MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Interpolation-Based Model Reduction for Polynomial Systems

Peter Benner

Joint work with
Pawan Goyal (MPI DCTS, Magdeburg)

SIAM Conference on Computational Science and Engineering
MS309: Data-Driven Modeling and Model Reduction:
Honoring the 70th Birthday of Thanos Antoulas

March 5, 2021

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NETWORK
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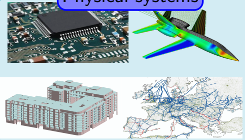
materials science

Motivation

Physical systems



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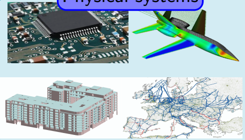

Circuits laws, mechanics, etc.

Partial differential equations

e.g., Navier-Stokes equations

$$\dot{v} + (v \cdot \nabla)v - \frac{1}{\mathrm{Re}}\Delta v + \nabla p = 0,$$
$$\nabla \cdot v = 0.$$

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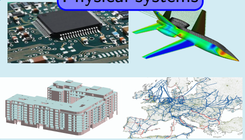

Using expert knowledge, experimental data

Circuits laws, mechanics, etc.

Differential (algebraic) equations

$$E\dot{x}(t) = f(x(t), u(t)),$$
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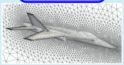

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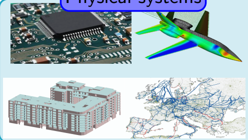

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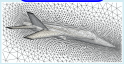

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simulations, control, optimization, UQ, etc.

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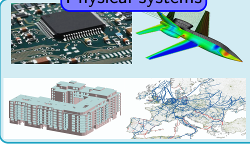

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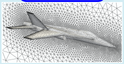

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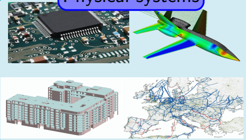

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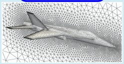

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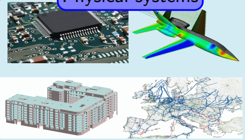

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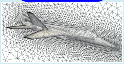

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Reduce numerical complexity

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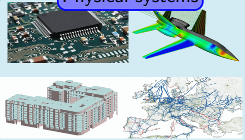

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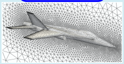

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MOR →

A solution

Reduced-order modeling

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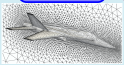

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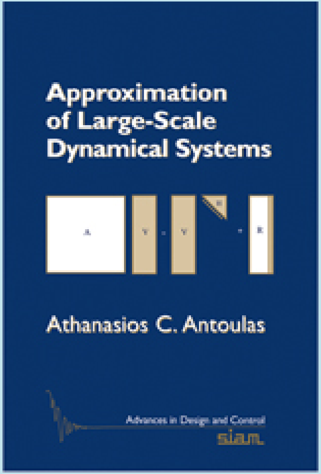
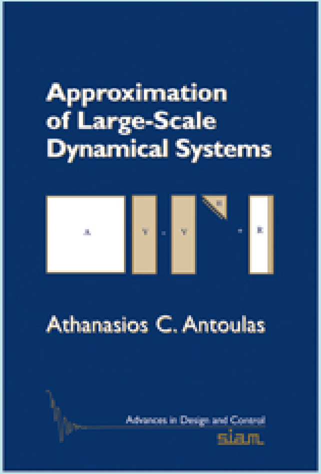
Reduced-order modeling

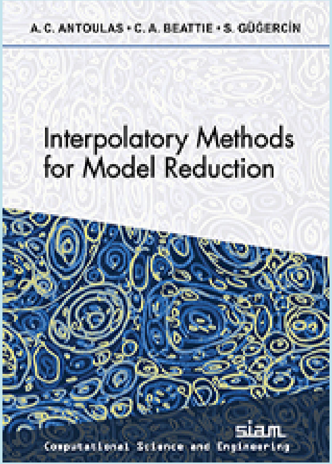
From 2005 . . .


Approximation of Large-Scale Dynamical Systems
Athanasios C. Antoulas
Advances in Design and Control
siam

Cornerstones of Model Order Reduction

From 2005 . . . to 2020





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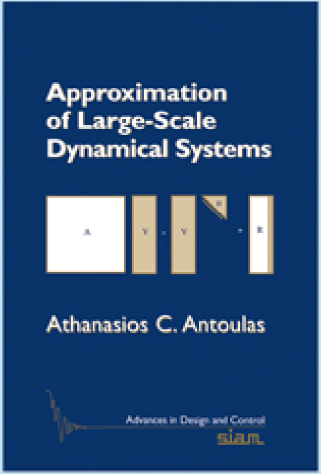

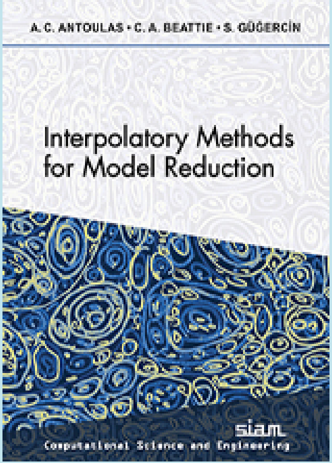

Polynomial systems

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \sum_{\xi=2}^{d} \mathbf{H}_\xi \mathbf{x}^{ⓧ\xi}(t) + \sum_{\eta=2}^{d} \mathbf{N}_\xi \left(\mathbf{u}(t) \otimes \mathbf{x}^{ⓧ\eta}(t)\right) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad \mathbf{x}(0) = 0,$$

where

- d is the degree of the polynomial term in the system;
- (generalized) states $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{x}^{ⓖ} := \underbrace{\mathbf{x}(t) \otimes \cdots \otimes \mathbf{x}(t)}_{g-\text{times}}$;
- inputs (controls) $\mathbf{u}(t) \in \mathbb{R}^m$;
- outputs (measurements, quantity of interest) $\mathbf{y}(t) \in \mathbb{R}^q$.

Polynomial systems

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Lifting

- A large class of nonlinear systems can be lifted to polynomial systems using auxiliary variables.
- Integral part of McCormick Relaxation in optimization. $\rightsquigarrow$ No approximation! [McCormick '76, Gu '09]

A toy example

$$\dot{\mathbf{x}}_1(t) = -\mathbf{x}_1(t) + \mathbf{x}_2^3(t) + e^{-\mathbf{x}_2(t)}, \tag{1a}$$

$$\dot{\mathbf{x}}_2(t) = -\mathbf{x}_1(t) + \mathbf{u}(t). \tag{1b}$$

- To write system (1) in polynomial form, define $\mathbf{z}(t) := e^{-\mathbf{x}_2(t)}$.
- Then, derive ODE for $\mathbf{z}(t)$:

$$\dot{\mathbf{z}}(t) = -e^{-\mathbf{x}_2(t)} \cdot \dot{\mathbf{x}}_2(t) = -\mathbf{z}(t)\left(-\mathbf{x}_1(t) + \mathbf{u}(t)\right).$$

Resulting polynomial(-bilinear) system

$$\begin{aligned}
\dot{\mathbf{x}}_1(t) &= -\mathbf{x}_1(t) + \mathbf{x}_2^3(t) + \mathbf{z}(t),\\
\dot{\mathbf{x}}_2(t) &= -\mathbf{x}_1(t) + \mathbf{u}(t),\\
\dot{\mathbf{z}}(t) &= \mathbf{x}_1(t)\mathbf{z}(t) - \mathbf{z}(t)\mathbf{u}(t).
\end{aligned}$$

Construction of ROMs

Full-order system

$$
\begin{aligned}
\mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \sum_{\xi=2}^{d} \mathbf{H}_\xi x^{\textcircled{\scriptsize$\xi$}}(t) + \sum_{\eta=2}^{d} \mathbf{N}_\xi \left(\mathbf{u}(t) \otimes \mathbf{x}^{\textcircled{\scriptsize$\eta$}}(t)\right) + \mathbf{B}\mathbf{u}(t),\\
\mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \quad \mathbf{x}(0) = 0.
\end{aligned}
$$

(Petrov-)Galerkin

Reduced-order system

$$
\begin{aligned}
\widehat{\mathbf{E}}\dot{\widehat{\mathbf{x}}}(t) &= \widehat{\mathbf{A}}\widehat{\mathbf{x}}(t) + \sum_{\xi=2}^{d} \widehat{\mathbf{H}}_\xi \widehat{\mathbf{x}}^{\textcircled{\scriptsize$\xi$}}(t) + \sum_{\eta=2}^{d} \widehat{\mathbf{N}}_\xi \left(\mathbf{u}(t) \otimes \widehat{\mathbf{x}}^{\textcircled{\scriptsize$\eta$}}(t)\right) + \widehat{\mathbf{B}}\mathbf{u}(t),\\
\widehat{\mathbf{y}}(t) &= \widehat{\mathbf{C}}\widehat{\mathbf{x}}(t), \quad \widehat{\mathbf{x}}(0) = 0.
\end{aligned}
$$

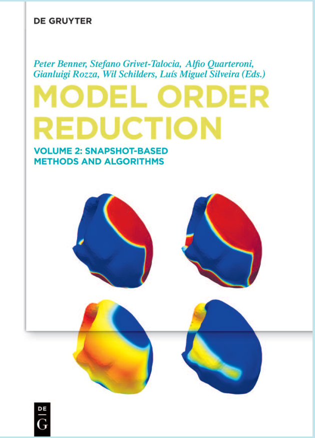
$$
\begin{aligned}
&\widehat{\mathbf{E}} = \mathbf{W}^\top\mathbf{E}\mathbf{V}, \quad \widehat{\mathbf{A}} = \mathbf{W}^\top\mathbf{A}\mathbf{V}, \quad \widehat{\mathbf{H}}_\xi = \mathbf{W}^\top\mathbf{H}_\xi V^{\textcircled{\scriptsize$\xi$}}, \quad \xi \in \{2,\dots,d\},\\
&\widehat{\mathbf{B}} = \mathbf{W}^\top\mathbf{B}, \quad \widehat{\mathbf{C}} = \mathbf{C}\mathbf{V}, \quad \widehat{\mathbf{N}}_\eta = \mathbf{W}^\top\mathbf{H}_\eta\mathbf{V}^{\textcircled{\scriptsize$\eta$}}, \quad \eta \in \{1,\dots,d\}.
\end{aligned}
$$

Common Approaches for Nonlinear MOR

Snapshot-based methods

- Proper orthogonal decomposition — e.g., recent tutorial [Grässle/Hinze/Volkwein '20]
- Reduced basis methods — e.g., recent tutorial [Maday/Patera '20]
- Non-intrusive reduced-order modeling — e.g, [Peherstorfer/Willcox '16]

Commercial break: recent tutorials on snapshot-based methods (full open access!)



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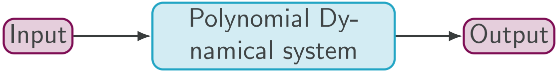
System-theoretic methods

No transient simulation of full-order systems, rather utilize concepts from systems and control theory.

- For order 2 polynomial systems (known as quadratic-bilinear systems)
  - Balanced truncation [B./Goyal '17]
  - Interpolation-based methods [Gu '11, B./Breiten '15, Ahmad/B./Jaimoukha 16, B./Goyal/Gugercin '18, Ahmad/Feng/B. 19]
  - Loewner approach (based on suitable input-output data) [Ionita '13, Antoulas/Goesa '18]
- For order $d \geq 3$ polynomial systems
  - Balanced truncation [B./Goyal/Pontes '19]
  - Here: interpolation based approach.

Input-Output Mapping



- Like for linear systems, we can define input-output mapping by generalized transfer functions.
- Instead of a single transfer function, we have a series of transfer functions.

Input-Output Mapping

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\mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \sum_{\xi=2}^{d} \mathbf{H}_{\xi}\mathbf{x}^{\textcircled{\scriptsize$\xi$}}(t) + \sum_{\eta=2}^{d} \mathbf{N}_{\xi}\left(\mathbf{u}(t) \otimes \mathbf{x}^{\textcircled{\scriptsize$\eta$}}(t)(t)\right) + \mathbf{B}\mathbf{u}(t),\\
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- Like for linear systems, we can define input-output mapping by generalized transfer functions.
- Instead of a single transfer function, we have a series of transfer functions.
- Generalized transfer functions (a few leading ones) [B./Goyal '21]

$$
\begin{aligned}
\mathbf{F}_{\mathrm{L}}(s_1) &:= \mathbf{C}\Phi(s_1)\mathbf{B},\\
\mathbf{F}_{\mathrm{H}}^{(\xi)}(s_1, \ldots, s_{\xi+1}) &:= \mathbf{C}\Phi(s_{\xi+1})\mathbf{H}_{\xi}\left(\Phi(s_{\xi})\mathbf{B} \otimes \cdots \otimes \Phi(s_1)\mathbf{B}\right),\\
\mathbf{F}_{\mathrm{N}}^{(\eta)}(s_1, \ldots, s_{\eta+1}) &:= \mathbf{C}\Phi(s_{\eta+1})\mathbf{N}_{\eta}\left(\mathbf{I}_m \otimes \Phi(s_{\eta})\mathbf{B} \otimes \cdots \otimes \Phi(s_1)\mathbf{B}\right),
\end{aligned}
$$

where $\Phi(s) = (s\mathbf{E} - \mathbf{A})^{-1}$.

Interpolating Reduced System

Goal

Construct projection matrices $\mathbf{V}$ and $\mathbf{W}$ such that generalized transfer functions (GTF) satisfy interpolation conditions, i.e.,

$$(\mathsf{GTF}(\sigma))_{\text{original}} = (\mathsf{GTF}(\sigma))_{\text{reduced}},$$

and reduced matrices are constructed via Petrov-Galerkin projection:

$$\begin{aligned}
\widehat{\mathbf{E}} &= \mathbf{W}^\top \mathbf{E}\mathbf{V}, & \widehat{\mathbf{A}} &= \mathbf{W}^\top \mathbf{A}\mathbf{V}, & \widehat{\mathbf{H}}_\xi &= \mathbf{W}^\top \mathbf{H}_\xi \mathbf{V}^{\textcircled{\xi}}, & \xi &\in \{2,\dots,d\},\\
\widehat{\mathbf{B}} &= \mathbf{W}^\top \mathbf{B}, & \widehat{\mathbf{C}} &= \mathbf{C}\mathbf{V}, & \widehat{\mathbf{N}}_\eta &= \mathbf{W}^\top \mathbf{N}_\eta \mathbf{V}^{\textcircled{\eta}}, & \eta &\in \{1,\dots,d\}.
\end{aligned}$$

- Extended ideas from linear systems to polynomial systems.
- Choice of good interpolation points (open question).
- Moreover, we discuss construction of dominant subspaces by combining interpolation and Loewner framework.

Theorem (simplified) [B./Goyal '21]

Let σ_i and μ_i, $i \in \{1, \ldots, \tilde{r}\}$, be interpolation points, and define subspaces $\mathcal{V}$ and $\mathcal{W}$ as follows:

$$\mathcal{V}_{\mathrm{L}} = \operatorname{range}\left(\mathbf{\Phi}(\sigma_1)\mathbf{B}, \ldots, \mathbf{\Phi}(\sigma_{\tilde{r}})\mathbf{B}\right)$$

$$\mathcal{V}_{\mathrm{N}}^{(\eta)} = \bigcup_{i=1}^{\tilde{r}} \operatorname{range}\left(\mathbf{\Phi}(\sigma_i)\mathbf{N}_\eta\left(\mathbf{\Phi}(\sigma_i)\mathbf{B} \otimes \cdots \otimes \mathbf{\Phi}(\sigma_i)\mathbf{B}\right)\right)$$

$$\mathcal{V}_{\mathrm{H}}^{(\xi)} = \bigcup_{i=1}^{\tilde{r}} \operatorname{range}\left(\mathbf{\Phi}(\sigma_i)\mathbf{H}_\xi\left(\mathbf{\Phi}(\sigma_i)\mathbf{B} \otimes \cdots \otimes \mathbf{\Phi}(\sigma_i)\mathbf{B}\right)\right)$$

$$\mathcal{V} \subset \operatorname{range}\left(\mathbf{V}_{\mathrm{L}}, \mathbf{V}_{\mathrm{N}}^{(\eta)}, \mathbf{V}_{\mathrm{H}}^{(\xi)}\right)$$

$$\mathcal{W}_{\mathrm{L}} = \operatorname{range}\left(\mathbf{\Phi}(\mu_1)^\top\mathbf{B}, \ldots, \mathbf{\Phi}(\mu_{\tilde{r}})^\top\mathbf{C}^\top\right)$$

$$\mathcal{W}_{\mathrm{N}}^{(\eta)} = \bigcup_{i=1}^{\tilde{r}} \operatorname{range}\left(\mathbf{\Phi}(\sigma_i)^\top(\mathbf{N}_\eta)_{(2)}\left(\mathbf{\Phi}(\sigma_i)\mathbf{B} \otimes \cdots \otimes \mathbf{\Phi}(\mu_i)^\top\mathbf{C}^\top\right)\right)$$

$$\mathcal{W}_{\mathrm{H}}^{(\xi)} = \bigcup_{i=1}^{\tilde{r}} \operatorname{range}\left(\mathbf{\Phi}(\sigma_i)^\top(\mathbf{H}_\xi)_{(2)}\left(\mathbf{\Phi}(\sigma_i)\mathbf{B} \otimes \cdots \otimes \mathbf{\Phi}(\mu_i)^\top\mathbf{C}^\top\right)\right)$$

$$\mathcal{W} \subset \operatorname{range}\left(\mathbf{W}_{\mathrm{L}}, \mathbf{W}_{\mathrm{N}}^{(\eta)}, \mathbf{W}_{\mathrm{H}}^{(\xi)}\right)$$

where $\Phi(s) := (s\mathbf{E} - \mathbf{A})^{-1}$. If the ROM is constructed using Petrov-Galerkin projection with compatible basis matrices $\mathbf{V}$ of $\mathcal{V}$ and $\mathbf{W}$ of $\mathcal{W}$, then GTFs of original model and ROM match at σ, μ.

Remarks

- Quality of the ROM depends on the interpolation points. Optimal selection is an open problem.
- Thus, we propose an approach to determine dominant subspaces for MOR.

Algorithm: Construction of Dominant Subspaces for MOR

1. Take $\sigma_i, \mu_i,\ i = 1, \ldots, \mathcal{N}$.
2. Compute $\mathcal{R} := \begin{cases} \operatorname{range}\left(\Phi(\sigma_1)\mathbf{B}, \ldots, \Phi(\sigma_\mathcal{N})\mathbf{B}\right) \\ \bigcup_{\eta=1}^{d} \bigcup_{i=1}^{\mathcal{N}} \operatorname{range}\left(\Phi(\sigma_i)\mathbf{N}_\eta \left(\Phi(\sigma_i)\mathbf{B} \otimes \cdots \otimes \Phi(\sigma_i)\mathbf{B}\right)\right) \\ \bigcup_{\xi=2}^{d} \bigcup_{i=1}^{\mathcal{N}} \operatorname{range}\left(\Phi(\sigma_i)\mathbf{H}_\xi \left(\Phi(\sigma_i)\mathbf{B} \otimes \cdots \otimes \Phi(\sigma_i)\mathbf{B}\right)\right) \end{cases}$
3. Compute $\mathcal{O} = \begin{cases} \operatorname{range}\left(\Phi(\mu_1)^\top \mathbf{C}^\top, \ldots, \Phi(\mu_\mathcal{N})^\top \mathbf{C}^\top\right), \\ \bigcup_{\eta=1}^{d} \bigcup_{i=1}^{\mathcal{N}} \operatorname{range}\left(\Phi(\sigma_i)\left(\mathbf{N}_\eta\right)_{(2)} \left(\Phi(\sigma_i)\mathbf{B} \otimes \cdots \otimes \Phi(\sigma_i)\mathbf{B} \otimes \Phi(\mu_i)^\top \mathbf{C}^\top\right)\right) \\ \bigcup_{\xi=2}^{d} \bigcup_{i=1}^{\mathcal{N}} \operatorname{range}\left(\Phi(\sigma_i)\left(\mathbf{H}_\xi\right)_{(2)} \left(\Phi(\sigma_i)\mathbf{B} \otimes \cdots \otimes \Phi(\sigma_i)\mathbf{B} \otimes \Phi(\mu_i)^\top \mathbf{C}^\top\right)\right) \end{cases}$
4. Determine matrices (alike Loewner and shifted-Loewner): $\mathbb{L} = -\mathcal{O}^\top \mathbf{E}\mathcal{R}, \quad \mathbb{L}_s = -\mathcal{O}^\top \mathbf{A}\mathcal{R}$.
5. Compute singular value decomposition:
$$\left[\mathbf{Y}_1, \Sigma_1, \mathbf{X}_1\right] = \operatorname{svd}\left(\left[\mathbb{L}, \mathbb{L}_s\right]\right), \quad \left[\mathbf{Y}_2, \Sigma_2, \mathbf{X}_2\right] = \operatorname{svd}\left(\begin{bmatrix} \mathbb{L} \\ \mathbb{L}_s \end{bmatrix}\right).$$
6. Determine projection matrices: $\mathbf{V} := \mathcal{R}X_2(:, 1:r), \quad \mathbf{W} := \mathcal{O}Y_2(:, 1:r)$.

Algorithm to Determine Dominant Subspaces

Algorithm: Construction of Dominant Subspaces for MOR

1. Take $\sigma_i, \mu_i, i = 1, \ldots, \mathcal{N}$.

2. Compute $\mathcal{R} := \begin{cases} \operatorname{range}\left(\Phi(\sigma_1)\mathbf{B}, \ldots, \Phi(\sigma_\mathcal{N})\mathbf{B}\right) \\ \bigcup_{\eta=1}^{d} \bigcup_{i=1}^{\mathcal{N}} \operatorname{range}\left(\Phi(\sigma_i)\mathbf{N}_\eta\left(\Phi(\sigma_i)\mathbf{B} \otimes \cdots \otimes \Phi(\sigma_i)\mathbf{B}\right)\right) \\ \bigcup_{\xi=2}^{d} \bigcup_{i=1}^{\mathcal{N}} \operatorname{range}\left(\Phi(\sigma_i)\mathbf{H}_\xi\left(\Phi(\sigma_i)\mathbf{B} \otimes \cdots \otimes \Phi(\sigma_i)\mathbf{B}\right)\right) \end{cases}$

3. Compute $\mathcal{O} = \begin{cases} \operatorname{range}\left(\Phi(\mu_1)^\top \mathbf{C}^\top, \ldots, \Phi(\mu_\mathcal{N})^\top \mathbf{C}^\top\right), \\ \bigcup_{\eta=1}^{d} \bigcup_{i=1}^{\mathcal{N}} \operatorname{range}\left(\Phi(\sigma_i)\left(\mathbf{N}_\eta\right)_{(2)}\left(\Phi(\sigma_i)\mathbf{B} \otimes \cdots \otimes \Phi(\sigma_i)\mathbf{B} \otimes \Phi(\mu_i)^\top \mathbf{C}^\top\right)\right) \\ \bigcup_{\xi=2}^{d} \bigcup_{i=1}^{\mathcal{N}} \operatorname{range}\left(\Phi(\sigma_i)\left(\mathbf{H}_\xi\right)_{(2)}\left(\Phi(\sigma_i)\mathbf{B} \otimes \cdots \otimes \Phi(\sigma_i)\mathbf{B} \otimes \Phi(\mu_i)^\top \mathbf{C}^\top\right)\right) \end{cases}$

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Construct reduced-order system

$\hat{\mathbf{E}} = \mathbf{W}^\top \mathbf{A}\mathbf{V}, \ \hat{\mathbf{A}} = \mathbf{W}^\top \mathbf{A}\mathbf{V}, \ \widehat{\mathbf{H}}_\xi = \mathbf{W}^\top \mathbf{H}_\xi V^{\textcircled{\xi}},$

$\hat{\mathbf{B}} = \mathbf{W}^\top \mathbf{B}, \ \hat{\mathbf{C}} = \mathbf{C}\mathbf{V}, \ \widehat{\mathbf{N}}_\eta = \mathbf{W}^\top \mathbf{H}_\eta \mathbf{V}^{\textcircled{\eta}}.$

Algorithm to Determine Dominant Subspaces

- Embarrassingly parallel.
- No need to solve *all* shifted systems, can employ low-rank techniques.

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Algorithm to Determine Dominant Subspaces

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6. Determine projection matrices: $\mathbf{V} := \mathcal{R}X_2(:, 1:r), \quad \mathbf{W} := \mathcal{O}Y_2(:, 1:r)$.

- Efficient variants of SVD can be applied, including randomized SVD.

Governing equation

$$v_t = v_{xx} + v(1 - v^2)$$

with boundary conditions

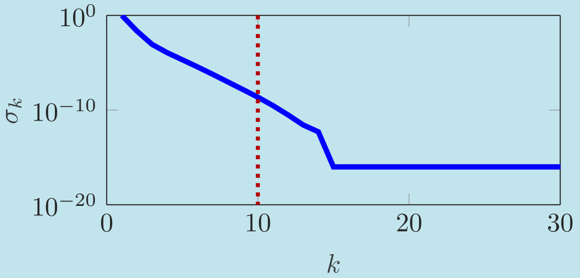
$$v(0, \cdot) = u(t) \qquad v_x(L, \cdot) = 0,\ t \in (0, T) \qquad v(x, 0) = 0, \quad x \in (0, L).$$

- To construct dominant subspaces, we take 200 points on $\jmath\omega$ axis in the frequency range $\left[10^{-3}, 10^{3}\right]$.

Governing equation

$$v_t = v_{xx} + v(1 - v^2)$$

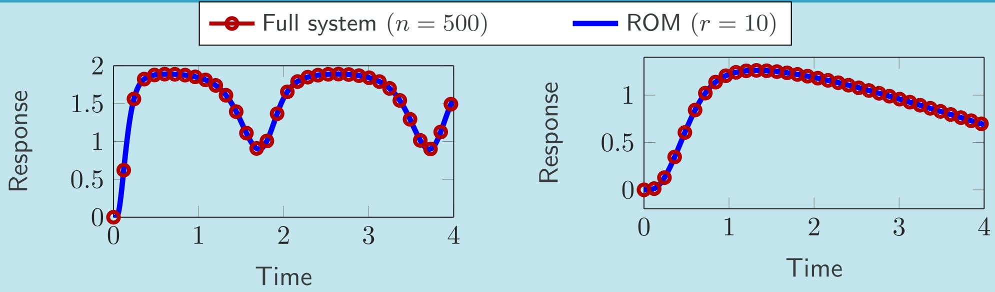
Decay of "Loewner" singular values



Governing equation

$$v_t = v_{xx} + v(1 - v^2)$$

Construction of reduced systems



$\mathbf{u}(t) = 10\sin(\pi t)$

$\mathbf{u}(t) = 5te^{-t}$

Governing coupled PDE-ODE system

$$
\begin{aligned}
\epsilon v_t &= \epsilon^2 v_{xx} + v(v-0.1)(1-v) - w + q,\\
w_t &= hv - \gamma w + q,
\end{aligned}
$$

with boundary conditions

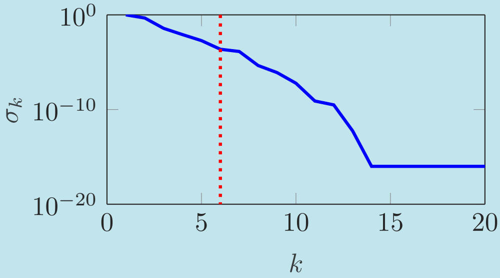
$$
v(x,0) = 0, \qquad w(x,0) = 0, \qquad x \in (0,L), \qquad v_x(0,t) = i_0(t), \qquad v_x(1,t) = 0, \qquad t \geq 0.
$$

- To construct dominant subspaces, we take 200 points on $\jmath\omega$ axis in the frequency range $\left[10^{-2}, 10^{2}\right]$.

Governing coupled PDE-ODE system

$$\epsilon v_t = \epsilon^2 v_{xx} + v(v-0.1)(1-v) - w + q,$$
$$w_t = hv - \gamma w + q,$$

Decay of singular values



Governing coupled PDE-ODE system

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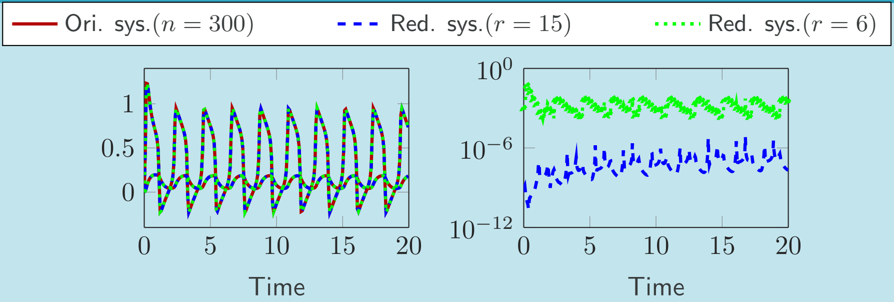
Construction of reduced systems



Outlook and Conclusions

Contributions of the talk

- Employed generalized transfer functions of polynomial systems.
- Construction of interpolatory reduced-order models.
- Proposed an algorithm to determine dominant subspaces.
- Computational aspects in a large-scale setting (low-rank factors, randomized SVDs).
- Application to benchmark examples.
- Not in the talk: use of CUR for cheaper tensor calculus, extension to parametric systems.

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Open questions and future work

- Extension to structured systems such as delay and second-order systems $\rightsquigarrow$ for $d = 1, 2$, see [B./Gugercin/Werner '21]!
- Stability of reduced-order systems $\rightsquigarrow$ for $d = 2$: talk by Boris Kramer in MS31!
- Non-homogeneous initial conditions.
- Construct polynomial systems directly from input-output data $\rightsquigarrow$ for $d = 1, 2$, [Antoulas et al].

Selected References

Anderson, B. D. O. and Antoulas, A. C. (1990).
Rational interpolation and state-variable realizations.
Linear Algebra Appl., 137/138:479–509.

Antoulas, A. C., Lefteriu, S., and Ionita, A. C. (2017).
A tutorial introduction to the Loewner framework for model reduction.
In Benner, P., Cohen, A., Ohlberger, M., and Willcox, K., editors, *Model Reduction and Approximation: Theory and Algorithms*, pages 335–376. SIAM.

Benner, P. and Breiten, T. (2015).
Two-sided projection methods for nonlinear model order reduction.
SIAM J. Sci. Comput., 37(2):B239–B260.

Benner, P. and Goyal, P. (2021).
Interpolation-based model order reduction for polynomial systems.
SIAM J. Sci. Comp, 43(1):A84–A108.

Benner, P., Goyal, P., and Gugercin, S. (2018).
$\mathcal{H}_2$-quasi-optimal model order reduction for quadratic-bilinear control systems.
SIAM J. Matrix Anal. Appl., 39(2):983–1032.

Benner, P., Goyal, P., and Pontes Duff, I. (2019).
Identification of dominant subspaces for linear structured parametric systems and model reduction.
e-prints 1910.13945, arXiv.

Kramer, B. and Willcox, K. (2019).
Nonlinear model order reduction via lifting transformations and proper orthogonal decomposition.
AIAA Journal, 57(6):2297–2307.

Peherstorfer, B. and Willcox, K. (2016).
Data-driven operator inference for nonintrusive projection-based model reduction.
Comp. Meth. Appl. Mech. Eng., 306:196–215.