MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Physics-Informed Learning for Low-Dimensional Nonlinear Dynamical Systems Using Operator Inference

Part I: From Projection-based to Data-driven Model Order Reduction — an Overview

Peter Benner Pawan Goyal

IMPRS ProEng Summer School 2021
Magdeburg, September 27, 2021

Supported by:

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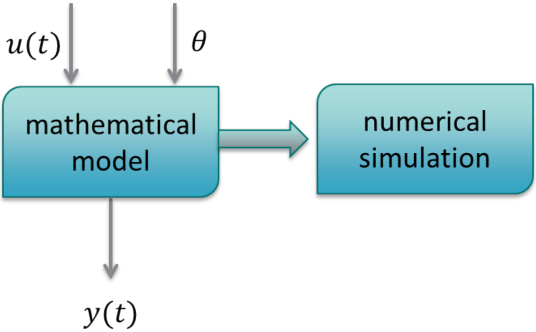

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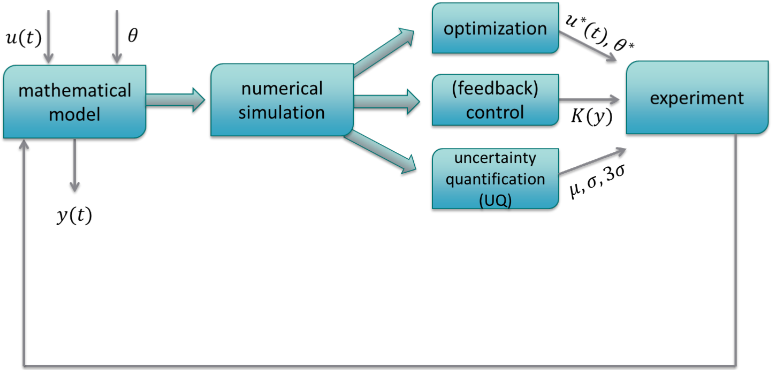

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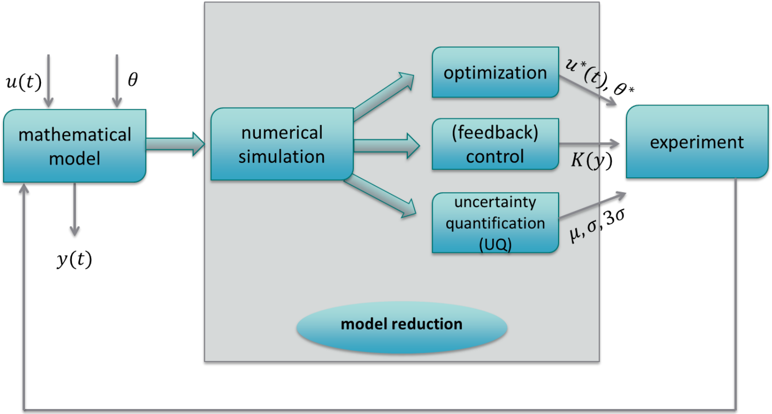
experiment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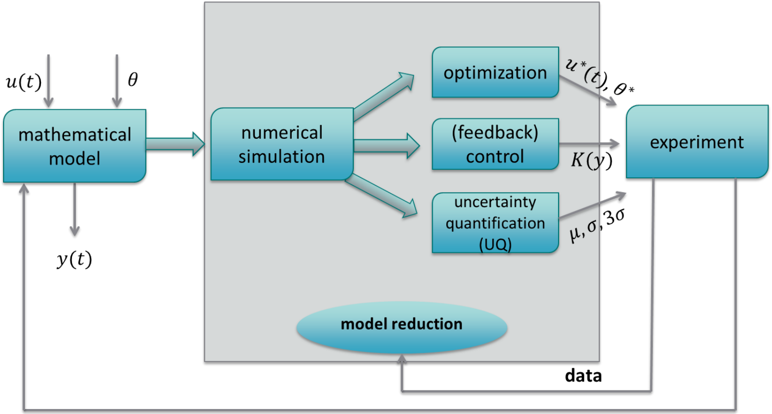
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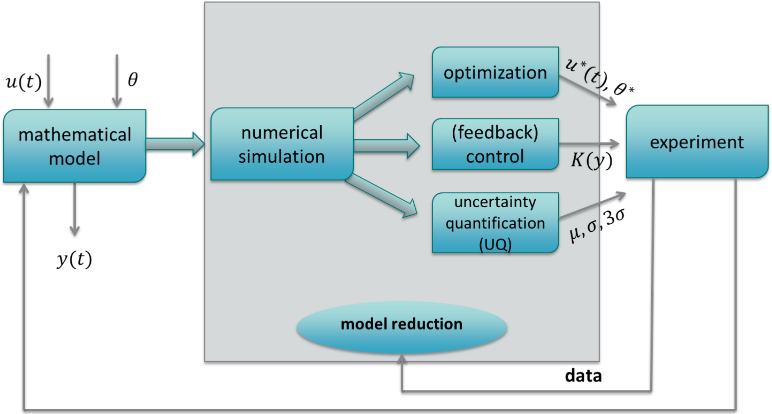
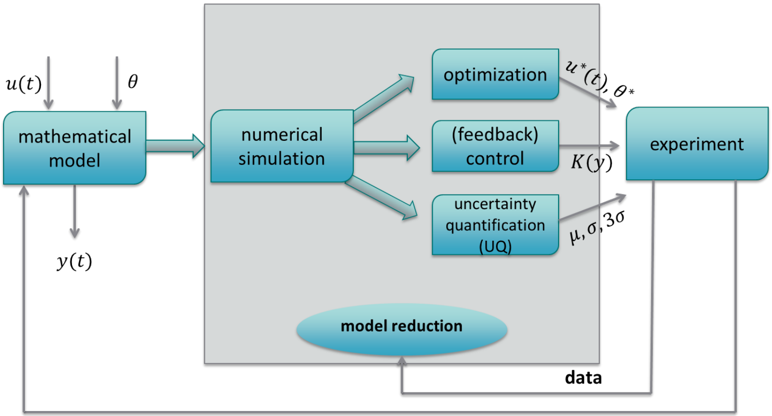


Goal: *Use all acquired knowledge about the model during the CSE process chain in the design of the reduced-order model, including experimental data.*



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$\rightsquigarrow$ *Data-enhanced model reduction methods.*

Outline

1. Model Order Reduction of Dynamical Systems

2. Data-driven/-enhanced Model Reduction

1. Model Order Reduction of Dynamical Systems
   - Model Order Reduction of Linear Systems
   - MOR Methods Based on Projection

2. Data-driven/-enhanced Model Reduction

Original System

$$\Sigma : \begin{cases} \dot{x}(t) & = & f(t, x(t), u(t)), \\ y(t) & = & g(t, x(t), u(t)), \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
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Reduced-Order Model (ROM)

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$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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Secondary goal: reconstruct approximation of x from $\hat{x}$.

Original System

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases}$$

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$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{A}\hat{x}(t) + \widehat{B}u(t), \\ \hat{y}(t) = \widehat{C}\hat{x}(t) + \widehat{D}u(t). \end{cases}$$

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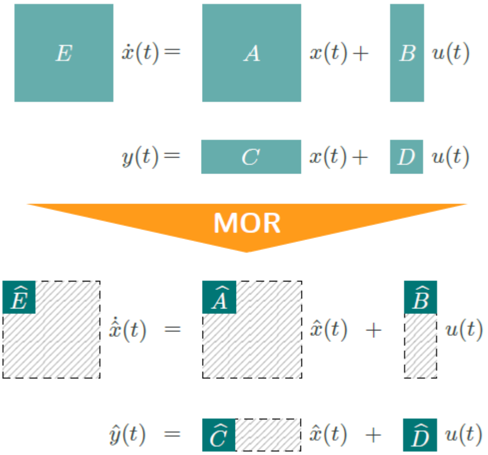

- $E, A \in \mathbb{R}^{n \times n}$
- $B \in \mathbb{R}^{n \times m}$
- $C \in \mathbb{R}^{p \times n}$
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For linear systems, the reduced-order model is

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- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

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Extends to nonlinear systems with some effort:

$$\begin{aligned} \dot{\hat{x}} &= W^T f(t, V\hat{x}, u), \\ \hat{y} &= g(t, V\hat{x}, u). \end{aligned}$$

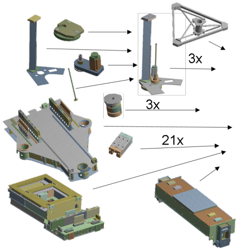
Needs hyperreduction if the cost for evaluation of the functions $W^T f, g$ is not reduced!

MOR Methods Based on Projection

Classes of Projection-based MOR Methods

1. Modal Truncation
2. Rational Interpolation / Moment Matching
   (Padé-Approximation and (rational) Krylov Subspace Methods)
3. Balanced Truncation
4. Proper Orthogonal Decomposition (POD) / Principal Component Analysis (PCA)
5. Reduced Basis Method
6. ...

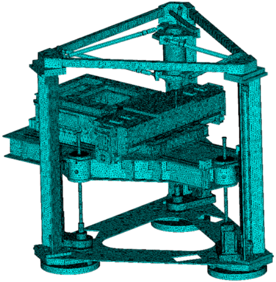
50 subassemblies



CAD model

$\overset{FEM}{\rightsquigarrow}$

FE-Model: 1.2M DOFs



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FE-coupled

method	red. order tol 10^{-3}	t_{red}
2phase	196	6.5h
BTX0	174	4.5h

output-coupled

method	red. order tol 10^{-3}	t_{red}
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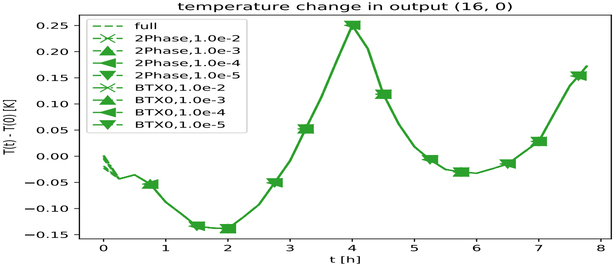
→ *Required storage for reduced matrices just 1MB!*
→ *Simulation speed-up factors range from* ≈ 8–$2,000$.



Vettermann, J., Sauerzapf, S., Naumann, A., Beitelschmidt, M., Herzog, R., Benner, P., Saak, J. (2021): Model order reduction methods for coupled machine tool models. MM Science Journal, pp. 4652–4659.

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$=$ *learning (compact, surrogate) models from (full, detailed) models.*

This is often impossible!

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= *LEARNING (compact, surrogate) MODELS FROM DATA!*

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Some methods:

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- Loewner interpolation: frequency and time domain
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Data-driven/-enhanced Model Reduction
Learning Models from Data

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- Koopman/Dynamic Mode Decomposition (DMD): time domain
  [Mezič 2005; Schmid 2008; Brunton, Kevrekidis, Kutz, Rowley, Noé, Nüske, Schütte, Peitz, ...],
  for control systems [Kaiser/Kutz/Brunton 2017, B./Himpe/Mitchell 2018]

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  [Ho/Kalman 1966; Ljung 1987/1999; Van Overschee/De Moor 1994; Verhaegen 1994; De Wilde, Eykhoff, Moonen, Sima, ...]
- Neural networks: time domain [Narendra/Parthasarathy 1990; Lee/Carlberg 2019; ...]
- Loewner interpolation: frequency and time domain
  [Antoulas/Anderson 1986; Mayo/Antoulas 2007; Gosea, Gugercin, Ionita, Lefteriu, Peherstorfer, ...]
- Koopman/Dynamic Mode Decomposition (DMD): time domain
  [Mezič 2005; Schmid 2008; Brunton, Kevrekidis, Kutz, Rowley, Noé, Nüske, Schütte, Peitz, ...],
  for control systems [Kaiser/Kutz/Brunton 2017, B./Himpe/Mitchell 2018]
- Operator inference (OpInf): time domain [Peherstorfer/Willcox 2016; Kramer, Qian, B., Goyal,...]

- System identification tries to infer discrete linear time-invariant (LTI) systems

$$x_{k+1} = Ax_k + Bu_k + Kw_k,$$
$$y_k = Cx_k + Du_k + v_k.$$

from input-output data, given as time series $(u_0, y_0), (u_1, y_1), \ldots, (u_K, y_K)$, where v_k, w_k are uncorrelated Gaussian white noise processes.

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A few Remarks on System Identification and DNNs

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- Continuous-time system can be identified, e.g., by "inverse" Euler method.
- Many extensions to nonlinear systems, imposing certain structural assumptions, including artificial neural networks...

A paper from 1990. . .

4 IEEE TRANSACTIONS ON NEURAL NETWORKS, VOL. 1, NO. 1, MARCH 1990

Identification and Control of Dynamical Systems Using Neural Networks

KUMPATI S. NARENDRA FELLOW, IEEE, AND KANNAN PARTHASARATHY

Abstract—The paper demonstrates that neural networks can be used effectively for the identification and control of nonlinear dynamical systems. The emphasis of the paper is on models for both identification and control. Static and dynamic back-propagation methods for the adjustment of parameters are discussed. In the models that are introduced, multilayer and recurrent networks are interconnected in novel configurations and hence there is a real need to study them in a unified fashion. Simulation results reveal that the identification and adaptive control schemes suggested are practically feasible. Basic concepts and definitions are introduced throughout the paper, and theoretical questions which have to be addressed are also described.

are well known for such systems [1]. In this paper our interest is in the identification and control of nonlinear dynamic plants using neural networks. Since very few results exist in nonlinear systems theory which can be directly applied, considerable care has to be exercised in the statement of the problems, the choice of the identifier and controller structures, as well as the generation of adaptive laws for the adjustment of the parameters.

Two classes of neural networks which have received considerable attention in the area of artificial neural net-

Narendra, K.S., Parthasarathy, K. (1990): Identification and control of dynamical systems using neural networks. IEEE Transactions on Neural Networks 1(1):4–27.

A paper from 1990...

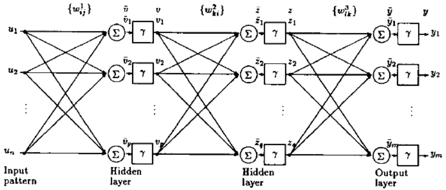

Fig. 2. A three layer neural network.

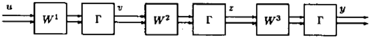

Fig. 3. A block diagram representation of a three layer network.

Narendra, K.S., Parthasarathy, K. (1990): Identification and control of dynamical systems using neural networks. IEEE Transactions on Neural Networks 1(1):4–27.

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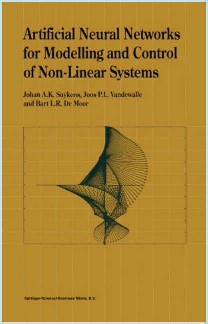

Narendra, K.S., Parthasarathy, K. (1990): Identification and control of dynamical systems using neural networks. IEEE Transactions on Neural Networks 1(1):4–27.

Suykens, J.A.K., Vandewalle, J.P.L., de Moor, B.L. (1996): Artificial Neural Networks for Modelling and Control of Non-Linear Systems. Springer US.

Given a smooth dynamical system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in \mathbb{R}^n.$$

Take snapshots $x_k := x(t_k)$ on grid $t_k := kh$ for $k = 0, 1, \ldots, K$ and fixed $h > 0$ (using simulation software, or measurements from real life experiment $\rightsquigarrow$ nonintrusive!), and find "best possible" A_* such that

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Motivation: Koopman theory

- $\exists$ a linear, infinite-dimensional operator describing the evolution of $f(x(\cdot))$ in an appropriate function space setting.
- Can be considered as lifting of a finite-dimensional, nonlinear problem to a infinite-dimensional, linear problem.

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Basic DMD Algorithm

Set $X_0 := [\,x_0, x_1, \ldots, x_{K-1}\,] \in \mathbb{R}^{n \times K}$, $X_1 := [\,x_1, x_2, \ldots, x_K\,] \in \mathbb{R}^{n \times K}$ and note that $X_1 = AX_0$ is desired $\rightsquigarrow$ over-/underdetermined linear system, solved by linear least-squares problem (regression):

$$A_* := \arg\min_{A \in \mathbb{R}^{n \times n}} \|X_1 - AX_0\|_F + \beta \|A\|_q$$

with a potential regularization term choosing $\beta > 0$, $q = 0, 1, 2$.

Computation usually via singular value decomposition (SVD), many variants.

Given a smooth control system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \in \mathbb{R}^n, \qquad y(t) = g(x(t), u(t)),$$

with control $u(t) \in \mathbb{R}^m$ and output $y(t) \in \mathbb{R}^p$.

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Take state, control, and output snapshots

$$x_k := x(t_k), \quad u_k := u(t_k), \quad y_k := y(t_k), \qquad k = 0, 1, \ldots, K$$

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Basic ioDMD Algorithm ($\equiv$ N4SID)

Let $\mathbb{S} := \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$. Set X_0, X_1 as before and

$$U_0 := [\, u_0, u_1, \ldots, u_{K-1} \,] \in \mathbb{R}^{m\times K}, \qquad Y_0 := [\, y_0, y_1, \ldots, y_{K-1} \,] \in \mathbb{R}^{p\times K}.$$

Solve the linear least-squares problem (regression):

$$(A_*, B_*, C_*, D_*) := \arg\min_{(A,B,C,D)\in\mathbb{S}} \left\| \begin{bmatrix} X_1 \\ Y_0 \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} \right\|_F + \beta \| [\, A\ B\ C\ D \,] \|_q$$

with a potential regularization term choosing $\beta > 0$, $q = 0, 1, 2$.

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Can be combined with ioDMD to obtain reduced-order LTI system.

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where $P \otimes Q := [p_{ij}Q]_{ij}$ denotes the Kronecker (tensor) product, from data

$$X := [\, x_0, x_1, \ldots, x_K \,] \in \mathbb{R}^{n \times (K+1)}, \quad U := [\, u_0, u_1, \ldots, u_K \,] \in \mathbb{R}^{m \times (K+1)}.$$

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- Use compressed trajectories (via POD / PCA) $\rightsquigarrow \hat{X}$.
- Compress snapshot matrix of time derivatives: if residuals $f(t_j, u_j)$ are available

  $$\dot{\hat{X}} := [\, \dot{x}(0), \dot{x}(t_1), \ldots, \dot{x}(t_K) \,] \approx [\, f(x_0, u_0), f(x_1, u_1), \ldots, f(x_K, u_K) \,] \in \mathbb{R}^{n \times (K+1)},$$

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  with potential regularization as before and $\widehat{X^2} := [x_0 \otimes x_0, \ldots, x_K \otimes x_K]$.

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Conclusions — Part I

- DMD and operator inference (OpInf) are regression-based powerful methods to infer linear and certain nonlinear dynamical systems from data.
- Both look simple, but the devil is in the details.
- Choice of good observables? (Learning to learn?)
- Statistical aspects are not too well understood: impact of noise in the data on inferred models?
- Recent work combines OpInf with neural networks to solve nonlinear identification problems **($\rightsquigarrow$ Part II).**
- Error bounds for non-intrusive MOR not well developed yet, but theoretic results indicate that the OpInf model asymptotically (when increasing the number of snapshots) yields the POD model. Then, intrusive MOR error bounds can be applied.