MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

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CSC COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Numerical Methods for Feedback Stabilization of Unsteady PDE Problems

Peter Benner

joint work with Jens Saak, Eberhard Bänsch,
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Optimal Control

is used for the optimization of **dynamical processes**,

described by ordinary or partial differential equations.

This is achieved by minimizing a **cost functional**

(penalizing, e.g. energy consumption, deviation from reference trajectory),

such that a prescribed target
is reached in given or **minimal time**

whilst complying with given control and state constraints.

Let (x_*, u_*) solve optimal control problem

$$\min_{u \in \mathcal{U}_{ad}} J(x,u) \textbf{ s.t. } \dot{x}(t) = f(x(t), u(t)).$$

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Fundamental observation

Optimized trajectory $x_*(t; u_*)$ and precomputed optimal control $u_*(t)$ will not be attainable in practice due to

- modeling errors and/or unmodeled dynamics,
- model uncertainties,
- external perturbations,
- measurement errors.

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- external perturbations,
- measurement errors.

Consequence: need **feedback control**

$$u(t) = u_*(t) + U(t, x(t) - x_*(t))$$

in order to attenuate perturbations/errors!

Example: Optimal control of a simple transport model

Burgers' equation:

$$\begin{aligned}
\partial_t x(t,\xi) &= \nu\, \partial_{\xi\xi} x(t,\xi) - x(t,\xi)\, \partial_\xi x(t,\xi) + B(\xi)u(t), \\
x(t,0) &= x(t,1) = 0, \quad x(0,\xi) = x_0(\xi), \quad \xi \in (0,1), \\
y(t,\xi) &= C\, x(t,\xi).
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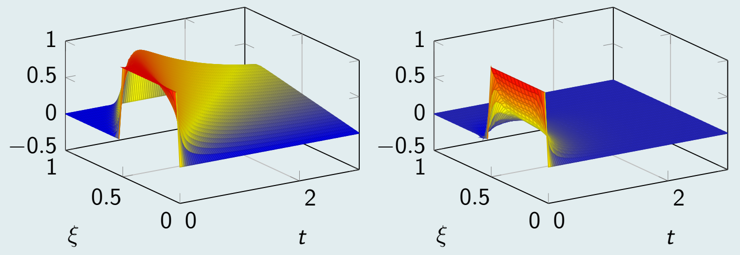

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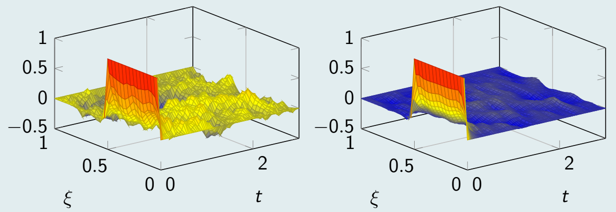
Nonlinear control (here: MPC-LQG):



Reduction of tracking error $\int_0^T \|x(t) - x_*(t)\|_2^2\, dt$ by factor > 10.

[Benner/Görner, PAMM 2006]; [Benner/Görner/Saak, Springer LNCSE 2006].

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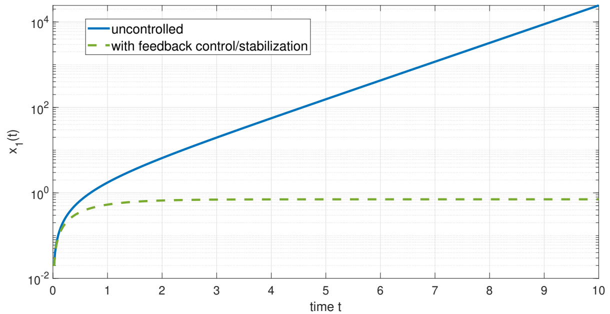
Illustration

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$



Outline

The Linear-Quadratic Regulator (LQR) Problem

Minimize $\mathcal{J}(u) = \frac{1}{2}\int_0^\infty \left(y^T Q y + u^T R u\right) dt$ for $u \in \mathcal{L}_2(0,\infty;\mathbb{R}^m)$,

subject to

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t), \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$.

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Solution of finite-dimensional LQR problem: **feedback control**

$$u_*(t) = -B^T X_* x(t) =: -K_* x(t),$$

where $X_* = X_*^T \geq 0$ is unique **stabilizing**[1] solution of **algebraic Riccati equation (ARE)**

$$0 = \mathcal{R}(X) := C^T Q C + A^T X + XA - XBR^{-1}B^T X.$$

[1] X is stabilizing $\Leftrightarrow \Lambda\left(A - BB^T X\right) \subset \mathbb{C}^-$.

Given Hilbert spaces

$\mathbb{X}$ – **state space**,
$\mathbb{U}$ – **control space**,
$\mathbb{Y}$ – **output space**,

and operators

$$\mathcal{A} : \mathrm{dom}(\mathcal{A}) \subset \mathbb{X} \to \mathbb{X}, \quad \mathcal{B} : \mathbb{U} \to \mathbb{X}, \quad \mathcal{C} : \mathbb{X} \to \mathbb{Y}.$$

LQR Problem in Hilbert Space

Minimize

$$\mathcal{J}(x_0, u) = \frac{1}{2} \int_0^\infty \left(\|y\|_{\mathcal{Y}}^2 + \|u\|_{\mathcal{U}}^2 \right) dt, \qquad \text{for} \quad u \in \mathbb{L}_2(0, \infty; \mathbb{U})$$

subject to

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \qquad x(0) = x_0 \in \mathbb{X},$$
$$y = \mathcal{C}x.$$

Theorem (Gibson '79)

Assumptions:

- $\mathcal{A}$ *infinitesimal generator of a strongly continuous* (C_0)*-semigroup;* $\mathcal{B}, \mathcal{C}$ *linear, bounded.*
- $(\mathcal{A}, \mathcal{B})$ stabilizable, *i.e.,* $\exists \mathcal{K} : \mathbb{X} \rightarrow \mathbb{U}$ *linear, bounded, such that* C_0*-semigroup generated by* $\mathcal{A} + \mathcal{B}\mathcal{K}$ *is exponentially stable.*
- $(\mathcal{C}, \mathcal{A})$ detectable, *i.e.,* $(\mathcal{A}^*, \mathcal{C}^*)$ *stabilizable.*
- $\forall x_0 \in \mathbb{X}$ *there exists* admissible control u. *(*$u \in \mathbb{L}_2(0, \infty; \mathbb{U})$ *admissible* $\iff$ $\mathcal{J}(x_0, u) < \infty$.*)*

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Then: algebraic operator Riccati equation

$$0 = \mathcal{R}(\mathcal{P}) := \mathcal{C}^*\mathcal{C} + \mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A} - \mathcal{P}\mathcal{B}\mathcal{B}^*\mathcal{P}$$

has unique, selfadjoint solution $\mathcal{P}_\infty : \mathrm{dom}(\mathcal{A}) \to \mathrm{dom}(\mathcal{A}^*)$ *that is linear, bounded, and positive semidefinite* $(\mathcal{P} \geq 0)$.

The optimal control solving the LQR problem is given by the feedback control

$$u_\infty(t) = -\mathcal{B}^*\mathcal{P}_\infty x(t) = \mathcal{K}_\infty x(t).$$

$\mathcal{P}_\infty$ *is* stabilizing, *that is, the* C_0*-semigroup generated by* $\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P}_\infty$ *is exponentially stable.*

Parabolic PDE in domain $\Omega \subset \mathbb{R}^d$ (heat equation, convection-diffusion equation)

$$
\begin{aligned}
\frac{\partial x}{\partial t} - \nabla_\xi \left(A(\xi)\nabla_\xi x\right) + d(\xi)\nabla_\xi x + r(\xi)x &= Bu(t), \quad \xi \in \Omega,\ t > 0,\\
y &= Cx, \quad t \geq 0,
\end{aligned}
$$

with initial and boundary conditions ($\partial\Omega = \Gamma_D \cup \Gamma_N$)

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\begin{aligned}
x(\xi, t) &= B_D u_1(t), \quad \xi \in \Gamma_D,\\
\alpha x(\xi, t) + \beta \frac{\partial}{\partial \eta} x(\xi, t) &= B_N u_2(t), \quad \xi \in \Gamma_N,\\
x(\xi, 0) &= x_0(\xi), \quad \xi \in \Omega.
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- $B = 0$ $\Longrightarrow$ **boundary control problem**
- $B_D = B_N = 0$ $\Longrightarrow$ **point control problem**

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Weak formulation, use test functions $v \in \mathbb{V} = \mathbb{H}_0^1(\Omega) \implies$ LQR Problem.

Consider sequence of subspaces $\mathbb{X}_n \subset \mathbb{X}$, $\dim(\mathbb{X}_n) < \infty$, such that $\forall \varphi \in \mathbb{X}$ there exists $\varphi_n \in \mathbb{X}_n$ with

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{\mathbb{X}} = 0.$$

Define **Galerkin projection** $\Pi_n : \mathbb{X} \to \mathbb{X}_n$ and

$$\begin{aligned} < A_n\varphi_n, \psi_n >_{\mathbb{X}_n} &:= -< \mathcal{A}\varphi_n, \psi_n >_{\mathbb{X}} \quad \forall \varphi_n, \psi_n \in \mathbb{X}_n, \\ B_n &:= \Pi_n B, \qquad C_n := C|_{\mathbb{X}_n}. \end{aligned}$$

$\Longrightarrow$ **finite dimensional LQR problem/LQR(n)**

Minimize $\quad \mathcal{J}_n(P_n x_0, u_n) = \frac{1}{2} \int\limits_0^\infty \left(\|C_n x_n\|^2_{\mathcal{Y}} + \|u_n\|^2_{\mathcal{U}}\right) dt \quad$ for $u_n \in \mathbb{L}_2(0, \infty; \mathbb{U})$

subject to $\quad \dot{x}_n = A_n x_n + B_n u_n, \qquad x(0) = \Pi_n x_0.$

Corresponding **ARE(n):** $0 = \mathcal{R}_n(P_n) := C_n^* C_n + A_n^* P_n + P_n A_n - P_n B_n B_n^* P_n.$

Theorem (Gibson '79, Banks/Kunisch '84)

Under given assumptions, the optimizing solution of LQR(n) is given by feedback control

$$u_{n,*}(t) = -B_n^* P_{n,*} x_n(t) = \mathcal{K}_{n,*} x_n(t),$$

where $P_{n,*}$ is the stabilizing solution of ARE(n).

Furthermore,

$$\lim_{n\to\infty} \| P_{n,*} \Pi_n \varphi_n - \mathcal{P}_\infty \varphi \|_{\mathbb{X}} = 0 \qquad \forall \varphi \in \mathbb{X},$$

i.e., strong convergence $P_{n,*}\Pi_n \to \mathcal{P}_\infty$ *in* $\mathbb{X}$.

Algebraic Riccati equation (ARE)

For $A, G = G^T, W = W^T \in \mathbb{R}^{n \times n}$ given and $X \in \mathbb{R}^{n \times n}$ unknown:

$$0 = \mathcal{R}(X) := A^T X + XA - XGX + W.$$

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Many applications:

- model reduction of (unstable) linear time-invariant (LTI) systems,
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Typical situation in LQR control:

- G, W **low-rank** with $G, W \in \{BB^T, C^T C\}$, where $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$, $m, p \ll n$.
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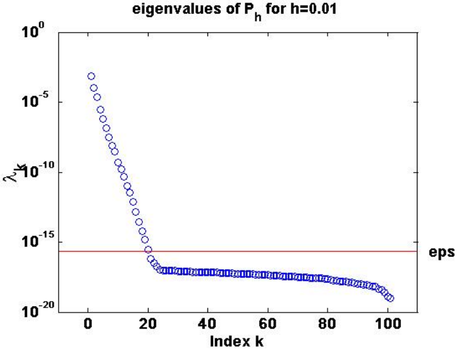
- **Want:** solution with $X = X^T \geq 0$ (and $\Lambda(A - GX) \subset \mathbb{C}^-$).
- $n = 10^3 - 10^6$ $\Longrightarrow$ X has $10^6 - 10^{12}$ unknowns
  $\Longrightarrow$ as X is dense in general, we face a storage problem!

Example:

- Linear 1D heat equation with point control,
- $\Omega = [0, 1]$,
- FEM discretization using linear B-splines,
- $h = 1/100 \implies n = 101$.



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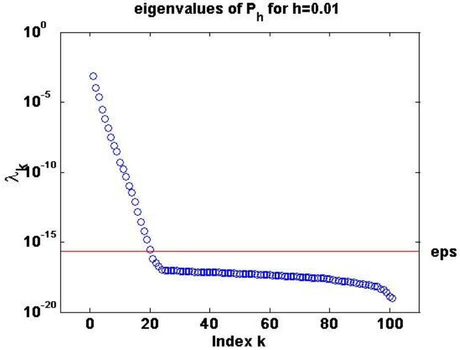

Idea: $X = X^T \geq 0 \implies$

$$X = ZZ^T = \sum_{k=1}^{n} \lambda_k z_k z_k^T \approx \sum_{k=1}^{r} \lambda_k z_k z_k^T = \sum_{k=1}^{r} \left(\sqrt{\lambda_k} z_k\right) \left(\sqrt{\lambda_k} z_k\right)^T =: Z^{(r)}(Z^{(r)})^T.$$

$\implies$ **Goal:** compute $Z^{(r)} \in \mathbb{R}^{n \times r}$ directly w/o ever forming X!

- Consider $0 = \mathcal{R}(X) = C^T C + A^T X + XA - XBB^T X.$

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- Newton-Kantorovich method:

$$X_{j+1} = X_j - \left(\mathcal{R}'_{X_j}\right)^{-1} \mathcal{R}(X_j), \quad j = 0, 1, 2, \ldots$$

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Newton's method (with line search) for AREs

FOR $j = 0, 1, \ldots$

1. $A_j \leftarrow A - BB^T X_j =: A - BK_j.$
2. Solve the Lyapunov equation $A_j^T N_j + N_j A_j = -\mathcal{R}(X_j).$
3. $X_{j+1} \leftarrow X_j + t_j N_j.$

END FOR j

- **Convergence for K_0 stabilizing:**
  - $A_j = A - BK_j = A - BB^T X_j$ is stable $\forall\, j \geq 0$.
  - $\lim_{j\to\infty} \|\mathcal{R}(X_j)\|_F = 0$ (monotonically).
  - $\lim_{j\to\infty} X_j = X_* \geq 0$ (locally quadratic).

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- Need large-scale Lyapunov solver; here, **ADI iteration**:
  linear systems with dense, but "sparse+low rank" coefficient matrix A_j:

$$A_j = A - B \cdot K_j$$

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$=$ 

- $m \ll n \implies$ efficient "inversion" using **Sherman-Morrison-Woodbury formula:**

$$(A - BK_j + p_k^{(j)} I)^{-1} = (I_n + (A + p_k^{(j)} I)^{-1} B (I_m - K_j (A + p_k^{(j)} I)^{-1} B)^{-1} K_j)(A + p_k^{(j)} I)^{-1}.$$

Newton's Method for AREs

Properties and Implementation

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- **BUT:** $X = X^T \in \mathbb{R}^{n\times n} \Longrightarrow n(n+1)/2$ **unknowns!**

Re-write Newton's method for AREs

$$A_j^T N_j + N_j A_j = -\mathcal{R}(X_j)$$

$$\Longleftrightarrow$$

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + \underbrace{(X_j + N_j)}_{=X_{j+1}} A_j = \underbrace{-C^T C - X_j B B^T X_j}_{=:-W_j W_j^T}$$

Set $X_j = Z_j Z_j^T$ for $\operatorname{rank}(Z_j) \ll n \Longrightarrow$

$$A_j^T \left(Z_{j+1} Z_{j+1}^T \right) + \left(Z_{j+1} Z_{j+1}^T \right) A_j = -W_j W_j^T$$

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Factored Newton Iteration [B./Li/Penzl 1999/2008]

Solve Lyapunov equations for Z_{j+1} directly by factored ADI iteration and use **'sparse + low-rank'** structure of A_j.

Optimal feedback

$$K_* = B^T X_* = B^T Z_* Z_*^T$$

can be computed by **direct feedback iteration:**

- jth Newton iteration:

$$K_j = B^T Z_j Z_j^T = \sum_{k=1}^{k_{\max}} (B^T V_{j,k}) V_{j,k}^T \quad \xrightarrow{j\to\infty} \quad K_* = B^T Z_* Z_*^T$$

- K_j can be updated in ADI iteration, no need to even form Z_j, need only fixed workspace for $K_j \in \mathbb{R}^{m\times n}$!

Related to earlier work by [Banks/Ito 1991].

- Mathematical model: boundary control for non-linear 2D heat equation,

$$
\begin{aligned}
c(\Theta)\rho(\Theta)\partial_t\Theta &= \nabla.(\lambda(\Theta)\nabla\Theta) \text{ in } [0, t_f] \times \Omega, \\
\lambda(\Theta)\partial_\nu\Theta &= \alpha(\Theta - \Theta_{ext}) + \beta(\Theta^4 - \Theta_{ext}^4) \\
& \qquad \text{on } \Gamma_k,\ k = 1, \dots, 7, \\
\partial_n\Theta &= 0 \quad \text{on } \Gamma_8.
\end{aligned}
$$

- λ, c, ρ: linear-affine functions, valid for **austenite phase;** linearization about their mean.
- FE discretization with linear elements $\rightsquigarrow$ Model hierarchy: $n = 1357, 5177, 20209, 79841$.
- Goal: fast cooling (improved production rate), **avoiding the formation of perlite** requires bounded gradients.
- Approach: adaptive LQR.

Numerical Example: Optimal Cooling

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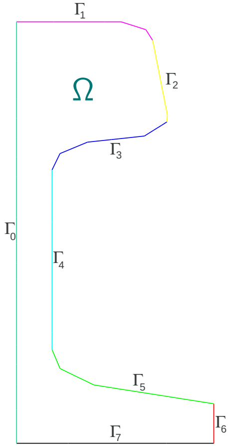
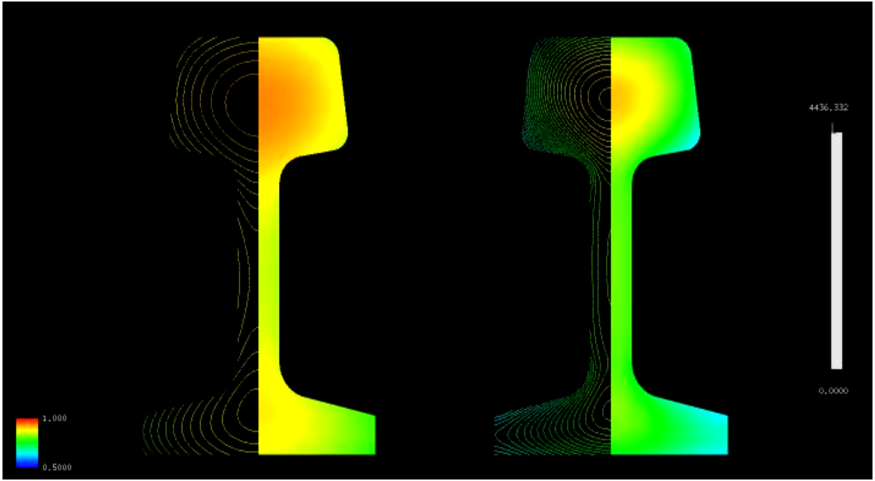
\end{aligned}
$$

- λ, c, ρ: linear-affine functions, valid for **austenite phase;** linearization about their mean.
- FE discretization with linear elements $\rightsquigarrow$ Model hierarchy: $n = 1357, 5177, 20209, 79841$.
- Goal: fast cooling (improved production rate), **avoiding the formation of perlite** requires bounded gradients.
- Approach: adaptive LQR.



uncontrolled controlled

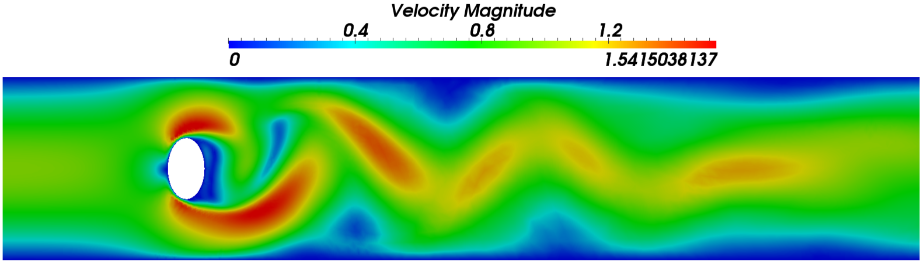


Outline

- **Physical transport** is one of the most fundamental dynamical processes in nature.
- **Prediction** and **manipulation** of transport processes are important research topics, e.g., to
  - avoid stall — for stable and safe flight;
  - save energy (or increase attainable speed) by minimizing drag coefficient;
  - use fluid flow for optimal transport (e.g., in blood veins).
- **Open-loop** controllers are widely used in various engineering fields.
  $\rightarrow$ **Not robust** regarding perturbation
- Dynamical systems are often influenced via so called **distributed control**.
  $\rightarrow$ **Unfeasible** in many real-world areas

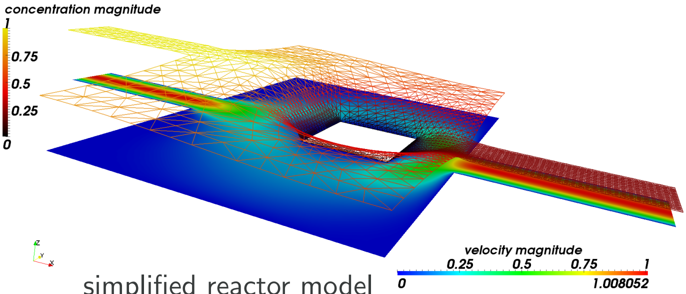
$\Rightarrow$ **Boundary feedback stabilization (closed-loop)**
should be used to increase robustness and feasibility.

- Consider 2D flow problems described by **incompressible Navier–Stokes equations**.
- Riccati feedback approach requires the solution of an **algebraic Riccati equation**.
- Conservation of mass introduces a **divergence-freeness** condition $\rightsquigarrow$ problems with mathematical basis of control design schemes.



Kármán vortex street

- Consider 2D flow problems described by **incompressible Navier–Stokes equations**.
- Riccati feedback approach requires the solution of an **algebraic Riccati equation**.
- Conservation of mass introduces a **divergence-freeness** condition ⇝ problems with mathematical basis of control design schemes.
- **Coupling** flow problems with a **scalar reaction-advection-diffusion equation**.



simplified reactor model

Navier–Stokes Equations

$$\frac{\partial \vec{v}}{\partial t} - \frac{1}{\mathrm{Re}}\Delta \vec{v} + (\vec{v} \cdot \nabla)\vec{v} + \nabla p = \vec{f}$$

$$\operatorname{div} \vec{v} = 0$$

- defined for time $t \in (0, \infty)$ and space $\vec{x} \in \Omega \subset \mathbb{R}^2$ bounded with $\Gamma = \partial\Omega$
- $+$ boundary and initial conditions
- initial boundary value problem with additional algebraic constraints

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Linearize + Discretize → Index-2 DAE

$$M\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(t) = A\mathrm{v}(t) + \widehat{G}\mathrm{p}(t) + B\mathrm{u}(t)$$

$$0 = \widehat{G}^T \mathrm{v}(t)$$

$$\mathrm{y}(t) = C\mathrm{v}(t)$$

$$A,\ M \in \mathbb{R}^{n\times n},\ \widehat{G} \in \mathbb{R}^{n\times n_{\mathrm{p}}}$$

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$$\mathrm{u}(t) \in \mathbb{R}^{n_r},\ \mathrm{y}(t) \in \mathbb{R}^{n_a}$$

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Show that projection in [Hei/Sor/Sun '08] is discretized version of Leray projector in [Ray '06].

$$M\Pi^T = \Pi M \quad \wedge \quad \Pi^T\mathrm{v} = \mathrm{v}_{\mathrm{div},0}$$

[**Bänsch**/B./Saak/**Stoll**/Weichelt **'13**,**'15**]

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[**Bänsch**/B./SAAK/**Stoll**/WEICHELT **'13**,**'15**]

Extension to coupled flow case, i.e.,

$$\widehat{\Pi} := \begin{bmatrix}\Pi & 0\\ 0 & I\end{bmatrix} \quad \wedge \quad \begin{bmatrix}\Pi^T & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\mathrm{v}\\ \mathrm{c}\end{bmatrix} = \begin{bmatrix}\mathrm{v}_{\mathrm{div},0}\\ \mathrm{c}\end{bmatrix}.$$

[BÄNSCH/B./SAAK/WEICHELT '14]

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[Bänsch/B./Saak/Weichelt '14]

Helmholtz Decomposition [Girault/Raviart '86]

- Splitting:

$$(L^2(\Omega))^2 = \mathrm{H}(\mathrm{div}, 0) \perp \mathrm{H}(\mathrm{div}, 0)^{\perp}$$

Divergence-free: $\mathrm{H}(\mathrm{div}, 0) := \{\vec{v} \in (L^2(\Omega))^2 \mid \mathrm{div}\, \vec{v} = 0, \vec{v} \cdot \vec{n}_{|\Gamma} = 0\}$

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Leray Projector P

This splitting is equivalent to $\vec{v} = \vec{v}_{\mathrm{div},0} + \nabla p$, where $\vec{v}_{\mathrm{div},0}$ and p fulfill

$$\begin{aligned} \vec{v}_{\mathrm{div},0} + \nabla p &= \vec{v} \quad \text{in } \Omega, \\ \operatorname{div} \vec{v}_{\mathrm{div},0} &= 0 \quad \text{in } \Omega, \\ \vec{v}_{\mathrm{div},0} \cdot \vec{n} &= 0 \quad \text{on } \Gamma. \end{aligned}$$

$P : (L^2(\Omega))^2 \to \mathrm{H}(\mathrm{div},0)$ with $P : \vec{v} \mapsto \vec{v}_{\mathrm{div},0}$.

Projection Method

[Heinkenschloss/Sorensen/Sun '08]

- Index reduction for Lyapunov solver.
- Projector:

$$\Pi^T := I_{n_v} - M_v^{-1} G (G^T M_v^{-1} G)^{-1} G^T.$$

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Recall $P : \vec{v} \mapsto \vec{w}$:

$$\begin{aligned} \vec{w} + \nabla p &= \vec{v}, \\ \operatorname{div} \vec{w} &= 0 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} M_v & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} \mathrm{w} \\ \mathrm{p} \end{bmatrix} = \begin{bmatrix} M_v \mathrm{v} \\ 0 \end{bmatrix}$$

$$\mathrm{p} = (G^T M_v^{-1} G)^{-1} G^T \mathrm{v}$$

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Leray vs. Π^T

$$\begin{aligned} \vec{w} &= P(\vec{v}), \\ 0 &= \mathrm{div}\, \vec{w} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathrm{w} &= \Pi^T \mathrm{v}, \\ 0 &= G^T \mathrm{w} \end{aligned}$$

Minimize

$$\mathcal{J}(\mathrm{y}, \mathrm{u}) = \frac{1}{2} \int_0^\infty \lambda^2 ||\mathrm{y}||^2 + ||\mathrm{u}||^2 \, \mathrm{d}t$$

subject to

$$\widehat{\Theta}_r^T M \widehat{\Theta}_r \frac{\mathrm{d}}{\mathrm{d}t} \widetilde{\boldsymbol{x}}(t) = \widehat{\Theta}_r^T A \widehat{\Theta}_r \widetilde{\boldsymbol{x}}(t) + \widehat{\Theta}_r^T B \mathrm{u}(t)$$

$$\mathrm{y}(t) = C \widehat{\Theta}_r \widetilde{\boldsymbol{x}}(t)$$

with $\widehat{\Pi} = \widehat{\Theta}_l \widehat{\Theta}_r^T$ such that $\widehat{\Theta}_r^T \widehat{\Theta}_l = I \in \mathbb{R}^{(n-n_\mathrm{p})\times(n-n_\mathrm{p})}$ and $\widetilde{\boldsymbol{x}} = \widehat{\Theta}_l^T \mathrm{x}$.

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Riccati Based Feedback Approach

- Optimal control: $\mathrm{u}(t) = -\mathcal{K}\widetilde{\boldsymbol{x}}(t)$, with feedback: $\mathcal{K} = \mathcal{B}^T \mathcal{X} \mathcal{M}$,

  where $\mathcal{X}$ is the solution of the generalized continuous-time algebraic Riccati equation (GCARE)

$$\mathcal{R}(\mathcal{X}) = \lambda^2 \mathcal{C}^T \mathcal{C} + \mathcal{A}^T \mathcal{X} \mathcal{M} + \mathcal{M} \mathcal{X} \mathcal{A} - \mathcal{M} \mathcal{X} \mathcal{B} \mathcal{B}^T \mathcal{X} \mathcal{M} = 0.$$

Determine $\mathcal{X} = \mathcal{X}^T \succeq 0$ such that $\mathcal{R}(\mathcal{X}) = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T\mathcal{X}\mathcal{M} + \mathcal{M}\mathcal{X}\mathcal{A} - \mathcal{M}\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}\mathcal{M} = 0.$

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Kleinman–Newton method

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Kleinman–Newton method

Step $m+1$: Solve the Lyapunov equation

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)})^T\mathcal{X}^{(m+1)}\mathcal{M} + \mathcal{M}\mathcal{X}^{(m+1)}(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)}) = -(\mathcal{W}^{(m)})^T\mathcal{W}^{(m)} \tag{1}$$

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Step ℓ**:** Solve the projected and shifted linear system

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)} + q_\ell\mathcal{M})^T\mathcal{V}_\ell = \mathcal{Y} \quad (2)$$

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Replace (2) and **solve instead** the saddle point system (SPS)

$$\begin{bmatrix} A^T - (K^{(m)})^T B^T + q_\ell M & \widehat{G} \\ \widehat{G}^T & 0 \end{bmatrix} \begin{bmatrix} V_\ell \\ * \end{bmatrix} = \begin{bmatrix} Y \\ 0 \end{bmatrix}$$

for different ADI shifts $q_\ell \in \mathbb{C}^-$ for a couple of rhs Y.

Determine $\mathcal{X} = \mathcal{X}^T \succeq 0$ such that $\mathcal{R}(\mathcal{X}) = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T\mathcal{X}\mathcal{M} + \mathcal{M}\mathcal{X}\mathcal{A} - \mathcal{M}\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}\mathcal{M} = 0.$

Kleinman–Newton method

Step $\mathrm{m}+1$**:** Solve the Lyapunov equation

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)})^T\mathcal{X}^{(m+1)}\mathcal{M} + \mathcal{M}\mathcal{X}^{(m+1)}(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)}) = -(\mathcal{W}^{(m)})^T\mathcal{W}^{(m)} \qquad (1)$$

low-rank ADI method

Step ℓ**:** Solve the projected and shifted linear system

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)} + q_\ell\mathcal{M})^T\mathcal{V}_\ell = \mathcal{Y} \qquad (2)$$

linear solver

Avoid explicit projection using $\widehat{\Theta}_r\mathcal{V}_\ell = V_\ell$, $\mathcal{Y} = \widehat{\Theta}_r^T Y$, and [Hei/Sor/Sun '08]:
Replace (2) and **solve instead** the saddle point system (SPS)
(using *Sherman–Morrison–Woodbury* formula)

$$\begin{bmatrix} A^T - (K^{(m)})^T B^T + q_\ell M & \widehat{G} \\ \widehat{G}^T & 0 \end{bmatrix} \begin{bmatrix} V_\ell \\ * \end{bmatrix} = \begin{bmatrix} Y \\ 0 \end{bmatrix}$$

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Replace (2) and **solve instead** the saddle point system (SPS)
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$$\begin{bmatrix} A^T + q_\ell M & \widehat{G} \\ \widehat{G}^T & 0 \end{bmatrix} \begin{bmatrix} V_\ell \\ * \end{bmatrix} = \begin{bmatrix} \tilde{Y} \\ 0 \end{bmatrix}$$

for different ADI shifts $q_\ell \in \mathbb{C}^-$ for a couple of rhs $\tilde{Y}$.

Theorem [B./Heinkenschloss/Saak/Weichelt '16]

- assume $(\mathsf{A},\mathsf{B};\mathsf{M})$ stabilizable, $(\mathsf{C},\mathsf{A};\mathsf{M})$ detectable
- $\Rightarrow \exists$ unique, symmetric solution $X^{(*)} = \widehat{\Theta}_r \mathcal{X}^{(*)} \widehat{\Theta}_r^T$ with $\mathcal{R}(\mathcal{X}^{(*)}) = 0$ that stabilizes

$$\left(\begin{bmatrix} \mathsf{A} - \mathsf{B}\mathsf{B}^T X^{(*)}\mathsf{M} & \widehat{G} \\ \widehat{G}^T & 0 \end{bmatrix}, \begin{bmatrix} \mathsf{M} & 0 \\ 0 & 0 \end{bmatrix} \right)$$

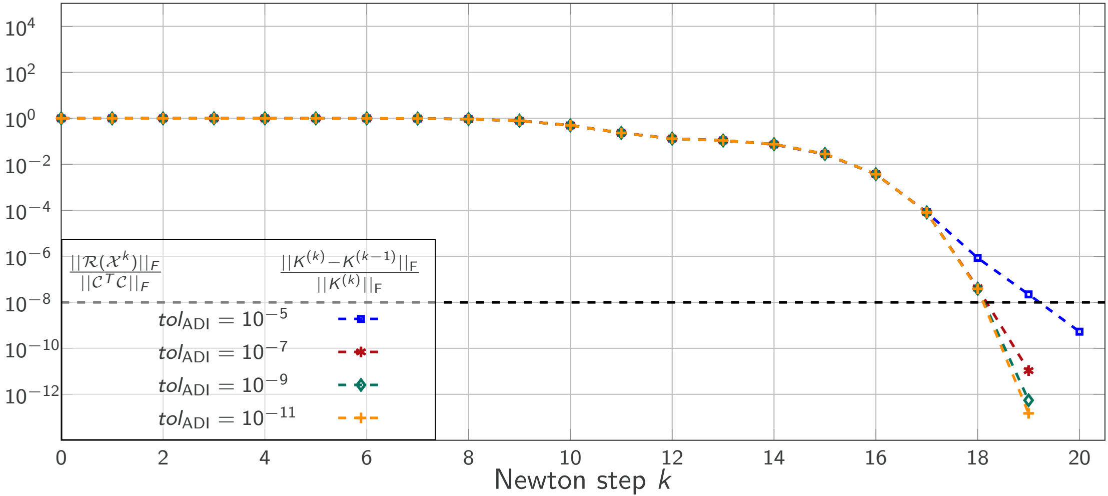
- for $\{X^{(k)}\}_{k=0}^{\infty}$ defined by $X^{(k)} := \widehat{\Theta}_r \mathcal{X}^{(k)} \widehat{\Theta}_r^T$, (1), and $X^{(0)}$ symmetric with $\left(\mathsf{A} - \mathsf{B}\left(\mathsf{K}^{(0)}\right)^T, \mathsf{M}\right)$ stable, it holds that, for $k \geq 1$,

$$X^{(1)} \succeq X^{(2)} \succeq \cdots \succeq X^{(k)} \succeq 0 \quad \text{and} \quad \lim_{k\to\infty} X^{(k)} = X^{(*)}$$

- $\exists\, 0 < \widetilde{\kappa} < \infty$ such that, for $k \geq 1$,

$$||X^{(k+1)} - X^{(*)}||_F \leq \widetilde{\kappa} ||X^{(k)} - X^{(*)}||_F^2$$

NSE scenario: Re $= 500$, $n = 5\,468$, $\lambda = 10^2$, $tol_{\text{Newton}} = 10^{-8}$



Feedback Stabilization for Index-2 DAE Systems

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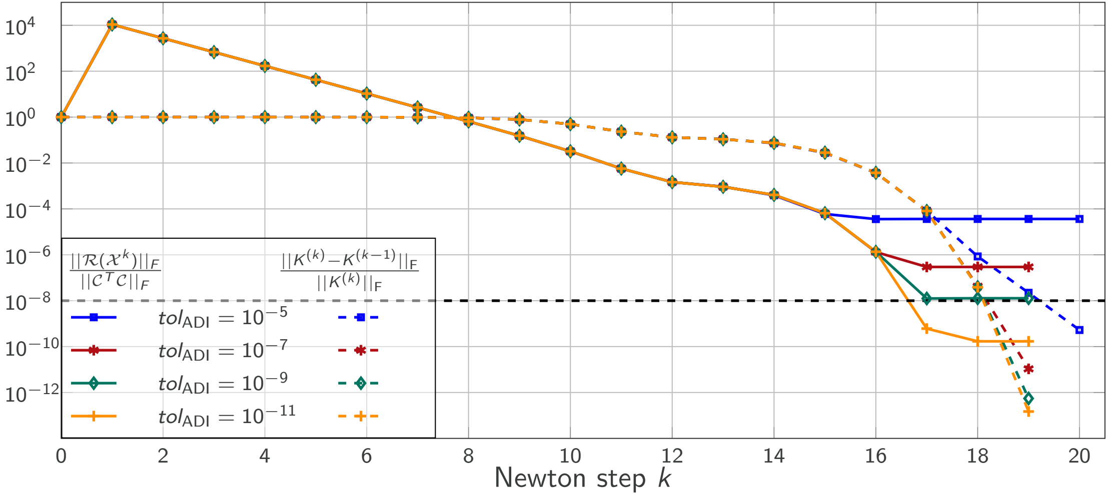

- Coefficients of GCARE are large-scale matrices (resulting from FE discretization).
- Quadratic system matrices A, $M = M^T \in \mathbb{R}^{n \times n}$ are sparse.

$$\mathcal{R}(X) = C^T C + A^T XM + MXA - MXBB^T XM$$

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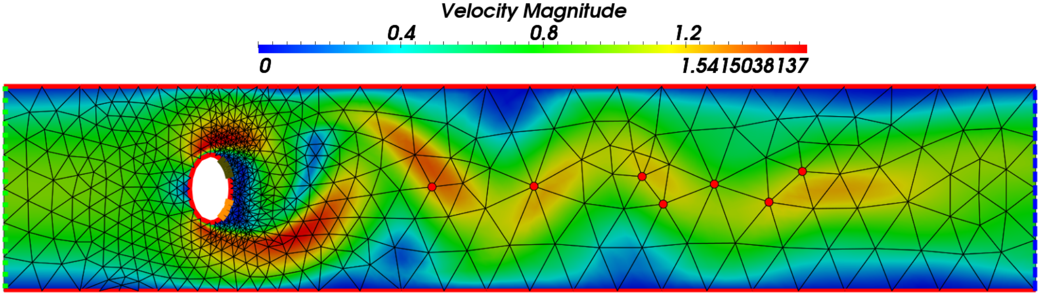

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- Coefficients of GCARE are large-scale matrices (resulting from FE discretization).
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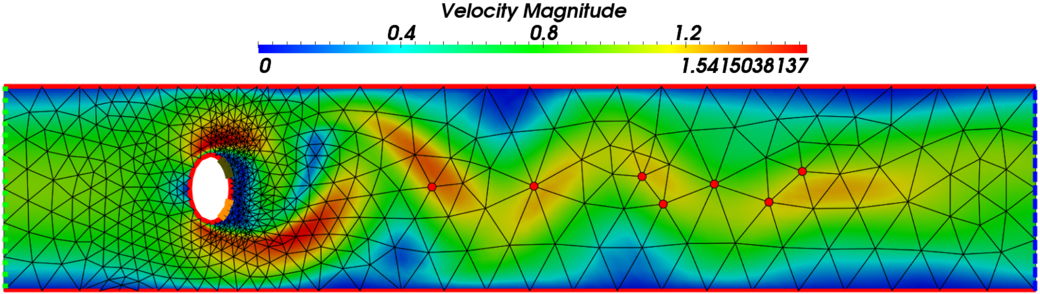

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Stabilization of Nonlinear Unsteady PDEs

Accelerated Solution of Riccati Equations

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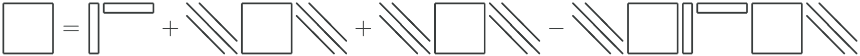

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Stabilization of Nonlinear Unsteady PDEs

Accelerated Solution of Riccati Equations

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- Residual is of low rank; $R(ZZ^T) = WW^T$, $W \in \mathbb{R}^{n \times k}$, $k \leq 2n_r + n_a \ll n$

$$WW^T = C^T C + A^T ZZ^T M + MZZ^T A - MZZ^T BB^T ZZ^T M$$

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- Step size computation in [B./Byers '98] involves dense residuals, therefore, it is not applicable in large-scale case.

Accelerated Solution of Riccati Equations

— Problems with Nested Iteration —

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- **extension to index-2 DAE case "straight forward"**

Theorem **[B./Heinkenschloss/Saak/Weichelt '16]**

Set $\tau_k \in (0,1)$ and assume: $(\mathcal{A}, \mathcal{B}; \mathcal{M})$ stabilizable, $(\mathcal{C}, \mathcal{A}; \mathcal{M})$ detectable, and $\exists\, \widetilde{\mathcal{X}}^{(k+1)} \succeq 0\ \forall k$ that solves

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(k)})^T \widetilde{\mathcal{X}}^{(k+1)} \mathcal{M} + \mathcal{M} \widetilde{\mathcal{X}}^{(k+1)} (\mathcal{A} - \mathcal{B}\mathcal{K}^{(k)}) = -\mathcal{C}^T\mathcal{C} - (\mathcal{K}^{(k)})^T \mathcal{K}^{(k)} + \mathcal{L}^{(k+1)}$$

such that

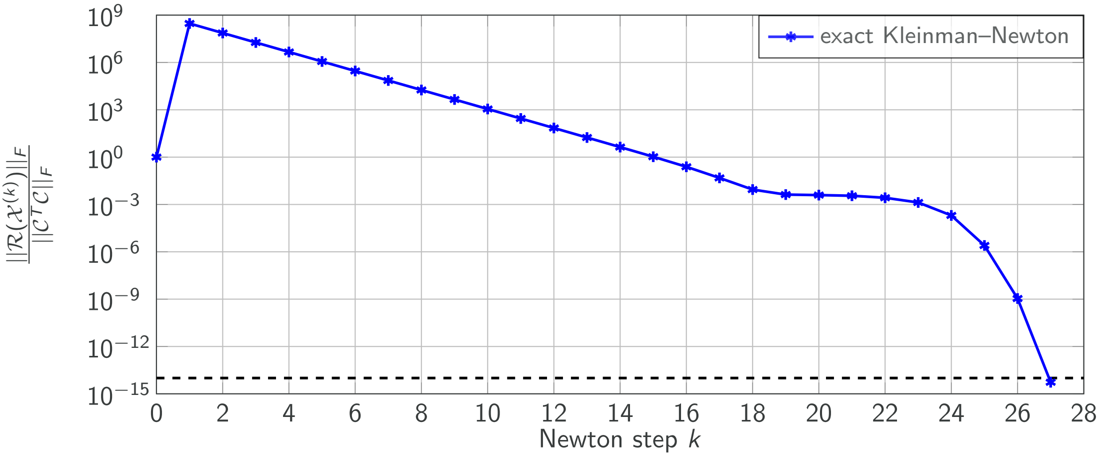
$$||\mathcal{L}^{(k+1)}||_F \leq \tau_k ||\mathcal{R}(\mathcal{X}^{(k)})||_F.$$

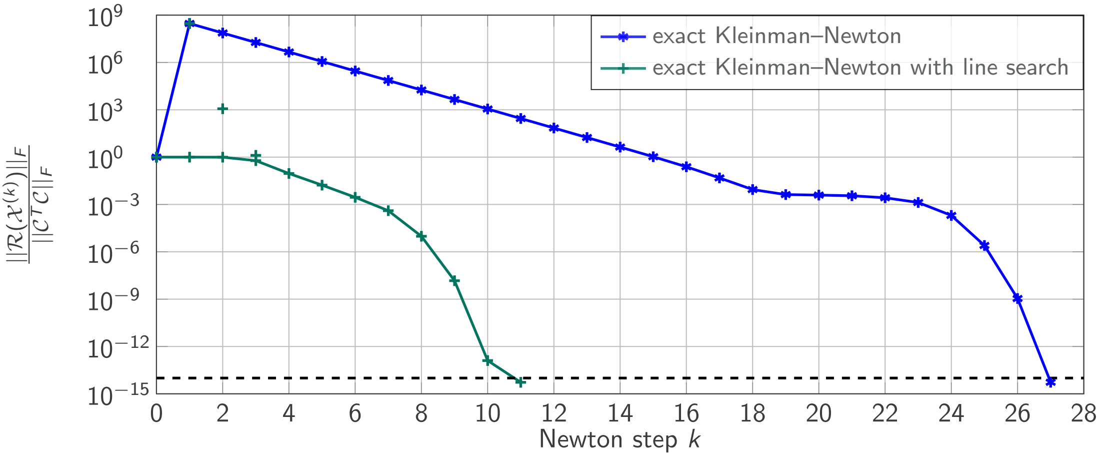
Find $\xi_k \in (0,1]$ such that $||\mathcal{R}(\mathcal{X}^{(k)} + \xi_k \mathcal{S}^{(k)})||_F \leq (1 - \xi_k \alpha)||\mathcal{R}(\mathcal{X}^{(k)})||_F$ and set

$$\mathcal{X}^{(k+1)} = (1 - \xi_k)\mathcal{X}^{(k)} + \xi_k \widetilde{\mathcal{X}}^{(k+1)}.$$

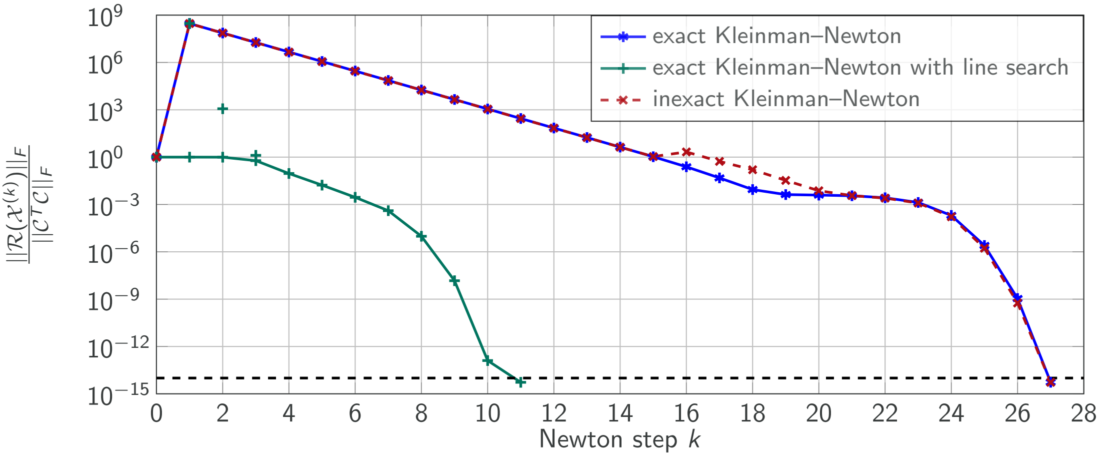
1. **IF** $\xi_k \geq \xi_{\min} > 0\ \forall k \ \Rightarrow\ \|\mathcal{R}(\mathcal{X}^{(k)})\|_F \to 0.$
2. **IF** $\mathcal{X}^{(k)} \succeq 0$, and $(\mathcal{A} - \mathcal{B}\mathcal{B}^T\mathcal{X}^{(k)}, \mathcal{M})$ stable for $k \geq K > 0 \ \Rightarrow\ \mathcal{X}^{(k)} \to \mathcal{X}^{(*)}$ ($\mathcal{X}^{(*)} \succeq 0$ the unique stabilizing solution).

Numerical Example: NSE scenario: Re $= 500$, Level 1, $\lambda = 10^4$, $tol_{\text{Newton}} = 10^{-14}$





exact Kleinman–Newton
exact Kleinman–Newton with line search
$\frac{\|\mathcal{R}(\mathcal{X}^{(k)})\|_F}{\|\mathcal{C}^T\mathcal{C}\|_F}$
10^9
10^6
10^3
10^0
10^{-3}
10^{-6}
10^{-9}
10^{-12}
10^{-15}
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28
Newton step k



$\frac{||\mathcal{R}(\mathcal{X}^{(k)})||_F}{||\mathcal{C}^T\mathcal{C}||_F}$
10^9
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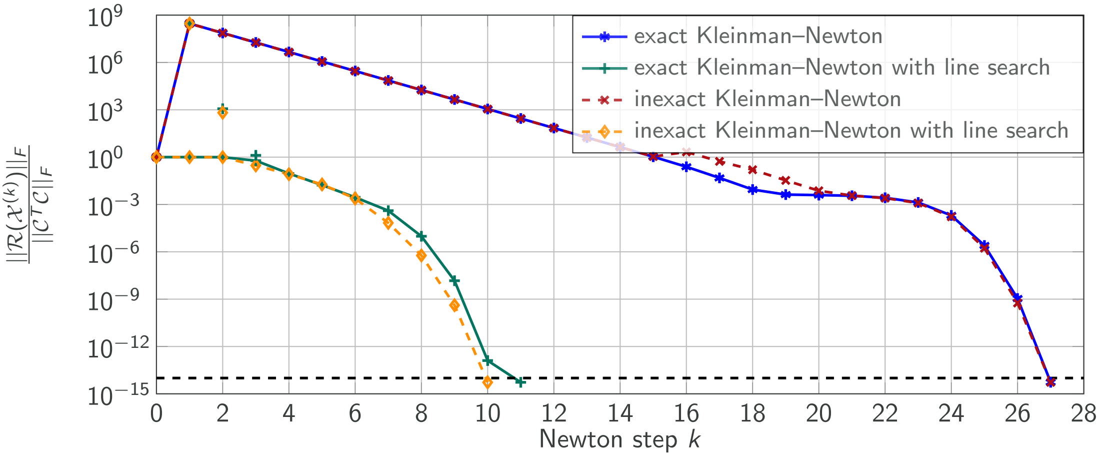

Accelerated Solution of Riccati Equations

Numerical Example: NSE scenario: Re $= 500$, Level 1, $\lambda = 10^4$, $tol_{\text{Newton}} = 10^{-14}$

	exact KN	exact KN+LS	inexact KN	inexact KN+LS
$\#Newt$	27	11	27	10
$\#ADI$	3185	1351	852	549
$t_{\text{Newt-ADI}}$	1304.769	540.984	331.871	222.295
t_{shift}	29.998	12.568	7.370	5.507
t_{LS}	–		–	
t_{total}	1334.767	553.581	339.241	227.824

Table: Numbers of steps and timings in seconds.

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Accelerated Solution of Riccati Equations

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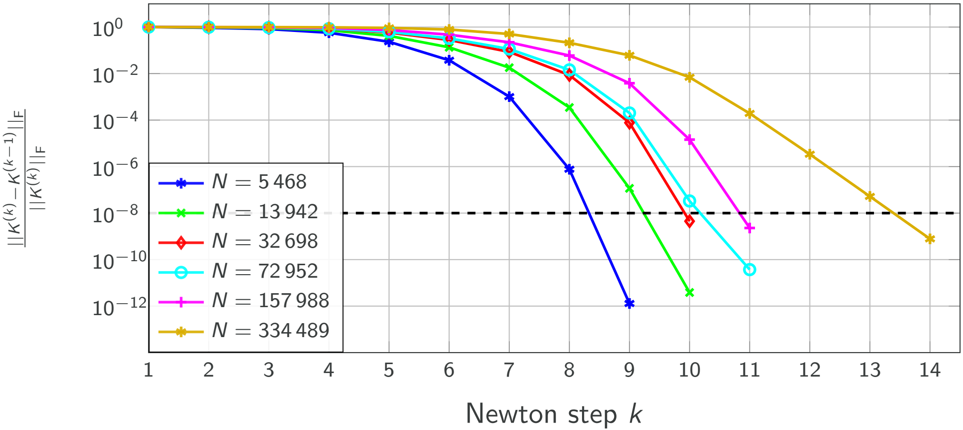
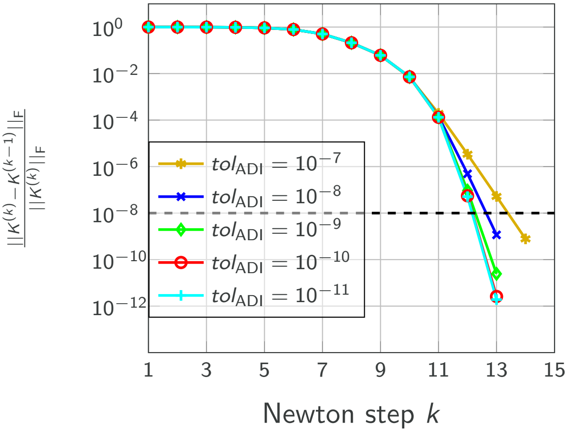
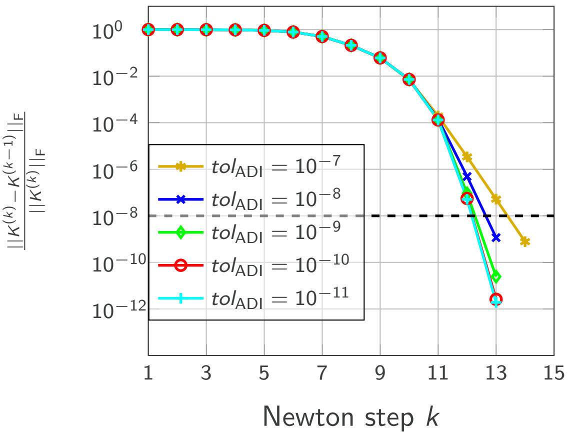
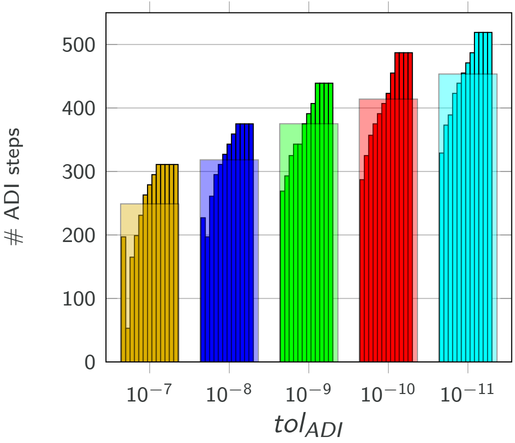


NSE scenario: Re $= 500$, $tol_{\text{Newton}} = 10^{-8}$, $N = 334\,489$





$\frac{\|K^{(k)}-K^{(k-1)}\|_F}{\|K^{(k)}\|_F}$
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
$tol_{ADI} = 10^{-7}$
$tol_{ADI} = 10^{-8}$
$tol_{ADI} = 10^{-9}$
$tol_{ADI} = 10^{-10}$
$tol_{ADI} = 10^{-11}$
1 3 5 7 9 11 13 15
Newton step k



ADI steps
0 100 200 300 400 500
10^{-7} 10^{-8} 10^{-9} 10^{-10} 10^{-11}
tol_{ADI}

Outline

Conclusions

- LQR control for PDEs benefits from advances in large-scale Lyapunov and Riccati solvers.
- Several recent Riccati solvers not mentioned here could be applied as well.
- Available in MATLAB toolbox M-M.E.S.S., https://zenodo.org/record/5938237.

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Contributions to Feedback Control of Flow Problems

- Analyzed **Riccati-based feedback** for **scalar** and **vector-valued transport** problems.
- Wide-spread usability tailored for standard **inf-sup stable finite element** discretizations.
- Established **specially tailored Kleinman–Newton-ADI** that **avoids explicit projections**.
- **Suitable preconditioners** for multi-field flow problems have been developed (not discussed here, see [Benner et al, SISC 2013]).
- Major run time improvements due to combination of **inexact Newton** and **line search**.
- Established **new convergence proofs** that were verified by **extensive numerical tests**.

- P. Benner and M. Hinze, *Feedback control of time-dependent nonlinear PDEs with applications in fluid dynamics*, **Handbook of Numerical Analysis**, 24 (2023), pp. 77–130. https://doi.org/10.1016/bs.hna.2022.12.002

- E. Bänsch and P. Benner, *Stabilization of incompressible flow problems by Riccati-based feedback*, in Constrained Optimization and Optimal Control for Partial Differential Equations, vol. 160 of International Series of Numerical Mathematics, Birkhäuser, 2012, pp. 5–20.
- P. Benner, J. Saak, M. Stoll, and H. K. Weichelt, *Efficient solution of large-scale saddle point systems arising in Riccati-based boundary feedback stabilization of incompressible Stokes flow*, **SIAM J. Sci. Comput.**, 35 (2013), pp. S150–S170.
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