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COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Rational Approximation of Passive Systems — Where to Interpolate?

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joint work with Chris Beattie,
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Honoring Christopher Beattie's 70th Birthday
Blacksburg / Virginia Tech
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  Interpolatory projection methods for parameterized model reduction.
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- U. Baur, C. Beattie, and P. Benner.
  Mapping Parameters Across System Boundaries: Parameterized Model Reduction with Low Rank Variability in Dynamics.
  Proceedings in Applied Mathematics and Mechanics 14:19–22, 2014.
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  H_2-Optimality Conditions for Structured Dynamical Systems.
  MPI Magdeburg Preprint MPIMD/14-18, October 2014.
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  In: *Realization and Model Reduction of Dynamical Systems*, pp. vii–xv, Springer International Publishing, 2022.

Definition

A linear time-invariant (LTI) system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad && A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= Cx(t) + Du(t), \qquad && C \in \mathbb{R}^{m\times n},\ D \in \mathbb{R}^{m\times m} \end{aligned}$$

is **passive** if

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \qquad \forall t \in \mathbb{R} \text{ and } \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

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Usual characterization via **positive realness** of transfer function $G(s) = C(sI_n - A)^{-1}B + D$.

Positive-real Transfer Functions

Definition (Cauer 1926, Brune 1931)

A *real*, rational matrix-valued function $G : \mathbb{C} \to \bar{\mathbb{C}}^{m\times m}$ is **positive real** if

1. G is analytic in $\mathbb{C}^+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$,
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- $\rightsquigarrow$ **Port-Hamiltonian** representation (if $C = B^T$) of passive systems:

$$\dot{x} = (J - R)Qx + Bu, \quad y = B^T x + Du.$$

Original System

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.

Reduced-Order Model (ROM)

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{A}\hat{x}(t) + \widehat{B}u(t), \\ \hat{y}(t) = \widehat{C}\hat{x}(t) + \widehat{D}u(t). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t) \in \mathbb{R}^p$.

Goals:

- $\|y - \hat{y}\| <$ **tolerance** $\cdot \|u\|$ for all admissible input signals.
- **Preserve passivity in ROM.**

Formulating model reduction in frequency domain

Approximate the **time-domain** dynamical system

$$\begin{array}{rcll} \dot{x} & = & Ax + Bu, & A \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times m}, \\ y & = & Cx + Du, & C \in \mathbb{R}^{p \times n},\ D \in \mathbb{R}^{p \times m}, \end{array}$$

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of **order** $r \ll n$, such that

$$\begin{aligned} \|y - \hat{y}\| \simeq \left\|Y - \hat{Y}\right\| &= \left\|GU - \hat{G}U\right\| \\ &\leq \left\|G - \hat{G}\right\| \cdot \|U\| \simeq \left\|G - \hat{G}\right\| \cdot \|u\| \leq \text{tolerance} \cdot \|u\| \,. \end{aligned}$$

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$\Longrightarrow$ **Rational approximation problem:** $\min\limits_{\text{order}\,(\hat{G}) \leq r} \left\|G - \hat{G}\right\|$, where, mostly, $\|\,.\,\| = \|.\|_{\mathcal{H}_\infty}$ or $\|\,.\,\| = \|\,.\,\|_{\mathcal{H}_2}$.

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Here: approximation by **rational interpolation:** $G^{(j)}(s_k) = \hat{G}^{(j)}(s_k),\ j = 0, \ldots, \ell_k$.

Literature

- Padé-type methods with post-processing [Bai/(Feldmann)/Freund 1998,2001].
- PRIMA [Odabasioglu et al.1996/97] preserves passivity for interconnect models, basically Arnoldi process.
- SyPVL preserves passivity for RLC circuits [Feldmann/Freund 1996/97].
- LR-ADI/dominant subspace approximation can preserve passivity [Li/White 2001].
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- IRKA-PH [Gugercin/Polyuga/**Beattie**/Van Der Schaft 2009/12], IRKA iteration for port-Hamiltonian systems.

  Remaining issue: IRKA-PH does not satisfy necessary optimality conditions.

  $\rightsquigarrow$ Starting point of 2014 BB preprint.

Algorithm 1. *(IRKA-PH)* **IRKA for MIMO port-Hamiltonian systems.**

Let $\mathbf{G}(s) = \mathbf{B}^T\mathbf{Q}(s\mathbf{I} - (\mathbf{J}-\mathbf{R})\mathbf{Q})^{-1}\mathbf{B}$ as in (14).

(1) Choose initial interpolation points $\{s_1, \ldots, s_r\}$ and tangent directions $\{\mathbf{b}_1, \ldots, \mathbf{b}_r\}$. Both sets closed under conjugation.

(2) Construct a (real) matrix (cf. Remark 4):
$$\mathbf{V}_r = [\![(s_1\mathbf{I} - (\mathbf{J}-\mathbf{R})\mathbf{Q})^{-1}\mathbf{B}\mathbf{b}_1, \ldots, \ldots, (s_r\mathbf{I} - (\mathbf{J}-\mathbf{R})\mathbf{Q})^{-1}\mathbf{B}\mathbf{b}_r]\!].$$

(3) Calculate $\mathbf{W}_r = \mathbf{Q}\mathbf{V}_r(\mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r)^{-1}$

(4) repeat until convergence

(a) Calculate $\mathbf{J}_r = \mathbf{W}_r^T\mathbf{J}\mathbf{W}_r$, $\mathbf{R}_r = \mathbf{W}_r^T\mathbf{R}\mathbf{W}_r$, $\mathbf{Q}_r = \mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r$ and $\mathbf{B}_r = \mathbf{W}_r^T\mathbf{B}$.

(b) For $\mathbf{A}_r = (\mathbf{J}_r - \mathbf{R}_r)\mathbf{Q}_r$, compute $\mathbf{A}_r\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $\mathbf{y}_i^*\mathbf{A}_r = \lambda_i\mathbf{y}_i^*$ with $\mathbf{y}_i^*\mathbf{x}_j = \delta_{ij}$ for left and right eigenvectors $\mathbf{y}_i^*$ and $\mathbf{x}_i$ associated with λ_i.

(c) $s_i \leftarrow -\lambda_i$ and $\mathbf{b}_i^T \leftarrow \mathbf{y}_i^*\mathbf{B}_r$ for $i = 1, \ldots, r$.

(d) Compute a (real) matrix (cf. Remark 4):
$$\mathbf{V}_r = [\![(s_1\mathbf{I} - (\mathbf{J}-\mathbf{R})\mathbf{Q})^{-1}\mathbf{B}\mathbf{b}_1, \ldots, \ldots, (s_r\mathbf{I} - (\mathbf{J}-\mathbf{R})\mathbf{Q})^{-1}\mathbf{B}\mathbf{b}_r]\!].$$

(e) Calculate $\mathbf{W}_r = \mathbf{Q}\mathbf{V}_r(\mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r)^{-1}$

(5) The final reduced model is given by
$$\mathbf{J}_r = \mathbf{W}_r^T\mathbf{J}\mathbf{W}_r,\ \mathbf{R}_r = \mathbf{W}_r^T\mathbf{R}\mathbf{W}_r,\ \mathbf{B}_r = \mathbf{W}_r^T\mathbf{B},\ \mathbf{Q}_r = \mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r, \text{ and } \mathbf{C}_r = \mathbf{B}_r^T\mathbf{Q}_r.$$

$\mathcal{H}_2$-optimal rational approximation problem

$$\hat{G} = \operatorname{argmin}_{\substack{\text{order}(\tilde{G})=r \\ \tilde{G} \text{ stable}}} \|G - \tilde{G}\|_{\mathcal{H}_2}, \qquad \text{where} \quad \|Z\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|Z(\jmath\omega)\|_F^2 \, d\omega \right)^{\frac{1}{2}}.$$

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Candidate solution combines two facts (here, $m = 1$ for simplicity):

1. **Necessary optimality conditions:**

$$G(-\mu_i) = \hat{G}(-\mu_i) \quad \text{and} \quad G'(-\mu_i) = \hat{G}'(-\mu_i), \quad i = 1, \ldots, r, \quad \text{where} \quad \Lambda(\hat{A}) = \{\mu_1, \ldots, \mu_r\}.$$

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2. **Interpolation via projection:**

   $$\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B} = CV\left(sI_r - W^T AV\right)^{-1} W^T B,$$

   where V and W are given as

   $$V = \left[(\nu_1 I - A)^{-1}B, \ldots, (\nu_r I - A)^{-1}B\right], \qquad W = \left[(\nu_1 I - A^T)^{-1}C^T, \ldots, (\nu_r I - A^T)^{-1}C^T\right],$$

   Hermite interpolates $G(s)$ at given $\{\nu_1, \ldots, \nu_r\}$.

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Hermite interpolates $G(s)$ at given $\{\nu_1, \ldots, \nu_r\}$.

Starting with an initial guess for $\hat{A}$, compute $\Lambda(\hat{A})$, set $\nu_i := -\mu_i$, compute $V, W, \hat{A}, \hat{B}, \hat{C}$, repeat
$\rightsquigarrow$ **iterative rational Krylov algorithm (IRKA)** that yields $\mathcal{H}_2$-(sub)optimal model.

$\mathcal{H}_2$-Optimal Port-Hamiltonian ROMs?

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Main questions:

1. What are the complete necessary optimality conditions for (1)?
   Do they come in the form of rational Hermite interpolation as in the standard LTI case?

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Theorem ((partial) answer to 1., [Beattie/B. 2014])

Suppose that $\hat{G}$ is a solution to (1) with a reduced dissipation matrix $\hat{R} \succ 0$. Suppose further that $\hat{G}$ has r distinct poles and is represented in pole-residue form as $\hat{G}(s) = \sum_{i=1}^{r} \frac{1}{s-\mu_i} \ell_i \wp_i^T$. Then

$$\mathbb{L}[G, \mathcal{S}] = \mathbb{L}[\hat{G}, \mathcal{S}].$$

where $\mathcal{S}$ here denotes the interpolation data: $\mathcal{S} = \{\{-\mu_i\}_1^r, \{\ell_i\}_1^r, \{\wp_i\}_1^r\}$ and

$$(\mathbb{L}[G, \mathcal{S}])_{i,j} := \begin{cases} \dfrac{\ell_i^T G(-\mu_i)\wp_j - \ell_i^T G(-\mu_j)\wp_j}{-\mu_i + \mu_j} & \text{if } i \neq j \\ \ell_i^T G'(-\mu_i)\wp_i & \text{if } i = j \end{cases}$$

is the associated Loewner matrix.

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Partial answer to 1. by [**Beattie**/B. 2014]

$$\ell_i^T \left(G(-\mu_i) - G(-\mu_j)\right) \wp_j = \ell_i^T \left(\hat{G}(-\mu_i) - \hat{G}(-\mu_j)\right) \wp_j \quad \text{for} \quad i \neq j, \qquad \ell_i^T G'(-\mu_i)\wp_i = \ell_i^T \hat{G}'(-\mu_i)\wp_i,$$

requires interpolation at mirror images of ROM poles as in LTI case, but has several drawbacks:

- Mismatch in number of conditions and degrees of freedom $\rightsquigarrow$ incomplete!
- Bi-variate (non-Hermitian) interpolation conditions, unclear how to satisfy.
- No algorithm known to achieve these conditions.

Based on a new look at $\mathcal{L}_2/\mathcal{H}_2$-optimal rational approximation and using the **Wirtinger calculus,** general necessary optimality conditions for structured dynamical systems could be derived.

Completing the Necessary Conditions [B./Gugercin/Mlinarić 2023]

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Theorem

Suppose that $\hat{G}$ is a solution to (1) with a reduced dissipation and energy matrices $\hat{R} \succ 0$ and $\hat{Q} \succ 0$. Suppose further that $\hat{G}$ has r distinct poles and is represented in pole-residue form as

$$\hat{G}(s) = \sum_{i=1}^{r} \frac{1}{s - \mu_i} \ell_i \wp_i^T.$$

Then

$$\begin{aligned} \ell_i^T \left(G(-\overline{\mu_i}) - G(-\overline{\mu_j})\right) \wp_j &= \ell_i^T \left(\hat{G}(-\overline{\mu_i}) - \hat{G}(-\overline{\mu_j})\right) \wp_j \quad \text{for} \quad i \neq j, \\ \ell_i^T G'(-\overline{\mu_i}) \wp_i &= \ell_i^T \hat{G}'(-\overline{\mu_i}) \wp_i, \\ \sum_{i=1}^{r} \left(G(-\overline{\mu_i}) \wp_i t_i^H + G(-\overline{\mu_i})^H \ell_i s_i^H \right) &= \sum_{i=1}^{r} \left(\hat{G}(-\overline{\mu_i}) \wp_i t_i^H + \hat{G}(-\overline{\mu_i})^H \ell_i s_i^H \right), \end{aligned}$$

where t_i and s_i are right and left eigenvectors of $\hat{J} - \hat{R}$, resp.

Completing the Necessary Conditions [B./Gugercin/Mlinarić 2023]

- The complete set of necessary optimality conditions is not a set of standard Hermite interpolation conditions.
- An algorithm for computing a ROM satisfying these optimality conditions is not known — so we still do not know where to interpolate.
- We can relate the interpolation conditions to **tangential Hermite interpolation** of the real part, though.

- The complete set of necessary optimality conditions is not a set of standard Hermite interpolation conditions.
- An algorithm for computing a ROM satisfying these optimality conditions is not known — so we still do not know where to interpolate.
- We can relate the interpolation conditions to **tangential Hermite interpolation** of the real part, though.

Corollary

Let $Z(s) = G(s) + G(s)^H$, $\hat{Z}(s) = \hat{G}(s) + \hat{G}(s)^H$. Then under the same conditions as for the theorem, and assuming $J - R$ to be normal, we have for $i = 1, \ldots, r$:

$$Z(-\overline{\mu_i})\wp_i = \hat{Z}(-\overline{\mu_i})\wp_i,$$

$$\ell_i^T Z(-\overline{\mu_i}) = \ell_i^T \hat{Z}(-\overline{\mu_i}),$$

$$\ell_i^T Z'(-\overline{\mu_i})\wp_i = \ell_i^T \hat{Z}'(-\overline{\mu_i})\wp_i.$$