MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Balanced Truncation for Linear and Nonlinear Truly Large-Scale Systems

Peter Benner   Igor Pontes Duff   Pawan K. Goyal   Jens Saak

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1. Introduction
   Model Order Reduction of Dynamical Systems
   System Classes
   Model Reduction of Linear Systems
   Model Reduction in Frequency Domain

2. Balanced Truncation for Linear Systems

3. Balanced Truncation for Polynomial Systems

4. Conclusions

Original System

$$\Sigma : \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)), \\ y(t) &= g(t, x(t), u(t)), \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
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System Classes

Control-Affine (Autonomous) Systems

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y(t) &= g(t,x,u) = \mathcal{C}(x(t)) + \mathcal{D}(x(t))u(t), \quad \mathcal{C} : \mathbb{R}^n \to \mathbb{R}^q,\ \mathcal{D} : \mathbb{R}^n \to \mathbb{R}^{q\times m}.
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Bilinear Systems

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Polynomial Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = Ax(t) + \sum_{j=2}^{n_p} H_j \left(\otimes^j x(t)\right) + \sum_{j=2}^{n_p} \sum_{k=1}^{m} A_j^k \left(\otimes^j x(t)\right) u_k(t) + Bu(t), \\ &\qquad H_j, A_j^k \text{ of "appropriate size"}, \\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}. \end{aligned}$$

Polynomial-bilinear systems can be obtained from smooth nonlinear systems by *lifting* without approximation error! [Gu 2011].

System Classes

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Written in control-affine form:

$$\begin{aligned} \mathcal{A}(x) &:= Ax + H\left(x\otimes x\right), & \mathcal{B}(x) &:= \left[A_1,\ldots,A_m\right]\left(I_m \otimes x\right) + B \\ \mathcal{C}(x) &:= Cx, & \mathcal{D}(x) &:= D. \end{aligned}$$

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Here, we focus on linear and polynomial systems.

Model Reduction of Linear Systems

Linear Time-Invariant (LTI) Systems

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Secondary goal: reconstruct approximation of x from $\hat{x}$.

Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s),\, \dot{x}(t) \mapsto sX(s) - x(0))$ to LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

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$\Longrightarrow$ I/O-relation in frequency domain:

$$Y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)} \Big) U(s).$$

$G(s)$ is the **transfer function** of Σ.

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Model reduction in frequency domain: Fast evaluation of mapping $U \rightarrow Y$.

Formulating model reduction in frequency domain

Approximate the time domain dynamical system

$$\begin{array}{rcll} \dot{x} & = & Ax + Bu, & A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y & = & Cx + Du, & C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}, \end{array}$$

by reduced-order system

$$\begin{array}{rcll} \dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}u, & \hat{A} \in \mathbb{R}^{r\times r},\ \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} & = & \hat{C}\hat{x} + \hat{D}u, & \hat{C} \in \mathbb{R}^{p\times r},\ \hat{D} \in \mathbb{R}^{p\times m} \end{array}$$

of order $r \ll n$, such that

$$\begin{aligned} \|y - \hat{y}\| \simeq \left\|Y - \hat{Y}\right\| = \left\|GU - \hat{G}U\right\| & \\ \leq \left\|G - \hat{G}\right\| \cdot \|U\| \simeq \left\|G - \hat{G}\right\| \cdot \|u\| & \\ \leq \text{tolerance} \cdot \|u\|\,. & \end{aligned}$$

Outline

1. Introduction

2. Balanced Truncation for Linear Systems
   Basic Concept
   Numerical Examples

3. Balanced Truncation for Polynomial Systems

4. Conclusions

Basic concept

- System $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions P, Q of the Lyapunov equations

  $$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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- Compute balanced realization (needs P, Q!) of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C) &\mapsto (TAT^{-1}, TB, CT^{-1}) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right). \end{aligned}$$

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}) = (A_{11}, B_1, C_1)$.

  Note: in efficient algorithms, truncation is achieved via projection:

  $$(\hat{A}, \hat{B}, \hat{C}) = (W^TAV, W^TB, CV), \quad \text{where} \quad W^TV = I_r.$$

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Balanced Truncation for Linear Systems

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- Adaptive choice of r via computable error bound:

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where $\|G\|_{\mathcal{H}_\infty} := \sup_{u \in \mathcal{L}_2 \setminus \{0\}} \frac{\|Gu\|_2}{\|u\|_2} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(\jmath\omega))$.

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Practical implementation

- Rather than solving Lyapunov equations for P, Q (n^2 unknowns!), find $S, R \in \mathbb{R}^{n \times s}$ with $s \ll n$ such that $P \approx SS^T$, $Q \approx RR^T$.
- Reduced-order model directly obtained via small-scale ($s \times s$) SVD of $R^T S$!

[a] http://www.mpi-magdeburg.mpg.de/projects/morlab

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- Two software packages:
  - MORLAB[a] (**M**odel **O**rder **R**eduction **LAB**oratory), based on spectral projection methods ($\rightsquigarrow$ small to medium size problems, up to $n \sim 5,000$.)
  - M-M.E.S.S.[b] provides solvers for large-scale matrix equations with sparse/low-rank coefficients and basic MOR functionality; no $\mathcal{O}(n^3)$ or $\mathcal{O}(n^2)$ computations necessary!

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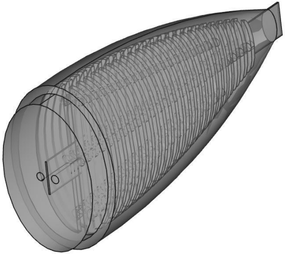
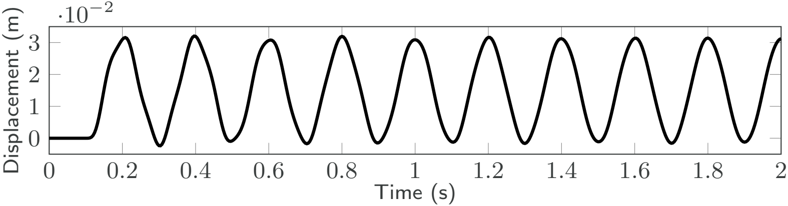
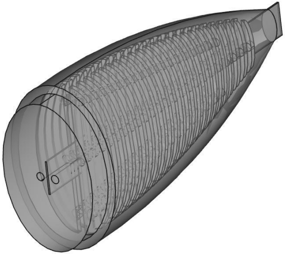
Original system:

- mechanical second-order structure
- $n = 779{,}232$, $m = 1$, $p = 3$
- test simulation time $\approx 4\,\mathrm{h}$







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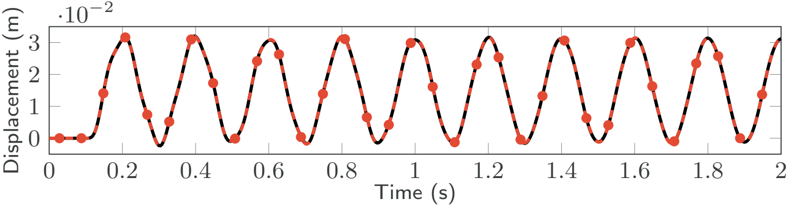
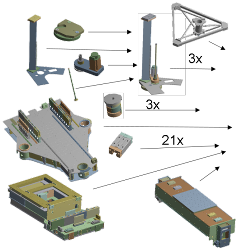
Second-Order Balanced Truncation:

- two-step method with MORLAB backend
- $r = 1$, wall time $\approx 21\,\text{h}$
- test simulation time $\approx 10\,\text{msec}$ (speed up $\approx 1.4 \cdot 10^6$)



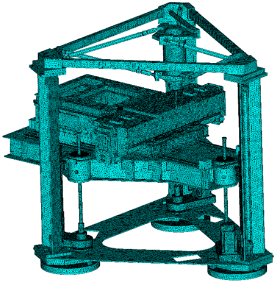
50 subassemblies



CAD model

$\overset{FEM}{\rightsquigarrow}$

FE-Model: **1.2M DOFs**



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FE-coupled

method	red. order tol 10^{-3}	t_{red}
2phase	196	6.5h
BTX0	174	4.5h

output-coupled

method	red. order tol 10^{-3}	t_{red}
2phase	3,005	2h
BTX0	2,515	1.8h

Numerical Examples

MAX: Results considering an inhomogeneous initial condition $x_0 \neq 0$

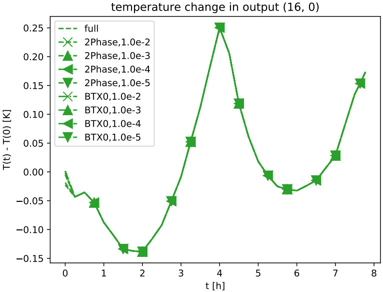
FE-coupled

method	red. order tol 10^{-3}	t_{red}
2phase	196	6.5h
BTX0	174	4.5h

output-coupled

method	red. order tol 10^{-3}	t_{red}
2phase	3,005	2h
BTX0	2,515	1.8h

$\rightarrow$ Required storage for reduced matrices just 1MB!



Vettermann, J., Sauerzapf, S., Naumann, A., Beitelschmidt, M., Herzog, R., Benner, P., Saak, J. (2021): Model order reduction methods for coupled machine tool models. MM Science Journal 2021:4652-4659.

1. Introduction

2. Balanced Truncation for Linear Systems

3. Balanced Truncation for Polynomial Systems
   Polynomial Control Systems
   Gramians for PC Systems
   Truncated Gramians
   Numerical Example

4. Conclusions

Now, consider the class of **polynomial control (PC) Systems**:

$$\boxed{\begin{aligned}\dot{x}(t) &= Ax(t) + \sum_{j=2}^{n_p} H_j \left(\otimes^j x(t)\right) + \sum_{j=2}^{n_p}\sum_{k=1}^{m} N_j^k \left(\otimes^j x(t)\right) u_k(t) + Bu(t),\\ y(t) &= Cx(t), \quad x(0) = 0,\end{aligned}}$$

where

- n_p is the degree of the polynomial part of the system,
- $x(t) \in \mathbb{R}^n$, $\otimes^j x(t) = \underbrace{x(t) \otimes \cdots \otimes x(t)}_{j\text{-times}}$,
- $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$, $n \gg m, p$.
- $A \in \mathbb{R}^{n\times n}$, $H_j, N_j^k \in \mathbb{R}^{n\times n^j}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$.
- Assumption: A is supposed to be Hurwitz $\Rightarrow$ local stability.

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- Assumption: A is supposed to be Hurwitz $\Rightarrow$ local stability.

Examples: FitzHugh-Nagumo and Chafee-Infante equations lead to cubic control systems; cubic-quintic Allen-Cahn equation to quintic control system.

Expanding the response of the PC system into a Volterra series representation and using the idea of iterated linear systems, we define the reachability Gramian as

$$\boxed{P = \sum_{k=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \bar{P}_k(t_1, \ldots, t_k) \bar{P}_k(t_1, \ldots, t_k)^T dt_1 \ldots dt_k,}$$

where $\bar{P}_1(t_1) = e^{At_1} B, \quad \bar{P}_2(t_1, t_2) = \sum_{k=1}^{m} e^{At_1} N_1^k e^{At_2} B,$

$\bar{P}_3(t_1, t_2, t_3) = e^{At_1} H_2 e^{At_2} B \otimes e^{At_3} B, \ldots$ are the kernels of the Volterra series expansion of the system output.

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Theorem [B./Goyal/Pontes Duff 2018]

The reachability Gramian P of a PC system solves the polynomial Lyapunov equation

$$AP + PA^T + BB^T + \sum_{j=2}^{n_p} H_j \left(\otimes^j P\right) H_j^T + \sum_{j=2}^{n_p} \sum_{k=1}^{m} N_j^k \left(\otimes^j P\right) \left(N_j^k\right)^T = 0.$$

The Observability Gramian is defined as follows:

- First, we write the adjoint system as [Fujimoto et al. 2002]

$$\begin{aligned}
&\dot{x}(t) = Ax(t) + \sum_{j=2}^{n_p} H_j x_j^{\otimes}(t) + \sum_{j=1}^{n_p}\sum_{k=1}^{m} N_j^k x_j^{\otimes}(t) u_k(t) + Bu(t),\\
&\dot{x_d}(t) = -A^T x_d(t) - \sum_{j=2}^{n_p} H_j^{(2)} x_{d,j}^{\otimes}(t) - \sum_{j=1}^{n_p}\sum_{k=1}^{m} \left(N_j^{k,(2)}\right) x_{d,j}^{\otimes}(t) u_{d,k}(t) - C^T u_d(t), \quad x_d(\infty) = 0,\\
&y_d(t) = B^T x_d(t).
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y_d(t) &= B^T x_d(t).
\end{aligned}
$$

- Then, by taking the kernel of Volterra series, one has

Theorem [B./Goyal/Pontes Duff 2018]

Let P be the reachability Gramian. Then, the observability Gramian Q of a PC system solves the polynomial Lyapunov equation

$$
A^T Q + QA + C^T C + \sum_{j=2}^{n_p} H_j^{(2)} \left(\otimes^{j-1} P \otimes Q\right) \left(H_j^{(2)}\right)^T + \sum_{j=2}^{n_p}\sum_{k=1}^{m} N_j^{k,(2)} \left(\otimes^{j-1} P \otimes Q\right) \left(N_j^{k,(2)}\right)^T = 0.
$$

- Polynomial Lyapunov equations are very expensive to solve, efficient algorithms have not yet been developed.

- We thus propose truncated Gramians that only involve a finite number of kernels and can be computed using the methods in MORLAB or M-M.E.S.S.:

$$\boxed{P_{\mathcal{T}} = \sum_{k=1}^{n_p+1} \int_0^\infty \cdots \int_0^\infty \bar{P}_k(t_1,\dots,t_k)\bar{P}_k(t_1,\dots,t_k)^T dt_1 \dots dt_k.}$$

Truncated Gramians

The reachability truncated Gramian solves

$$AP_{\mathcal{T}} + P_{\mathcal{T}}A^T + BB^T + \sum_{j=2}^{n_p} H_j \otimes^j P_l H_j^T + \sum_{j=2}^{n_p} \sum_{k=1}^{m} N_j^k \otimes^j P_l \left(N_j^k\right)^T = 0.$$

where $AP_l + P_l A^T + BB^T = 0$

- Advantage: Only need to solve a finite number of (linear) Lyapunov equations.

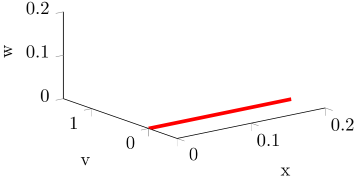
$$\epsilon v_t(x,t) = \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + q,$$
$$w_t(x,t) = hv(x,t) - \gamma w(x,t) + q,$$

with a nonlinear function

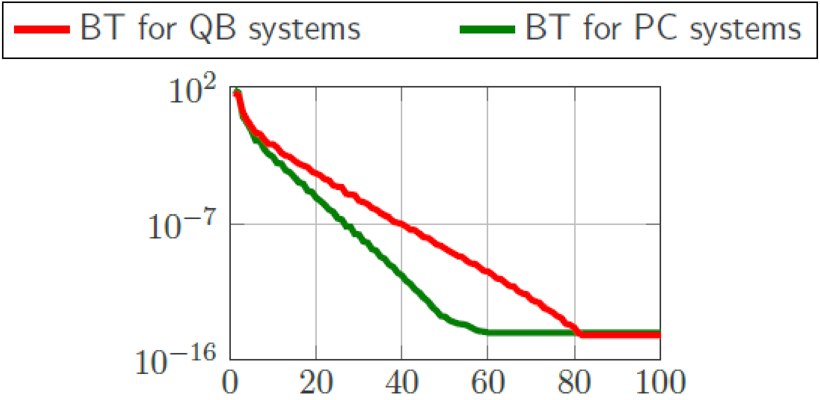
$$f(v(x,t)) = v(v-0.1)(1-v),$$

$\epsilon = 0.015,\ h = 0.5,\ \gamma = 2,\ q = 0.05,\ L = 0.2,$
and boundary conditions:

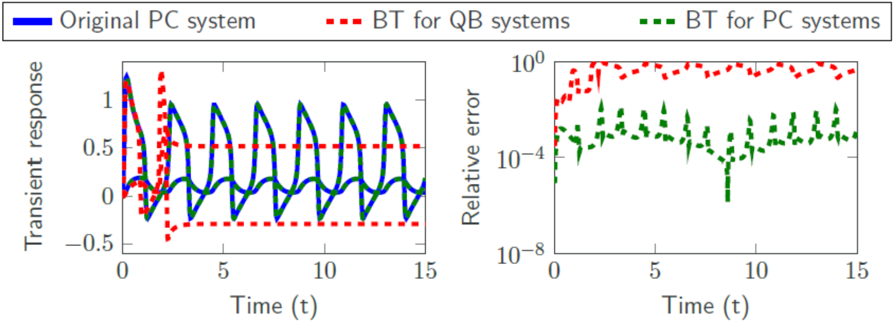
$$v_x(0,t) = i_0(t), \quad v_x(L,t) = 0, \quad t \geq 0,$$



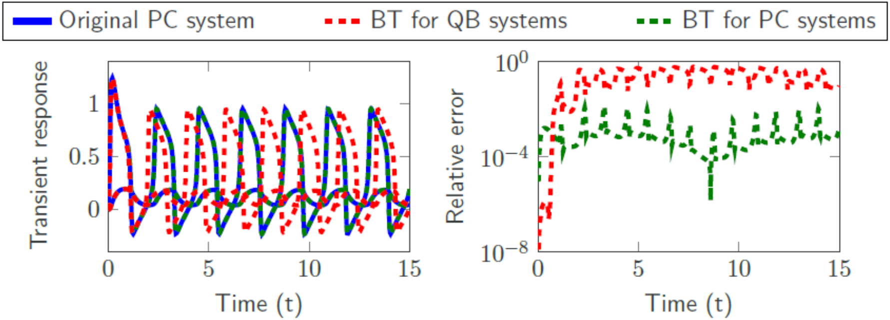
- Spatial discretization leads to PC system with cubic nonlinearity of order $n_{pc} = 600$.
- The transformed quadratic-bilinear (QB) system is of order $n_{qb} = 900$.
- Outputs of interest $v(0,t), w(0,t)$ are the responses at the left boundary.
- We compare balanced truncation for PC and QB systems.



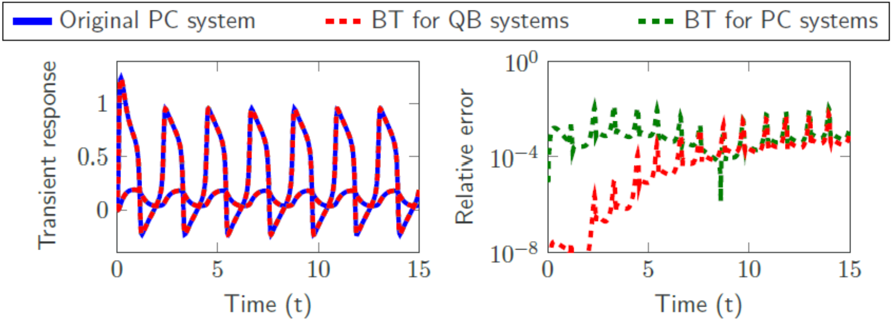
- Decay singular values for PC systems is faster $\Rightarrow$ smaller reduced order model!



- Original PC system of order 600. Original QB system of order 900.
- Reduced PC system of order 10. Reduced QB system of order 10.



- Original PC system of order 600. Original QB system of order 900.
- Reduced PC system of order 10. Reduced QB system of order 30.



- Original PC system of order 600. Original QB system of order 900.
- Reduced PC system of order 10. Reduced QB system of order 43.

Conclusions

- BT extended to bilinear, quadratic-bilinear, and polynomial systems.
- Not in this talk: local Lyapunov stability is preserved.
- As of yet, only weak motivation by local bounds of energy functionals.
- No error bounds in terms of "Hankel" singular values.
- Computationally efficient (as compared to nonlinear balancing), and input independent.
- **To do:**
  - improve efficiency of Lyapunov solvers with many right-hand sides further;
  - error bound;
  - conditions for existence of new PC Gramians;
  - extension to descriptor systems.

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Coming soon(er or later).