MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

An Alternative Algorithm for Unstable Balanced Truncation

Peter Benner

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Model Reduction of Linear Systems
Linear Time-Invariant (LTI) Systems

Original System

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.

Reduced-Order Model (ROM)

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{A}\hat{x}(t) + \widehat{B}u(t), \\ \hat{y}(t) = \widehat{C}\hat{x}(t) + \widehat{D}u(t). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
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Goals:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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Secondary goal: reconstruct approximation of x from $\hat{x}$.

Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s), \dot{x}(t) \mapsto sX(s) - x(0))$ to LTI system

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$\Longrightarrow$ I/O-relation in frequency domain:

$$Y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)}\Big)U(s).$$

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Model reduction in frequency domain: Fast evaluation of mapping $U \rightarrow Y$.

Formulating model reduction in frequency domain

Approximate the time domain dynamical system

$$\begin{array}{rcll} \dot{x} & = & Ax + Bu, & A \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times m}, \\ y & = & Cx + Du, & C \in \mathbb{R}^{p \times n},\ D \in \mathbb{R}^{p \times m}, \end{array}$$

by reduced-order system

$$\begin{array}{rcll} \dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}u, & \hat{A} \in \mathbb{R}^{r \times r},\ \hat{B} \in \mathbb{R}^{r \times m}, \\ \hat{y} & = & \hat{C}\hat{x} + \hat{D}u, & \hat{C} \in \mathbb{R}^{p \times r},\ \hat{D} \in \mathbb{R}^{p \times m} \end{array}$$

of order $r \ll n$, such that

$$\begin{aligned} \|y - \hat{y}\| \simeq \left\|Y - \hat{Y}\right\| = \left\|GU - \hat{G}U\right\| & \\ \leq \left\|G - \hat{G}\right\| \cdot \|U\| \simeq \left\|G - \hat{G}\right\| \cdot \|u\| & \\ \leq \text{tolerance} \cdot \|u\| . & \end{aligned}$$

Balanced Truncation for Stable Systems

Basic concept

- System $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

  satisfy: $P = Q = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization (needs P, Q!) of the system via state-space transformation

  $$\begin{aligned} \mathcal{T} : (A, B, C) &\mapsto (TAT^{-1}, TB, CT^{-1}) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right). \end{aligned}$$

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}) = (A_{11}, B_1, C_1)$.
  Note: in efficient algorithms, truncation is achieved via projection:

$$(\hat{A}, \hat{B}, \hat{C}) = (W^TAV, W^TB, CV), \quad \text{where} \quad W^TV = I_r.$$

Properties

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.

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- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \|G - \hat{G}\|_{\mathcal{H}_\infty} \|u\|_2 \leq \left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right) \|u\|_2 ,$$

where $\|G\|_{\mathcal{H}_\infty} := \sup_{u \in \mathcal{L}_2 \setminus \{0\}} \frac{\|Gu\|_2}{\|u\|_2} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(\jmath\omega))$.

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Practical implementation

- Rather than solving Lyapunov equations for P, Q (n^2 unknowns!), find $S, R \in \mathbb{R}^{n \times s}$ with $s \ll n$ such that $P \approx SS^T$, $Q \approx RR^T$.
- Reduced-order model directly obtained via small-scale ($s \times s$) SVD of $R^T S$!

[a] https://www.mpi-magdeburg.mpg.de/projects/morlab
[b] https://www.mpi-magdeburg.mpg.de/projects/mess, full MATLAB integration in progress.

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- Two software packages:
  - MORLAB[a] (**M**odel **O**rder **R**eduction **LAB**oratory), based on spectral projection methods ($\rightsquigarrow$ small to medium size problems, up to $n \sim 5,000$.)
  - M-M.E.S.S.[b] provides solvers for large-scale matrix equations with sparse/low-rank coefficients and basic MOR functionality; no $\mathcal{O}(n^3)$ or $\mathcal{O}(n^2)$ computations necessary!

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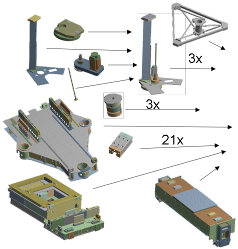
Numerical Example

Thermal Model of Experimental Machine Tool MAX

[Vettermann et al '21]

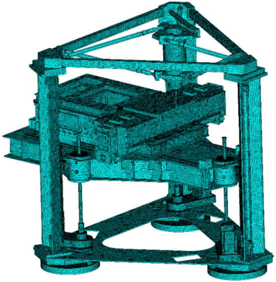
50 subassemblies



CAD model

FEM $\rightsquigarrow$

FE-Model: **1.2M DOFs**



Industrial challenges for virtual twin:

- **non-homogeneous initial conditions** (IC) — two approaches: augment input with IC ("BTX0") or use superposition ("2phase"),
- subsystem reduction ("output coupled") vs. holistic reduction ("FE-coupled").

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Numerical Example

MAX: Results considering an inhomogeneous initial condition $x_0 \neq 0$

FE-coupled

method	red. order tol 10^{-3}	t_{red}
2phase	196	6.5h
BTX0	174	4.5h

output-coupled

method	red. order tol 10^{-3}	t_{red}
2phase	3,005	2h
BTX0	2,515	1.8h

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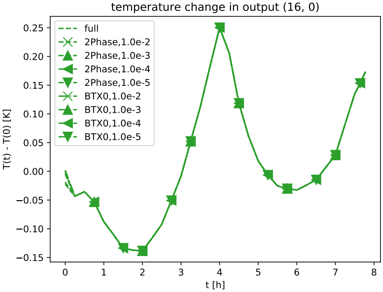
$\rightarrow$ Required storage for reduced matrices just 1MB!



Vettermann, J., Sauerzapf, S., Naumann, A., Beitelschmidt, M., Herzog, R., Benner, P., Saak, J. (2021): Model order reduction methods for coupled machine tool models. MM Science Journal 2021:4652-4659.

Basic Principle

Given some positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Balanced Truncation for Unstable Systems

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Classical Balanced Truncation (BT)

[Mullis/Roberts 1976, Moore 1981]

- $P =$ controllability Gramian of system given by (A, B, C, D).
- $Q =$ observability Gramian of system given by (A, B, C, D).
- If A is stable, P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0.$$

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LQG Balanced Truncation (LQGBT)

[Jonckheere/Silverman 1983]

- $P/Q =$ controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= AP + PA^T - PC^T CP + B^T B, \\ 0 &= A^T Q + QA - QBB^T Q + C^T C. \end{aligned}$$

- Computable error bound:

$$\|[\,N\ M\,] - [\,\hat{N}\ \hat{M}\,]\|_{\mathcal{H}_\infty} \leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{\mathrm{LQG}}}{\sqrt{1 + (\sigma_j^{\mathrm{LQG}})^2}},$$

  where σ_j^{LQG} are the singular values of $(PQ)^{\frac{1}{2}}$ and $G = M^{-1}N$ and $\hat{G} = \hat{M}^{-1}\hat{N}$ are left coprime factorizations.

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Closed-loop Balanced Truncation (CLBT)

[Wortelboer 1994]

- First, stabilize the system using LQR feedback:

$$\dot{x}_s(t) = (A - BB^T X_s)x_s(t) + Bu(t), \qquad y_s(t) = Cx_s(t), \qquad (*)$$

where X_s is the stabilizing solution of the LQR Riccati equation

$$A^T X + XA - XBB^T X + C^T C = 0.$$

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- Then apply BT to the closed-loop system $(*)$ by solving the Lyapunov equations

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Observation: P_s can be computed without ever forming X_s!

Theorem (Kenney/Laub/Jonckheere 1989, B. 2019)

Let (A, B) be stabilizable, (A, C) be detectable, and define the Hamiltonian matrix

$$\begin{bmatrix} A & -BB^T \\ -C^T C & -A^T \end{bmatrix}.$$

CLBT via the Matrix Sign Function

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Hence, $A - BB^TX_s$ is stable, the closed-loop Lyapunov equations

$$\begin{aligned} (A - BB^TX_s)P_s + P_s(A - BB^TX_s)^T + BB^T &= 0, \\ (A - BB^TX_s)^TQ_s + Q_s(A - BB^TX_s) + C^TC &= 0, \end{aligned}$$

have unique solutions $P_s = P_s^T \geq 0$, $Q_s = Q_s^T \geq 0$, resp., and it holds

$$\operatorname{sign}(H) = \begin{bmatrix} -I + 2P_sX_s & -2P_s \\ 2X_sP_sX_s - 2X_s & I - 2X_sP_s \end{bmatrix}.$$

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But: Q_s still would need X_s first!

Recall:

$$\operatorname{sign}(H) = \begin{bmatrix} -I + 2P_sX_s & -2P_s \\ 2X_sP_sX_s - 2X_s & I - 2X_sP_s \end{bmatrix}.$$

Can neither get Q_s from this, nor X_s without further computations.

[1] The proof is analogous to that of the previous theorem.

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But: using the stable[1] closed-loop matrix $A - Y_sC^TC$ obtained from the filter Riccati equation, we obtain another stable LTI system.

$$\dot{x}_f(t) = (A - Y_sC^TC)x_f(t) + Bu(t), \qquad y_f(t) = Cx_f(t). \qquad (\bullet)$$

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CLBT via the Matrix Sign Function

Where is Q_s?

Recall:

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$$\dot{x}_f(t) = (A - Y_s C^T C) x_f(t) + Bu(t), \qquad y_f(t) = C x_f(t). \qquad (\bullet)$$

The observability Gramian $\tilde{Q}_s$ of $(\bullet)$ solves the Lyapunov equation

$$(A - Y_s C^T C)^T \tilde{Q} + \tilde{Q}(A - Y_s C^T C) + C^T C = 0.$$

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Recall:

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$$\dot{x}_f(t) = (A - Y_sC^TC)x_f(t) + Bu(t), \qquad y_f(t) = Cx_f(t). \qquad (\bullet)$$

The observability Gramian $\tilde{Q}_s$ of $(\bullet)$ solves the Lyapunov equation

$$(A - Y_sC^TC)^T\tilde{Q} + \tilde{Q}(A - Y_sC^TC) + C^TC = 0.$$

Now, analogously to P_s, $\tilde{Q}_s$ can be read off from the $(1,2)$-block of the Hamiltonian matrix associated to the filter Riccati equation, i.e. $H^T \rightsquigarrow$

$$\begin{bmatrix} * & -2\tilde{Q}_s \\ * & * \end{bmatrix} = \operatorname{sign}\left(H^T\right)$$

[1] The proof is analogous to that of the previous theorem.

Recall:

$$\operatorname{sign}(H) = \begin{bmatrix} -I + 2P_sX_s & -2P_s \\ 2X_sP_sX_s - 2X_s & I - 2X_sP_s \end{bmatrix}.$$

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This suggests a new balanced truncation scheme **CLBT2** using $P_s, \tilde{Q}_s$ as Gramians.

[1] The proof is analogous to that of the previous theorem.

CD player data set from the SLICOT benchmark collection[a], with $n = 120$, $m = p = 2$.
We compute reduced-order models (ROMs) of order $r = 30$ using BT and LQGBT as implemented in MORLAB [B.2006, WERNER/B. 2020], and compare to the "new" variant CLBT2.

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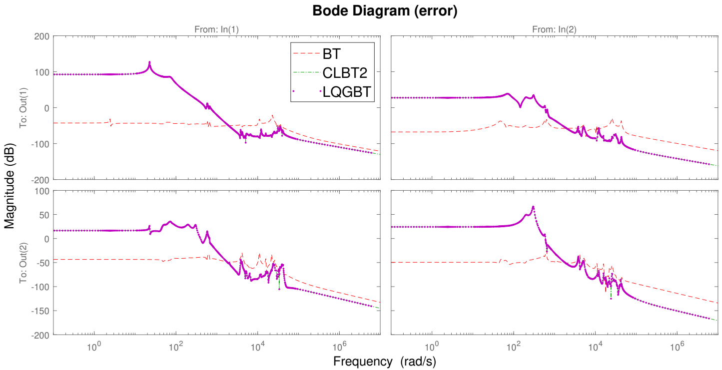

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EADY data set from the SLICOT benchmark collection[a], with $n = 598$, $m = p = 1$.
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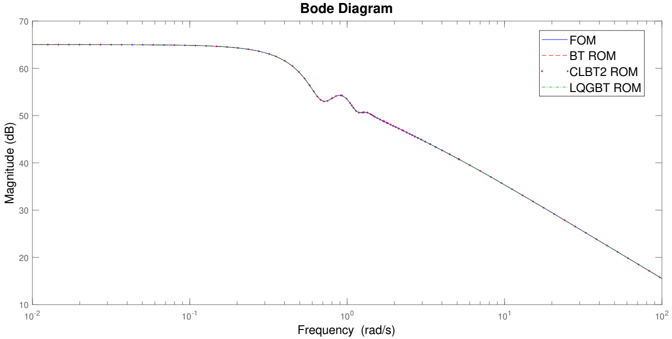

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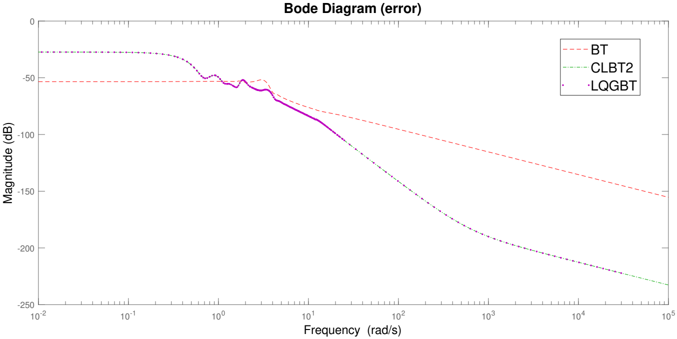

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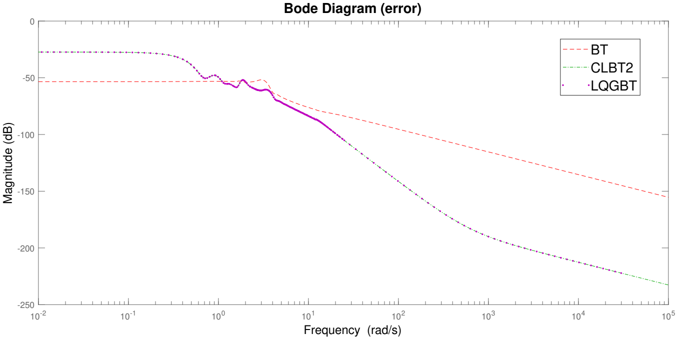

Both examples indicate that LGQ BT and CLBT2 yield the same ROMs.
If this holds, CLBT2 ROM satisfies same error bound as LQGBT.

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- Implementation of CLBT2 via sign function method is computationally more efficient than LQG BT.
  But: no implementation strategy for large-scale sparse systems so far!

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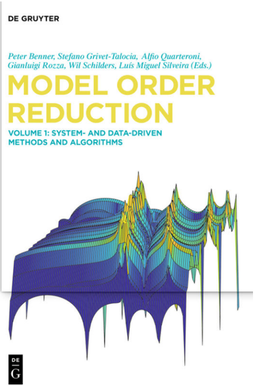

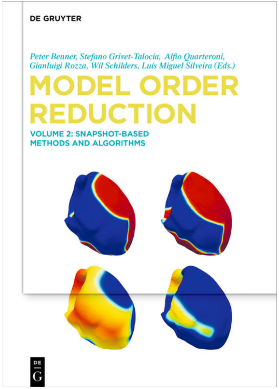

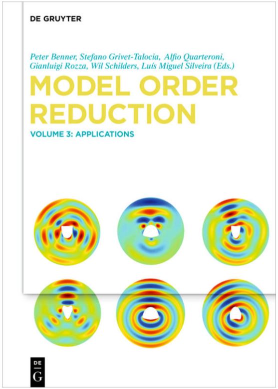

- Edited by Peter Benner, Stefano Grivet-Talocia, Alfio Quarteroni, Gianluigi Rozza, Wil Schilders, and Luís Miguel Silveira,
- contains 30 tutorial chapters on modern model reduction techniques, methods, applications, and software,
- published by DeGruyter in 2021, ebook is fully Open Access!