truncation

Properties:

General misconception: complexity $O(n^3)$ – true for several implementations! (e.g., MATLAB, SLICOT).

Use low-rank techniques ideas from numerical linear algebra:

**Sparse Balanced Truncation:**

- Implementation using sparse Lyapunov solver ($\rightarrow$ ADI+sparse LU).
- Complexity $O(n(k^2 + r^2))$.
- Software:
  + MATLAB toolbox LyaPack (Penzl 1999),
  + Software library M.E.S.S.$^a$ in C/MATLAB [B./SaaK/Köhler/uvm.],
  + pyMOR.

---

$^a$Matrix Equation Sparse Solvers
Recall Peaceman-Rachford ADI:
Consider $Au = s$ where $A \in \mathbb{R}^{n \times n}$ spd, $s \in \mathbb{R}^n$.

**ADI iteration idea:** decompose $A = H + V$ with $H, V \in \mathbb{R}^{n \times n}$ such that

\[
(H + pl)v = r \\
(V + pl)w = t
\]

can be solved easily/efficiently.
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$$(H + pl)v = r$$

$$(V + pl)w = t$$

can be solved easily/efficiently.

**ADI Iteration**

If $H, V$ spd $\Rightarrow \exists p_k, k = 1, 2, \ldots$, such that

$$u_0 = 0$$

$$(H + p_k l)u_{k-\frac{1}{2}} = (p_k l - V)u_{k-1} + s$$

$$(V + p_k l)u_k = (p_k l - H)u_{k-\frac{1}{2}} + s$$

converges to $u \in \mathbb{R}^n$ solving $Au = s$. 
The (linear) Lyapunov operator

\[ \mathcal{L} : \ X \mapsto AX + XA^T \]

can be decomposed into the linear operators

\[ \mathcal{L}_H : X \mapsto AX, \quad \mathcal{L}_V : X \mapsto XA^T. \]

In analogy to the standard ADI method we find the

**ADI iteration for the Lyapunov equation**

\[
\begin{align*}
X_0 &= 0, \\
(A + p_k I)X_{k-\frac{1}{2}} &= -W - X_{k-1}(A^T - p_k I), \\
(A + p_k I)X_k^T &= -W - X_{k-\frac{1}{2}}^T(A^T - p_k I).
\end{align*}
\]

[Wachspress 1988]
Consider \( AX + XA^T = -BB^T \) for stable \( A, B \in \mathbb{R}^{n \times m} \) with \( m \ll n \).

**ADI iteration for the Lyapunov equation**

For \( k = 1, \ldots, k_{\text{max}} \):

\[
\begin{align*}
X_0 & = 0 \\
(A + p_k I)X_{k-\frac{1}{2}} & = -BB^T - X_{k-1}(A^T - p_k I) \\
(A + p_k I)X_k^T & = -BB^T - X_{k-\frac{1}{2}}^T(A^T - p_k I)
\end{align*}
\]

[Wachspress 1988]
Consider $AX + XA^T = -BB^T$ for stable $A$, $B \in \mathbb{R}^{n \times m}$ with $m \ll n$.

**ADI iteration for the Lyapunov equation** [Wachspress 1988]

For $k = 1, \ldots, k_{\text{max}}$

\[
\begin{align*}
X_0 &= 0 \\
(A + p_k I)X_{k-\frac{1}{2}} &= -BB^T - X_{k-1}(A^T - p_k I) \\
(A + p_k I)X_k^T &= -BB^T - X_{k-\frac{1}{2}}^T (A^T - p_k I)
\end{align*}
\]

Rewrite as one step iteration and factorize $X_k = Z_kZ_k^T$, $k = 0, \ldots, k_{\text{max}}$

\[
\begin{align*}
Z_0Z_0^T &= 0 \\
Z_kZ_k^T &= -2p_k(A + p_k I)^{-1}BB^T(A + p_k I)^{-T} \\
&\quad + (A + p_k I)^{-1}(A - p_k I)Z_{k-1}Z_{k-1}^T(A - p_k I)^T(A + p_k I)^{-T}
\end{align*}
\]
Consider $AX + XA^T = -BB^T$ for stable $A, B \in \mathbb{R}^{n \times m}$ with $m \ll n$.

**ADI iteration for the Lyapunov equation** [Wachspress 1988]

For $k = 1, \ldots, k_{\text{max}}$

\[
\begin{align*}
X_0 &= 0 \\
(A + p_k I)X_{k-\frac{1}{2}} &= -BB^T - X_{k-1}(A^T - p_k I) \\
(A + p_k I)X_k^T &= -BB^T - X_k^T - p_k I
\end{align*}
\]

Rewrite as one step iteration and factorize $X_k = Z_k Z_k^T$, $k = 0, \ldots, k_{\text{max}}$

\[
\begin{align*}
Z_0 Z_0^T &= 0 \\
Z_k Z_k^T &= -2p_k (A + p_k I)^{-1}BB^T (A + p_k I)^{-T} \\
&\quad + (A + p_k I)^{-1} (A - p_k I) Z_{k-1} Z_{k-1}^T (A - p_k I)^T (A + p_k I)^{-T}
\end{align*}
\]

\[\cdots \rightsquigarrow \text{low-rank Cholesky factor ADI} \quad [\text{Penzl 1997/2000, Li/White 1999/2002, B./Li/Penzl 1999/2008, Gugercin/Sorensen/Antoulas 2003}]\]
\[ Z_k = \left[ \sqrt{-2p_k}(A + p_k I)^{-1}B, (A + p_k I)^{-1}(A - p_k I)Z_{k-1} \right] \quad \text{[Penzl 2000]} \]
\[ Z_k = \left[ \sqrt{-2p_k} (A + p_k I)^{-1} B, (A + p_k I)^{-1} (A - p_k I) Z_{k-1} \right] \]  

[Penzl 2000]

Observing that \((A - p_i I), (A + p_k I)^{-1}\) commute, we rewrite \(Z_{k_{\text{max}}}\) as

\[ Z_{k_{\text{max}}} = [z_{k_{\text{max}}}, P_{k_{\text{max}}-1} z_{k_{\text{max}}}, P_{k_{\text{max}}-2} (P_{k_{\text{max}}-1} z_{k_{\text{max}}}), \ldots, P_1 (P_2 \cdots P_{k_{\text{max}}-1} z_{k_{\text{max}}})], \]

where

\[ z_{k_{\text{max}}} = \sqrt{-2p_{k_{\text{max}}}} (A + p_{k_{\text{max}}} I)^{-1} B \]

and

\[ P_i := \frac{\sqrt{-2p_i}}{\sqrt{-2p_{i+1}}} \left[ I - (p_i + p_{i+1})(A + p_i I)^{-1} \right]. \]

[Li/White 2002]
\[ Z_k = [\sqrt{-2p_k} (A + p_k I)^{-1} B, (A + p_k I)^{-1}(A - p_k I)Z_{k-1}] \]  

[Penzl 2000]

Observing that \((A - p_i I), (A + p_k I)^{-1}\) commute, we rewrite \(Z_{k_{\text{max}}}\) as

\[ Z_{k_{\text{max}}} = [z_{k_{\text{max}}}, P_{k_{\text{max}} - 1}z_{k_{\text{max}}}, P_{k_{\text{max}} - 2}(P_{k_{\text{max}} - 1}z_{k_{\text{max}}}), \ldots, P_1(P_2 \cdots P_{k_{\text{max}} - 1}z_{k_{\text{max}}})], \]

where

\[ z_{k_{\text{max}}} = \sqrt{-2p_{k_{\text{max}}}} (A + p_{k_{\text{max}}} I)^{-1} B \]

and

\[ P_i := \frac{\sqrt{-2p_i}}{\sqrt{-2p_{i+1}}} [I - (p_i + p_{i+1})(A + p_i I)^{-1}] . \]

[Li/White 2002]

\[ \Rightarrow \text{Need to solve only one (sparse) linear system with } m \text{ right-hand sides per iteration!} \]
ADl Methods for Lyapunov Equations
Lyapunov equation $0 = AX + XA^T + BB^T$.


\[
V_1 \leftarrow \sqrt{-2 \text{re} p_1 (A + p_1 I)^{-1} B}, \quad Z_1 \leftarrow V_1 \\
\text{FOR } k = 2, 3, \ldots \\
V_k \leftarrow \sqrt{\frac{\text{re} p_k}{\text{re} p_{k-1}}} \left( V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1} V_{k-1} \right) \\
Z_k \leftarrow \begin{bmatrix} Z_{k-1} & V_k \end{bmatrix} \\
Z_k \leftarrow \text{rrlq}(Z_k, \tau) \quad \% \text{ column compression, optional}
\]
ADI Methods for Lyapunov Equations

Lyapunov equation $0 = AX + XA^T + BB^T$.


\[
V_1 \leftarrow \sqrt{-2 \text{re} p_1 (A + p_1 I)^{-1} B}, \quad Z_1 \leftarrow V_1 \\
\text{FOR } k = 2, 3, \ldots \\
V_k \leftarrow \sqrt{\frac{\text{re} p_k}{\text{re} p_{k-1}}} \left( V_{k-1} - (p_k + \overline{p_{k-1}}) (A + p_k I)^{-1} V_{k-1} \right) \\
Z_k \leftarrow \begin{bmatrix} Z_{k-1} & V_k \end{bmatrix} \\
Z_k \leftarrow \text{rrlq}(Z_k, \tau) \quad \% \text{column compression, optional}
\]

At convergence, $Z_{k_{\text{max}}} Z_{k_{\text{max}}}^T \approx X$, where (without column compression)

\[
Z_{k_{\text{max}}} = \begin{bmatrix} V_1 & \ldots & V_{k_{\text{max}}} \end{bmatrix}, \quad V_k = \begin{bmatrix} \end{bmatrix} \in \mathbb{C}^{n \times m}.
\]
Lyapunov equation \( 0 = AX + XA^T + BB^T \).


\[
V_1 \leftarrow \sqrt{-2 \text{re} p_1 (A + p_1 I)^{-1} B}, \quad Z_1 \leftarrow V_1
\]

FOR \( k = 2, 3, \ldots \)

\[
V_k \leftarrow \sqrt{\frac{\text{re} p_k}{\text{re} p_{k-1}}} \left( V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1} V_{k-1} \right)
\]

\[
Z_k \leftarrow \begin{bmatrix} Z_{k-1} & V_k \end{bmatrix}
\]

\[
Z_k \leftarrow \text{rrlq}(Z_k, \tau) \quad \% \text{ column compression, optional}
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At convergence, \( Z_{k_{\text{max}}} Z_{k_{\text{max}}}^T \approx X \), where (without column compression)

\[
Z_{k_{\text{max}}} = \begin{bmatrix} V_1 & \ldots & V_{k_{\text{max}}} \end{bmatrix}, \quad V_k = \begin{bmatrix} \end{bmatrix} \in \mathbb{C}^{n \times m}.
\]

Note: Implementation in real arithmetic is possible: combine two steps [B./Li/Penzl 1999/2008] or employ the relations of consecutive complex factors [B./Kürschner/Saak 2011].

Current implementations (pyMOR, M.E.S.S.) employ low-rank property of residual, update residual in each step, and compute new shifts on the fly!
Numerical Results for ADI
Optimal Cooling of Steel Profiles

- Mathematical model: boundary control for linearized 2D heat equation.

\[
c \cdot \rho \frac{\partial x}{\partial t} = \lambda \Delta x, \quad \xi \in \Omega
\]

\[
\lambda \frac{\partial x}{\partial n} = \kappa(u_k - x), \quad \xi \in \Gamma_k, \ 1 \leq k \leq 7,
\]

\[
\frac{\partial x}{\partial n} = 0, \quad \xi \in \Gamma_7.
\]

\[\implies m = 7, p = 6.\]

- FEM Discretization, different models for initial mesh \(n = 371\),
  1, 2, 3, 4 steps of mesh refinement \(\Rightarrow\)
  \(n = 1357, 5177, 20209, 79841.\)

Source: Physical model: courtesy of Mannesmann/Demag.
Numerical Results for ADI
Optimal Cooling of Steel Profiles

- Solve dual Lyapunov equations needed for balanced truncation, i.e.,

\[ APMT + MPAT + BB^T = 0, \quad A^TQM + MTAQ + CTC = 0, \]

for \( n = 79, 841 \).

- 25 shifts chosen by Penzl heuristic from 50/25 Ritz values of \( A \) of largest/smallest magnitude, no column compression performed.

- M.E.S.S. requires no factorization of mass matrix.

![Graph showing normalized residual norm vs. number of iterations for two systems](image-url)
Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis $\text{range}(Z)$, $Z \in \mathbb{R}^{n \times r}$, for subspace $Z \subset \mathbb{R}^n$, $\dim Z = r$.
2. Set $\hat{A} := Z^T AZ$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A}\hat{X} + \hat{X}\hat{A}^T + \hat{B}\hat{B}^T = 0$.
4. Use $X \approx Z\hat{X}Z^T$.

Examples:

- Krylov subspace methods, i.e., for $m = 1$:

$$Z = \mathcal{K}(A, B, r) = \text{span}\{B, AB, A^2B, \ldots, A^{r-1}B\}$$

Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis $\text{range}(Z) = Z \in \mathbb{R}^{n \times r}$, for subspace $\mathcal{Z} \subset \mathbb{R}^n$, $\dim \mathcal{Z} = r$.
2. Set $\hat{A} := Z^T AZ$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A} \hat{X} + \hat{X} \hat{A}^T + \hat{B} \hat{B}^T = 0$.
4. Use $X \approx Z \hat{X} Z^T$.

Examples:

- Krylov subspace methods, i.e., for $m = 1$:
  \[
  \mathcal{Z} = \mathcal{K}(A, B, r) = \text{span}\{B, AB, A^2 B, \ldots, A^{r-1} B\}
  \]

- Extended (and rational) Krylov method (EKSM, RKSM) [Simoncini 2007, Druskin/Knizhnerman/Simoncini 2011],
  \[
  \mathcal{Z} = \mathcal{K}(A, B, r) \cup \mathcal{K}(A^{-1}, B, r).
  \]
Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis $\text{range}(Z)$, $Z \in \mathbb{R}^{n \times r}$, for subspace $Z \subset \mathbb{R}^n$, $\dim Z = r$.
2. Set $\hat{A} := Z^T AZ$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A} \hat{X} + \hat{X} \hat{A}^T + \hat{B} \hat{B}^T = 0$.
4. Use $X \approx Z \hat{X} Z^T$.

Examples:

- ADI subspace [B./R.-C. Li/Truhar 2008]:

  $$Z = \text{colspan} \left[ V_1, \ldots, V_r \right].$$

Note:

1. ADI subspace is rational Krylov subspace [J.-R. Li/White 2002].
Balanced Truncation
Numerical example for BT: Optimal Cooling of Steel Profiles

\( n = 1357, \text{ Absolute Error} \)

- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.
Balanced Truncation
Numerical example for BT: Optimal Cooling of Steel Profiles

\( n = 1357, \text{ Absolute Error} \)

- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.

\( n = 79841, \text{ Absolute Error} \)

- BT model computed using M-M.E.S.S. in MATLAB,
By applying AC voltage to electrodes, wings are forced to vibrate in anti-phase in wafer plane.

Coriolis forces induce motion of wings out of wafer plane yielding sensor data.

- Vibrating micro-mechanical gyroscope for inertial navigation.
- Rotational position sensor.

Source: [http://modelreduction.org/index.php/Modified_Gyroscope](http://modelreduction.org/index.php/Modified_Gyroscope)
FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
\[ n = 34,722, \ m = 1, \ p = 12. \]

Reduced model computed using ADI-based balanced truncation, \( r = 30 \).
- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  \( \sim n = 34,722, m = 1, p = 12 \).
- Reduced model computed using ADI-based balanced truncation, \( r = 30 \).

**Bode Diagram**

- Frequency Response Analysis

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SVD-based MOR 32/37
Balanced Truncation
Numerical example for BT: Microgyroscope (Butterfly Gyro)

- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  \( n = 34,722, \ m = 1, \ p = 12. \)

- Reduced model computed using ADI-based balanced truncation, \( r = 30. \)

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**Frequency Repsonse Analysis**

**Hankel Singular Values**

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Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size $r$ with $\sigma_r > \sigma_{r+1}$. 
Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices \( P = S^T S \), \( Q = R^T R \), compute balancing state-space transformation so that

\[
P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,
\]

and truncate corresponding realization at size \( r \) with \( \sigma_r > \sigma_{r+1} \).

Classical Balanced Truncation (BT) [Mullis/Roberts 1976, Moore 1981]

- \( P = \) controllability Gramian of system given by \((A, B, C, D)\).
- \( Q = \) observability Gramian of system given by \((A, B, C, D)\).
- \( P, Q \) solve dual Lyapunov equations

\[
AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0.
\]
Balancing-Related Model Reduction

**Basic Principle**

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size $r$ with $\sigma_r > \sigma_{r+1}$.

**LQG Balanced Truncation (LQGBT)**

- $P/Q = \text{controllability/observability Gramian of closed-loop system based on LQG compensator}$.
- $P, Q$ solve dual algebraic Riccati equations (AREs)

$$0 = AP + PA^T - PC^T CP + B^T B,$$

$$0 = A^T Q + QA - QBB^T Q + C^T C.$$
Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size $r$ with $\sigma_r > \sigma_{r+1}$.

Balanced Stochastic Truncation (BST) \cite{Desai/Pal 1984, Green 1988}

- $P = \text{controllability Gramian of system given by } (A, B, C, D)$, i.e., solution of Lyapunov equation $AP + PA^T + BB^T = 0$.
- $Q = \text{observability Gramian of right spectral factor of power spectrum of system given by } (A, B, C, D)$, i.e., solution of ARE

$$\hat{A}^T Q + Q \hat{A} + QB_W (DD^T)^{-1} B_W^T Q + C^T (DD^T)^{-1} C = 0,$$

where $\hat{A} := A - B_W (DD^T)^{-1} C$, $B_W := BD^T + PC^T$. 

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Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices \( P = S^T S \), \( Q = R^T R \), compute balancing state-space transformation so that

\[
P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,
\]
and truncate corresponding realization at size \( r \) with \( \sigma_r > \sigma_{r+1} \).

Positive-Real Balanced Truncation (PRBT)

- Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.
- \( P, Q \) solve dual AREs

\[
0 = \tilde{A} P + P \tilde{A}^T + P C^T \tilde{R}^{-1} C P + B \tilde{R}^{-1} B^T,
\]
\[
0 = \tilde{A}^T Q + Q \tilde{A} + Q B \tilde{R}^{-1} B^T Q + C^T \tilde{R}^{-1} C,
\]

where \( \tilde{R} = D + D^T \), \( \tilde{A} = A - B \tilde{R}^{-1} C \).
Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size $r$ with $\sigma_r > \sigma_{r+1}$.

Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [Opdenacker/Jonckheere 1988];
- $H_\infty$ balanced truncation (HinfBT) – closed-loop balancing based on $H_\infty$ compensator [Mustafa/Glover 1991].

Both approaches require solution of dual AREs.
- Frequency-weighted versions of the above approaches.
Guaranteed preservation of physical properties like

- stability (all),
- passivity (PRBT),
- minimum phase (BST).

Computable error bounds, e.g.,

\[
\|G - G_r\|_{\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_{BT_j}, \\
\|G - G_r\|_{\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_{LQG_j} \sqrt{1 + (\sigma_{LQG_j})^2}, \\
\|G - G_r\|_{\infty} \leq \left( \prod_{j=r+1}^{n} 1 + \sigma_{BST_j} \right) \|G\|_{\infty},
\]

Can be combined with singular perturbation approximation (= Guyan reduction applied to balanced realization!) for improved steady-state performance.

Computations can be modularized ⇝ software packages M-M.E.S.S., MORLAB, see http://www.mpi-magdeburg.mpg.de/823508/software.
Guaranteed preservation of physical properties like
  – stability (all),
Guaranteed preservation of physical properties like
  – stability (all),
  – passivity (PRBT),
Balancing-Related Model Reduction

Properties

- Guaranteed preservation of physical properties like
  - stability (all),
  - passivity (PRBT),
  - minimum phase (BST).

\[ \| G - G_r \|_\infty \leq 2n \sum_{j=r+1}^{n} \sigma_{BT} \]

\[ \| G - G_r \|_\infty \leq 2n \sum_{j=r+1}^{n} \sigma_{LQG} \sqrt{1 + (\sigma_{LQG})^2} \]

\[ \| G - G_r \|_\infty \leq \left( n \prod_{j=r+1}^{n} \frac{1 + \sigma_{BST}}{1 - \sigma_{BST}} \right) \| G \|_\infty \]

Can be combined with singular perturbation approximation ( = Guyan reduction applied to balanced realization!) for improved steady-state performance.

Computations can be modularized ⇝ software packages M-M.E.S.S., MORLAB, see http://www.mpi-magdeburg.mpg.de/823508/software.
Balancing-Related Model Reduction

Properties

- Guaranteed preservation of physical properties like
  - stability (all),
  - passivity (PRBT),
  - minimum phase (BST).
- Computable error bounds, e.g.,

  \[
  \text{BT: } \| G - G_r \|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_j^{BT},
  \]
  \[
  \text{LQGBT: } \| G - G_r \|_\infty \leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}}
  \]
  \[
  \text{BST: } \| G - G_r \|_\infty \leq \left( \prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \| G \|_\infty,
  \]

Can be combined with singular perturbation approximation (= Guyan reduction applied to balanced realization!) for improved steady-state performance.

Computations can be modularized \(\rightarrow\) software packages M-M.E.S.S., MORLAB, see http://www.mpi-magdeburg.mpg.de/823508/software.
Balancing-Related Model Reduction

Properties

- Guaranteed preservation of physical properties like
  - stability (all),
  - passivity (PRBT),
  - minimum phase (BST).

- Computable error bounds, e.g.,

  \[ \| G - G_r \|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_j^{BT}, \]

  \[ \| G - G_r \|_\infty \leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}} \]

  \[ \| G - G_r \|_\infty \leq \left( \prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \| G \|_\infty, \]

- Can be combined with singular perturbation approximation (= Guyan reduction applied to balanced realization!) for improved steady-state performance.
Balancing-Related Model Reduction

Properties

- Guaranteed preservation of physical properties like
  - stability (all),
  - passivity (PRBT),
  - minimum phase (BST).

- Computable error bounds, e.g.,

  \[
  \text{BT: } \| G - G_r \|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_j^{BT},
  \]

  \[
  \text{LQGBT: } \| G - G_r \|_\infty \leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}}
  \]

  \[
  \text{BST: } \| G - G_r \|_\infty \leq \left( \prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \| G \|_\infty,
  \]

- Can be combined with singular perturbation approximation ( = Guyan reduction applied to balanced realization!) for improved steady-state performance.

- Computations can be modularized \( \leadsto \) software packages M-M.E.S.S., MORLAB, see [http://www.mpi-magdeburg.mpg.de/823508/software](http://www.mpi-magdeburg.mpg.de/823508/software).
1. Introduction to SVD-based Model Order Reduction

2. Model Reduction by Projection

3. Balanced Truncation

4. Final Remarks
Current Research Topics

- Special methods for second-order (mechanical), switched and delay systems.
- Time- and frequency-limited variants.
- Empirical variants using snapshots and integral representation of Gramians.
- Extensions to bilinear, quadratic-bilinear, polynomial, and stochastic systems.
- MOR methods for discrete-time systems.
- Extensions to descriptor systems $E\dot{x} = Ax + Bu$, $E$ singular.
- Parametric model reduction:
  \[
  \dot{x} = A(p)x + B(p)u, \quad y = C(p)x,
  \]
  where $p \in \mathbb{R}^d$ is a free parameter vector; parameters should be preserved in the reduced-order model.
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