MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Model Order Reduction at Industrial Scale

Peter Benner

with Jens Saak Steffen W.R. Werner ...

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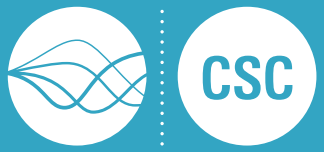

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   - Model Order Reduction of Dynamical Systems
   - Industrial Application Areas
   - System Classes
   - Model Reduction of Linear Systems
   - Model Reduction in Frequency Domain
2. Balanced Truncation for Linear Systems
3. Balanced Truncation for Polynomial Systems
4. Conclusions

Original System

$$\Sigma : \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)), \\ y(t) &= g(t, x(t), u(t)), \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
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$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\|$ for all admissible input signals.

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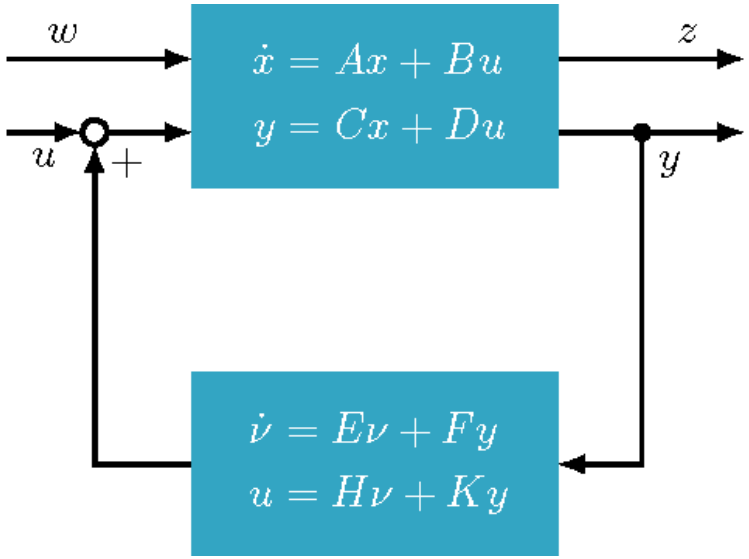
Feedback Control

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
- output $=$ input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



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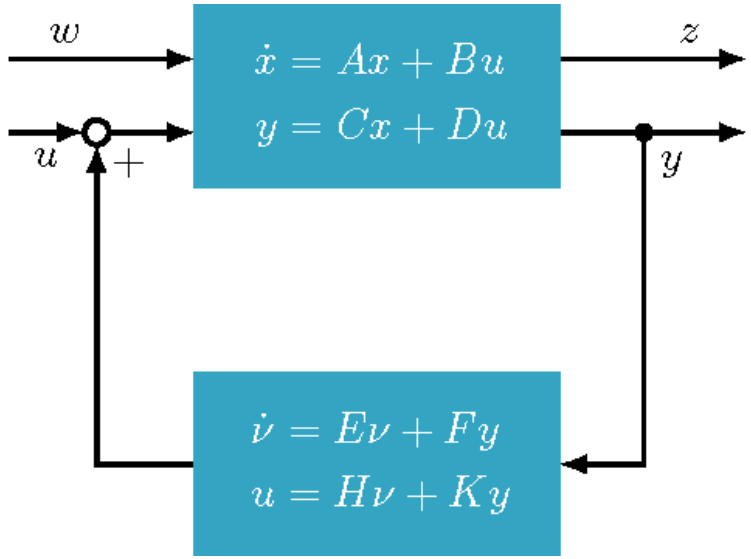
Practical controllers require small N ($N \sim 10$, say) due to

- real-time constraints,
- increasing fragility for larger N.



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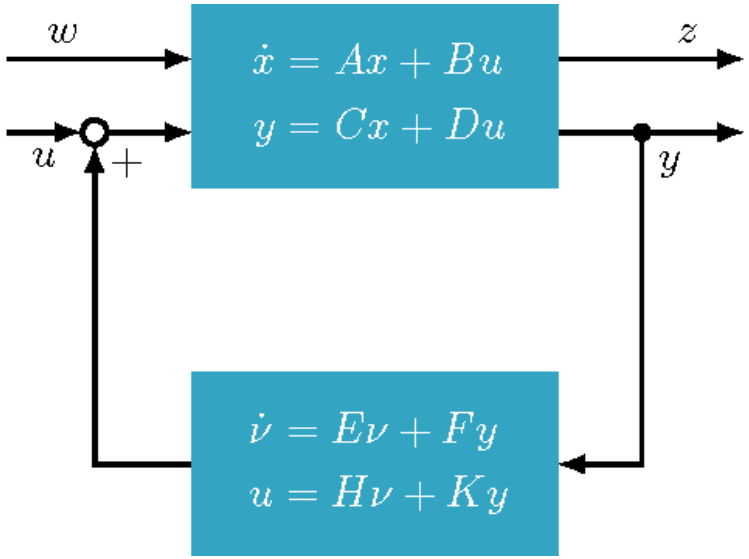
$\Longrightarrow$ **reduce order of plant (n) and/or controller (N).**



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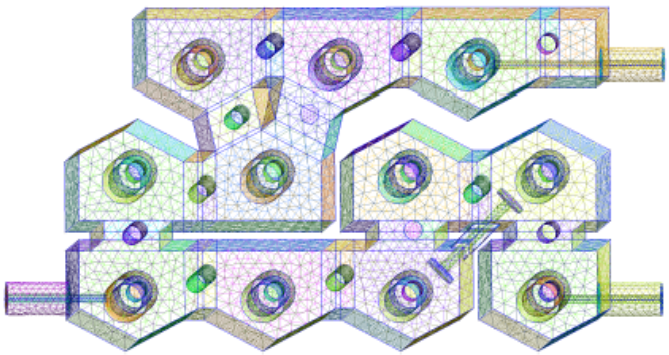
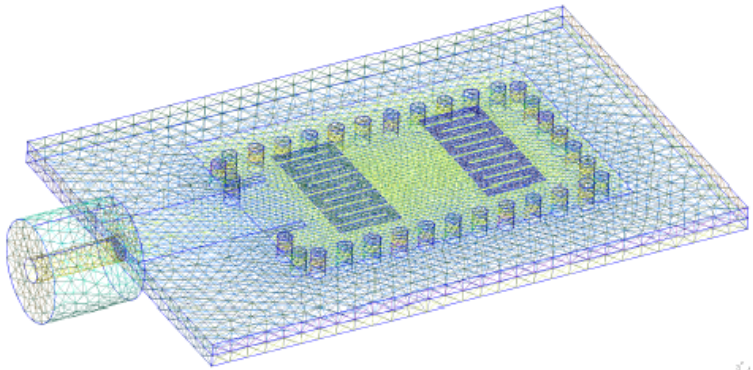
Industrial Application Areas

Nanoelectronics/Electromagnetics

- Progressive miniaturization: former **Moore's Law** stated that the number of on-chip transistors doubles each 12 months.
  No longer realistic, but miniaturization trend keeps on going, Intel timeline:
  10nm (2021) $\rightsquigarrow$ 7nm (now) $\rightsquigarrow$ 5nm (2025).





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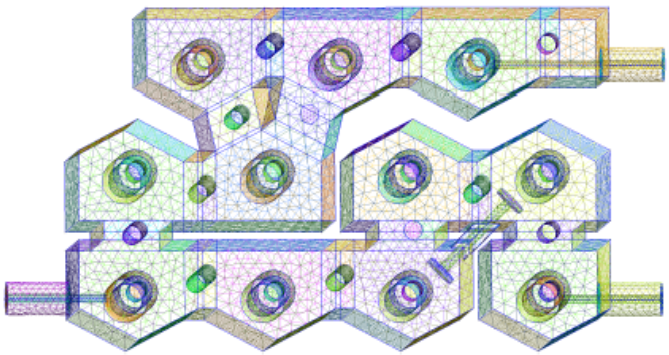

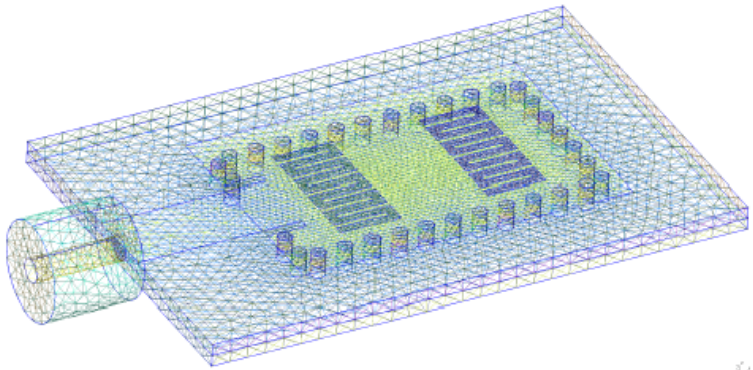

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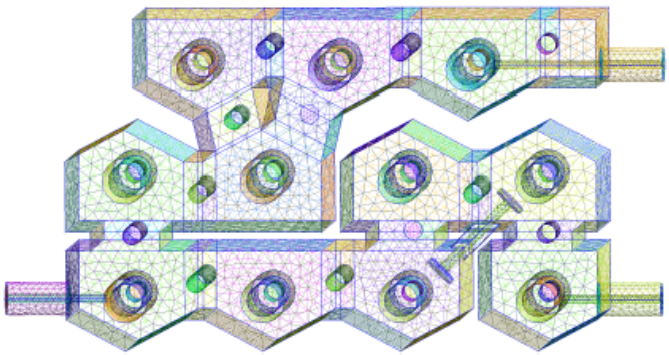

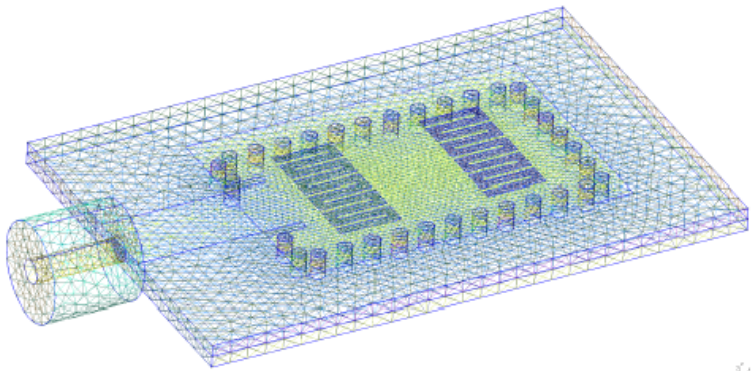

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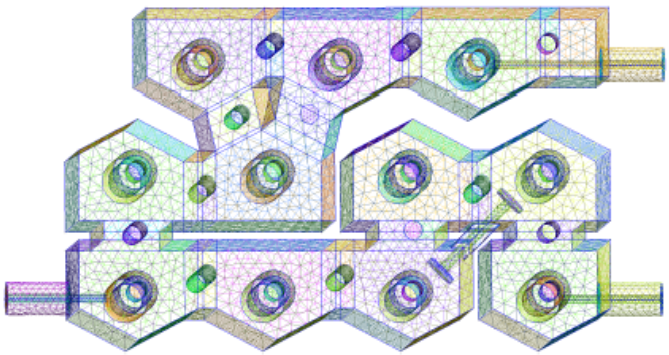
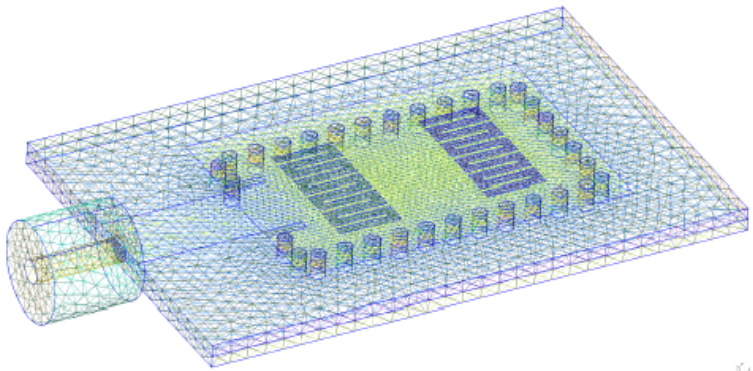
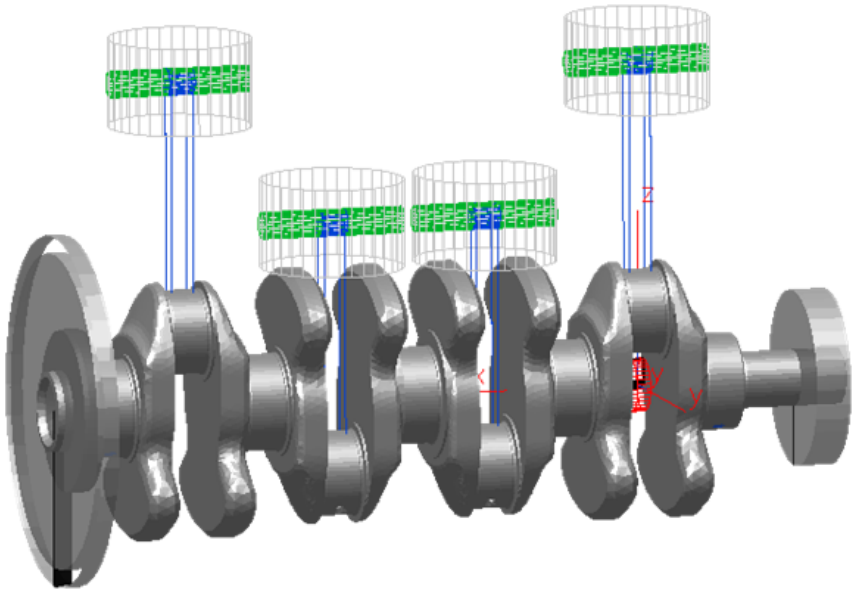
- Linear systems in nanoelectronics occur through
  - modified nodal analysis (MNA) for RLC networks, e.g., when decoupling large linear subcircuits,
  - modeling transmission lines (interconnect, powergrid), parasitic effects,
  - modeling circuit elements described by Maxwell's equation using partial element equivalent circuits (PEEC), e.g., microwave devices like splitters and diplexers, electromagnetic devices like antennas.

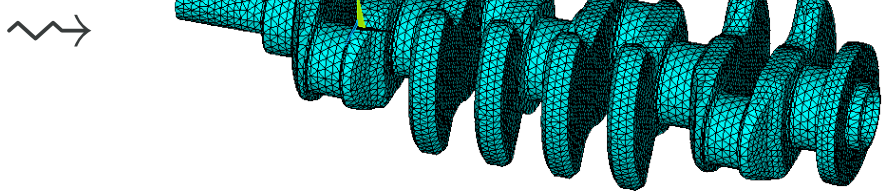




Industrial Application Areas

Structural Mechanics / Finite Element Modeling





- Resolving complex 3D geometries $\Rightarrow$ millions of degrees of freedom.
- Analysis of elastic deformations requires many simulation runs for varying external forces.
- Additional goal: Preserve second-order structure in reduced-order model, i.e.,

$$M\ddot{x}(t) + D\dot{x}(t) + \mathcal{K}(x(t)) = Bu(t)$$

for seamless integration into commercial EMBD simulation software.

System Classes

Control-Affine (Autonomous) Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = \mathcal{A}(x(t)) + \mathcal{B}(x(t))u(t), \quad \mathcal{A}:\mathbb{R}^n \to \mathbb{R}^n,\ \mathcal{B}:\mathbb{R}^n \to \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) = \mathcal{C}(x(t)) + \mathcal{D}(x(t))u(t), \quad \mathcal{C}:\mathbb{R}^n \to \mathbb{R}^{q},\ \mathcal{D}:\mathbb{R}^n \to \mathbb{R}^{q\times m}. \end{aligned}$$

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Linear Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = Ax(t) + Bu(t), & A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) = Cx(t) + Du(t), & C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}. \end{aligned}$$

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Bilinear Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = Ax(t) + \textstyle\sum_{i=1}^{m} u_i(t)A_ix(t) + Bu(t), & A, A_i \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) = Cx(t) + Du(t), & C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}. \end{aligned}$$

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$$\begin{aligned} \dot{x}(t) &= f(t,x,u) &&= Ax(t)+H\left(x(t)\otimes x(t)\right)+\textstyle\sum_{i=1}^{m} u_i(t)A_ix(t)+Bu(t), && \\ & && && A, A_i \in \mathbb{R}^{n\times n},\ H \in \mathbb{R}^{n\times n^2},\ B \in \mathbb{R}^{n\times m},\\ y(t) &= g(t,x,u) &&= Cx(t)+Du(t), && C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}. \end{aligned}$$

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Polynomial Systems

$$\begin{aligned}\dot{x}(t) &= f(t,x,u) = Ax(t) + \sum_{j=2}^{n_p} H_j\left(\otimes^j x(t)\right) + \sum_{j=2}^{n_p}\sum_{k=1}^{m} A_j^k\left(\otimes^j x(t)\right)u_k(t) + Bu(t),\\ &\qquad\qquad H_j, A_j^k \text{ of "appropriate size"},\\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}.\end{aligned}$$

Polynomial-bilinear systems can be obtained from smooth nonlinear systems by *lifting* without approximation error! [Gu 2011].

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Written in control-affine form:

$$\begin{aligned}\mathcal{A}(x) &:= Ax + H\left(x\otimes x\right), & \mathcal{B}(x) &:= [A_1,\ldots,A_m]\left(I_m\otimes x\right) + B\\ \mathcal{C}(x) &:= Cx, & \mathcal{D}(x) &:= D.\end{aligned}$$

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Here, we focus on linear and polynomial systems.

Model Reduction of Linear Systems

Linear Time-Invariant (LTI) Systems

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Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sX(s) - x(0))$ to LTI system

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$\Longrightarrow$ I/O-relation in frequency domain:

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Model reduction in frequency domain: Fast evaluation of mapping $U \ \rightarrow \ Y$.

Formulating model reduction in frequency domain

Approximate the time domain dynamical system

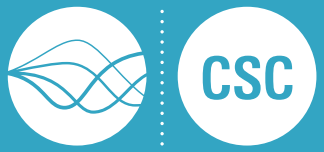
$$\begin{aligned} \dot{x} &= Ax + Bu, & A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y &= Cx + Du, & C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}, \end{aligned}$$

by reduced-order system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, & \hat{A} \in \mathbb{R}^{r\times r},\ \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, & \hat{C} \in \mathbb{R}^{p\times r},\ \hat{D} \in \mathbb{R}^{p\times m} \end{aligned}$$

of order $r \ll n$, such that

$$\begin{aligned} \|y - \hat{y}\| \simeq \left\|Y - \hat{Y}\right\| &= \left\|GU - \hat{G}U\right\| \\ &\leq \left\|G - \hat{G}\right\| \cdot \|U\| \simeq \left\|G - \hat{G}\right\| \cdot \|u\| \\ &\leq \text{tolerance} \cdot \|u\| \,. \end{aligned}$$



Outline

1. Introduction

2. Balanced Truncation for Linear Systems
   - Basic Concept
   - Software
   - Numerical Examples

3. Balanced Truncation for Polynomial Systems

4. Conclusions

Balanced Truncation for Linear Systems

Basic concept

- System Σ : $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization (needs P, Q!) of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C) &\mapsto (TAT^{-1}, TB, CT^{-1}) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right). \end{aligned}$$

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}) = (A_{11}, B_1, C_1)$.
  Note: in efficient algorithms, truncation is achieved via projection:

  $$(\hat{A}, \hat{B}, \hat{C}) = (W^TAV, W^TB, CV), \quad \text{where} \quad W^TV = I_r.$$

Balanced Truncation for Linear Systems

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- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.

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$$\|y - \hat{y}\|_2 \leq \|G - \hat{G}\|_{\mathcal{H}_\infty} \|u\|_2 \leq \left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right) \|u\|_2 ,$$

where $\|G\|_{\mathcal{H}_\infty} := \sup_{u \in \mathcal{L}_2 \setminus \{0\}} \frac{\|Gu\|_2}{\|u\|_2} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(\jmath\omega))$.

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Practical implementation

- Rather than solving Lyapunov equations for P, Q (n^2 unknowns!), find $S, R \in \mathbb{R}^{n \times s}$ with $s \ll n$ such that $P \approx SS^T$, $Q \approx RR^T$.
- Reduced-order model directly obtained via small-scale ($s \times s$) SVD of $R^T S$!

[a] http://www.mpi-magdeburg.mpg.de/projects/morlab

[b] https://www.mpi-magdeburg.mpg.de/projects/mess, partially integrated into MATLAB R2023b.

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- Reduced-order model directly obtained via small-scale ($s \times s$) SVD of $R^T S$!
- Two software packages:
  - MORLAB[a] (**M**odel **O**rder **R**eduction **LAB**oratory), based on spectral projection methods ($\rightsquigarrow$ small to medium size problems, up to $n \sim 5,000$.)
  - M-M.E.S.S.[b] provides solvers for large-scale matrix equations with sparse/low-rank coefficients and basic MOR functionality; no $\mathcal{O}(n^3)$ or $\mathcal{O}(n^2)$ computations necessary!

[a] http://www.mpi-magdeburg.mpg.de/projects/morlab

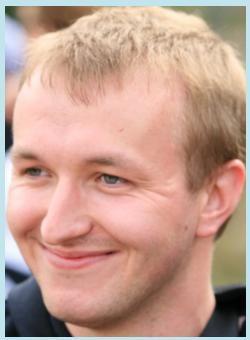
[b] https://www.mpi-magdeburg.mpg.de/projects/mess, partially integrated into MATLAB R2023b.

Core developer:

Jens Saak

Copyright holder:

Peter Benner



Martin Köhler

Further contributors (v1.0–v3.0)

- Björn Baran (former MPI Magdeburg)
- Maximilian Behr (former MPI Magdeburg)
- Zvonimir Bujanović (now University of Zagreb)
- Christian Himpe (ow WWU Münster)
- Manuela Hund (former MPI Magdeburg)
- Martin Köhler (MPI Magdeburg)
- Patrick Kürschner (now HTW Leipzig)
- Norman Lang (now IAV GmbH Ingenieurgesellschaft Auto und Verkehr)
- Tony Stillfjord (now Lund University)
- Heiko K. Weichelt (now The MathWorks, Inc.)
- Steffen W. R. Werner (now Virginia Tech)

Version History

2000	LyaPack-1.0	Thilo Penzl
2002–2006	LyaPack-1.1–1.9	Jens Saak, Peter Benner
2007–2016	M-M.E.S.S.-1.0(.1)	Jens Saak, Martin Köhler, Peter Benner
2017–2020	M-M.E.S.S.-2.0(.1)	Jens Saak, Martin Köhler, Peter Benner

Core algorithm: low-rank alternating directions implicit iteration.
LR-ADI

New and Planned Releases

M-M.E.S.S.-2.1	extended MOR features
M-M.E.S.S.-3.0	Krylov-projection based solvers / Inexact solves in LR-ADI / first Sylves
	released 22 Sept. 2023!
M-M.E.S.S.-4.0	non-symmetric matrix equations
long term	Lur'e equations / large-scale LMIs / parametric MOR?

C-M.E.S.S.

- C-language implementation of the M-M.E.S.S. core features with some additions.
- extended (multicore-)parallel capabilities.
- low memory footprint.
- BLAS & LAPACK abstraction.
- suitesparse and SuperLU support.

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MEX-wrapper supporting both complex number APIs

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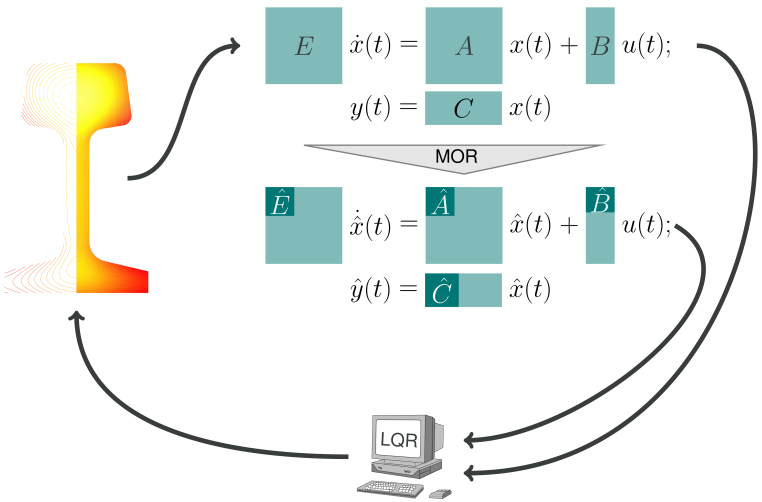
Ju-M.E.S.S.

Wrapper for the Julia language.



- All equations are associated to a possibly abstract **standard state-space system $\mathbf{\Sigma(E; A, B, C)}$** or its dual.
- Proper differential-algebraic equations (DAE) or second-order (SO) systems use **implicit index-reduction** or **implicit linearization** to achieve this.
- The transformed implicit systems are (preferably) never formed explicitly.
- Operations with systems are abstracted into **user-supplied functions sets** (USFS):

USFS	`default`	`so_1 / so_2`	`dae_1`	`dae_2`	`dae_1/2/3_so`
System	standard / generalized state-space form	second-order 1st/2nd companion form	index-1 DAE	index-2 Stokes-type DAEs	second-order index-1/2/3 using companion form
Demos	FDM, Rail	TripleChain	DAE1 (BIPS Power-systems model)	DAE2 Stokes, Kárman vortex shedding	constrained TripleChain

$$0 = \mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} + \mathbf{C}^T\mathbf{C} - \mathbf{X}\mathbf{B}\mathbf{B}^T\mathbf{X}$$

$$0 = \mathbf{A}^T\mathbf{X}\mathbf{E} + \mathbf{E}^T\mathbf{X}\mathbf{A} + \mathbf{C}^T\mathbf{C}$$

$$0 = \mathbf{A}^T\mathbf{X}\mathbf{E} + \mathbf{E}^T\mathbf{X}\mathbf{A} + \mathbf{C}^T\mathbf{C} - \mathbf{E}^T\mathbf{X}\left(\mathbf{B}_1\mathbf{B}_1^T - \mathbf{B}_2\mathbf{B}_2^T\right)\mathbf{X}\mathbf{E}$$

LyaPack

M-M.E.S.S.-1.0(.1) ...

M-M.E.S.S.-3.0

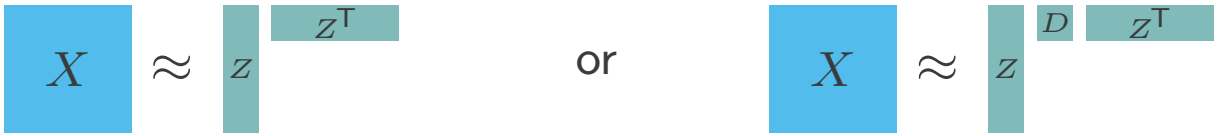
$$\mathbf{E}^T\dot{\mathbf{X}}(t)\mathbf{E} = \mathbf{A}^T\mathbf{X}(t)\mathbf{E} + \mathbf{E}^T\mathbf{X}(t)\mathbf{A} + \mathbf{C}^T\mathbf{C} - \mathbf{E}^T\mathbf{X}(t)\mathbf{B}\mathbf{B}^T\mathbf{X}(t)\mathbf{E}$$

$$\mathbf{E}(t)^T\dot{\mathbf{X}}(t)\mathbf{E}(t) = \left(\mathbf{A}(t) + \dot{\mathbf{E}}(t)\right)^T\mathbf{X}(t)\mathbf{E}(t) + \mathbf{E}(t)^T\mathbf{X}(t)\left(\mathbf{A}(t) + \dot{\mathbf{E}}(t)\right) + \mathbf{C}(t)^T\mathbf{C}(t) - \mathbf{E}(t)^T\mathbf{X}(t)\mathbf{B}(t)\mathbf{B}(t)^T\mathbf{X}(t)\mathbf{E}(t)$$

Algebraic Riccati equations $\mathcal{R}(X) = 0$:

where $A \in \mathbb{R}^{n\times n}$ (sparse), $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ with $m, p \ll n$ and n "large".

Common solution paradigm: compute low-rank approximation:

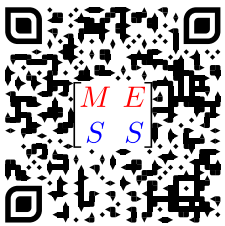


M-M.E.S.S. — facts & figures:

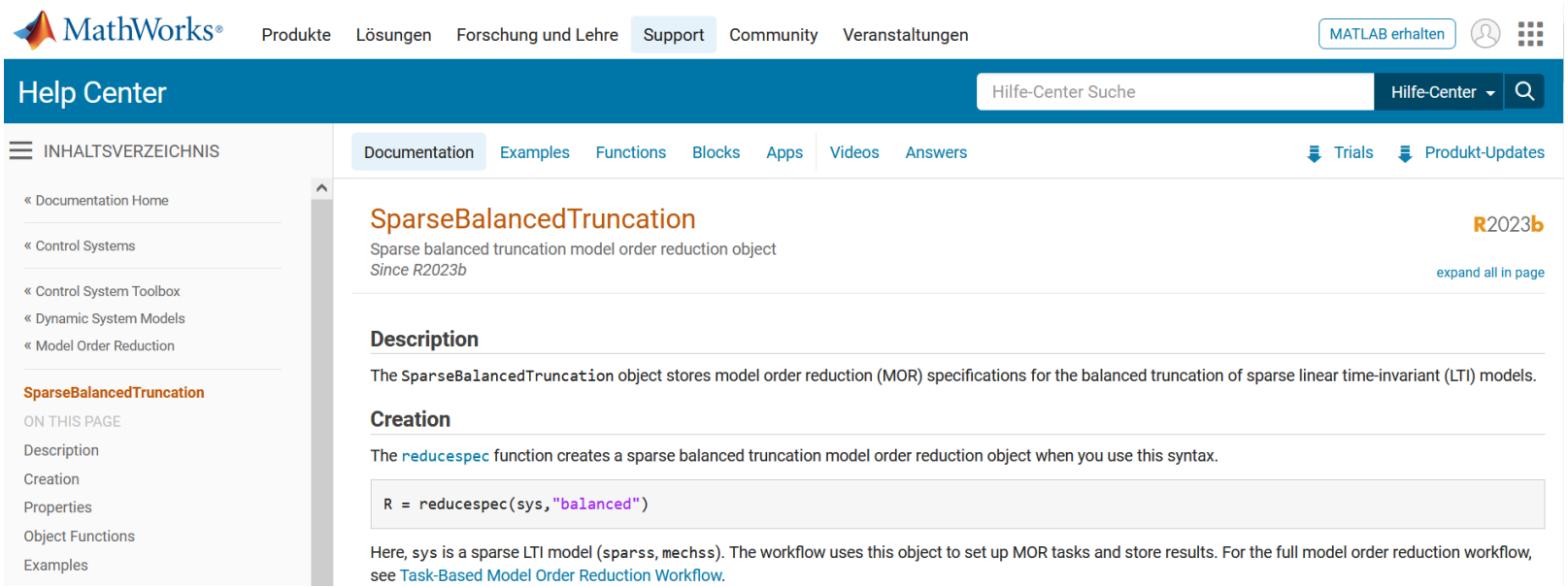
# contributors:	≈ 15
# code lines:	≈ 24,000 (≈ 32,000 including CI tests)
# comment and help lines:	≈ 14,000 (≈ 15,000 including CI tests)
history:	more than 20 years of active development
license:	BSD 2-Clause
current version:	3.0 (22 Sept. 2023)

Benchmarks

Fishtail	BT, second-order, ≈780k DoFs
Adaptive Spindle support	BT, second-order index-1, ≈250k DoFs
Stefan problem	LTV-LQR (non-autonomous DRE), ≈25k DoFs



E-Mail mess@mpi-magdeburg.mpg.de]
WWW https://www.mpi-magdeburg.mpg.de/projects/mess
GIT https://gitlab.mpi-magdeburg.mpg.de/mess/mmess-releases
Zenodo https://doi.org/10.5281/zenodo.632897



[. . .]

Algorithms

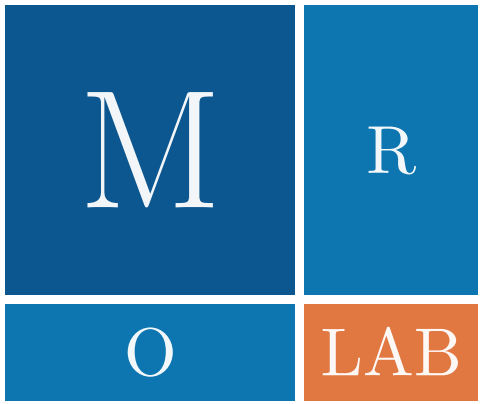
The sparse balanced truncation algorithm performs these steps to reduce the input model G to the desired order k.

1. Find the low-rank approximations L_r and L_o of the Gramian factors. This is based on the low-rank alternating directions implicit (LRADI) algorithm, which is an iterative method for solving the Lyapunov equations. For more details, see [1] and [2].
2. Compute the HSVs σ_j based on the approximate controllability and observability Gramians.
3. Obtain the reduced order model using the balanced model truncation with absolute error control [3] (see the Algorithms section of `BalancedTruncation`).

References

[1] Benner, Peter, Jing-Rebecca Li, and Thilo Penzl. "Numerical Solution of Large-Scale Lyapunov Equations, Riccati Equations, and Linear-Quadratic Optimal Control Problems." *Numerical Linear Algebra with Applications* 15, no. 9 (November 2008): 755–77. https://doi.org/10.1002/nla.622.

[2] Benner, Peter, Martin Köhler, and Jens Saak. "Matrix Equations, Sparse Solvers: M-M.E.S.S.-2.0.1—Philosophy, Features, and Application for (Parametric) Model Order Reduction." In *Model Reduction of Complex Dynamical Systems*, edited by Peter Benner, Tobias Breiten, Heike Faßbender, Michael Hinze, Tatjana Stykel, and Ralf Zimmermann, 171:369–92. Cham: Springer International Publishing, 2021. https://doi.org/10.1007/978-3-030-72983-7_18.

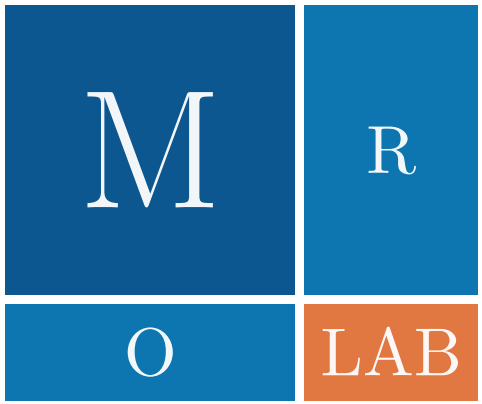


MORLAB (**M**odel **O**rder **R**eduction **LAB**oratory)

- model reduction in MATLAB/Octave
- based on spectral projection methods

https://www.mpi-magdeburg.mpg.de/projects/morlab

https://zenodo.org/record/7072831



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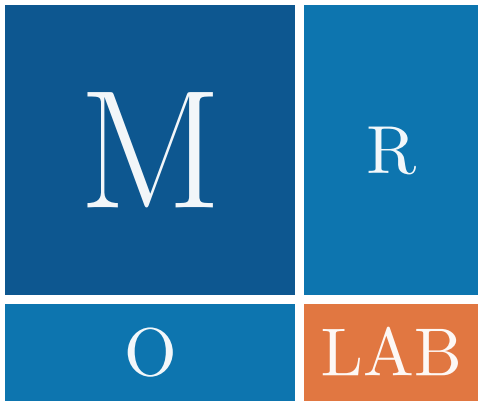
- basic solvers for standard factorized Lyapunov and Riccati equations



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- 10 model reduction methods for continuous-time and discrete-time standard, descriptor and second-order systems (balancing-related methods, . . .)
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- system-theoretic subroutines and frequency/time domain evaluation tools

MORLAB
Software (History)

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Version 6.0 (25 Sept. 2023) — P. Benner, S.W.R. Werner, J. Saak:

- Routines for sparse systems with M-M.E.S.S. backend.

Open source and free: Licensed under GNU Affero General Public License v3.0 and freely available on the project website.

Fast and exact: Spectral projection methods often outperform other established software in terms of accuracy and speed.

Unified framework: The same interface for all model reduction / system-theoretic routines allows for easy and quick exchange.

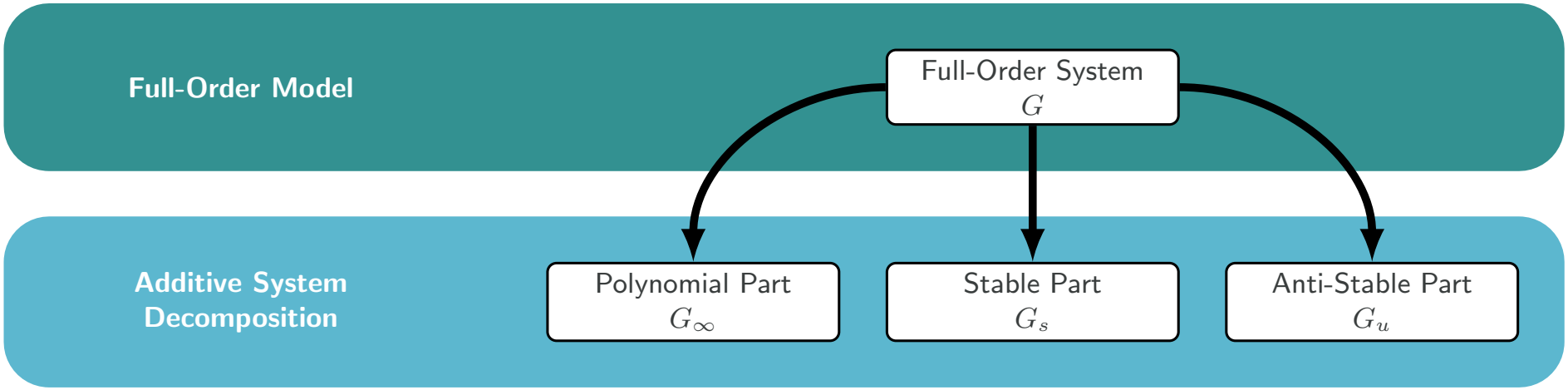
Modular: Individual subroutines can be combined by the user in various ways.

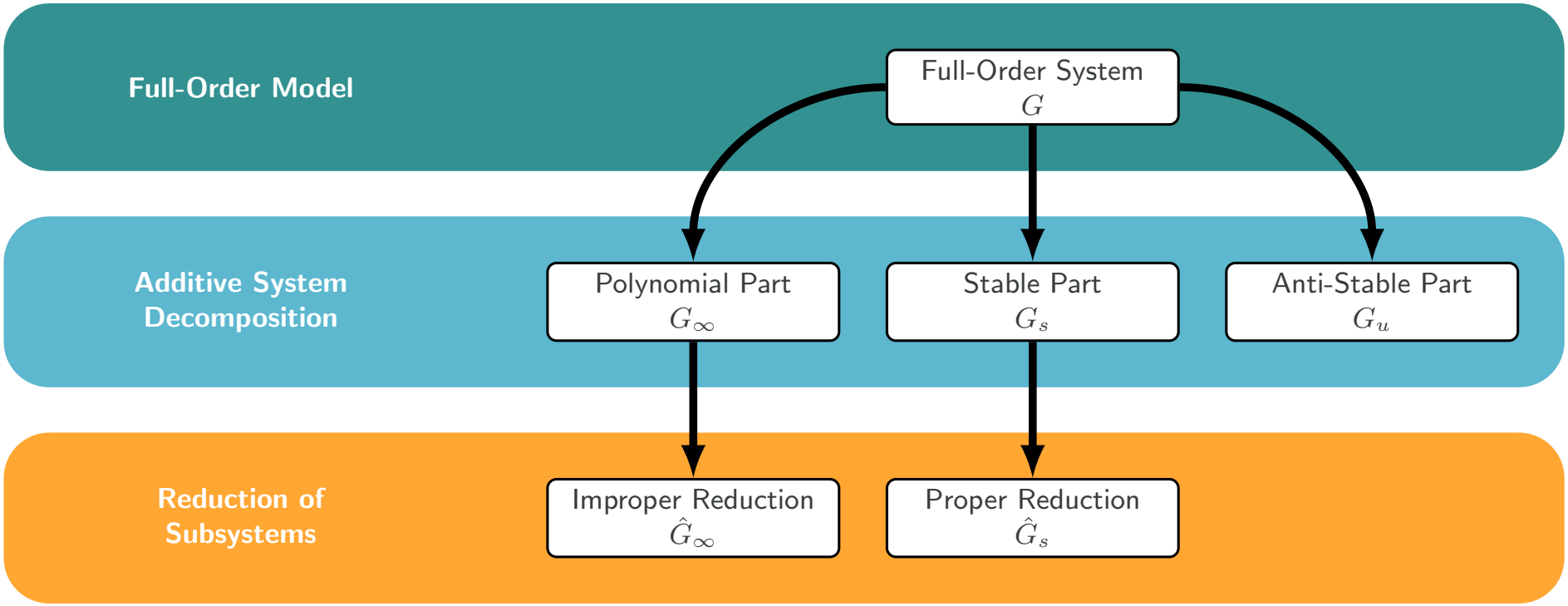
Portable: No binary extensions are required.

Dependencies: MATLAB ($\geq$ 2018a), Octave ($\geq$ 6.2.0)

Full-Order Model

Full-Order System
G


Full-Order Model
Full-Order System
G
Additive System
Decomposition
Polynomial Part
G_∞
Stable Part
G_s
Anti-Stable Part
G_u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Reduction of Subsystems
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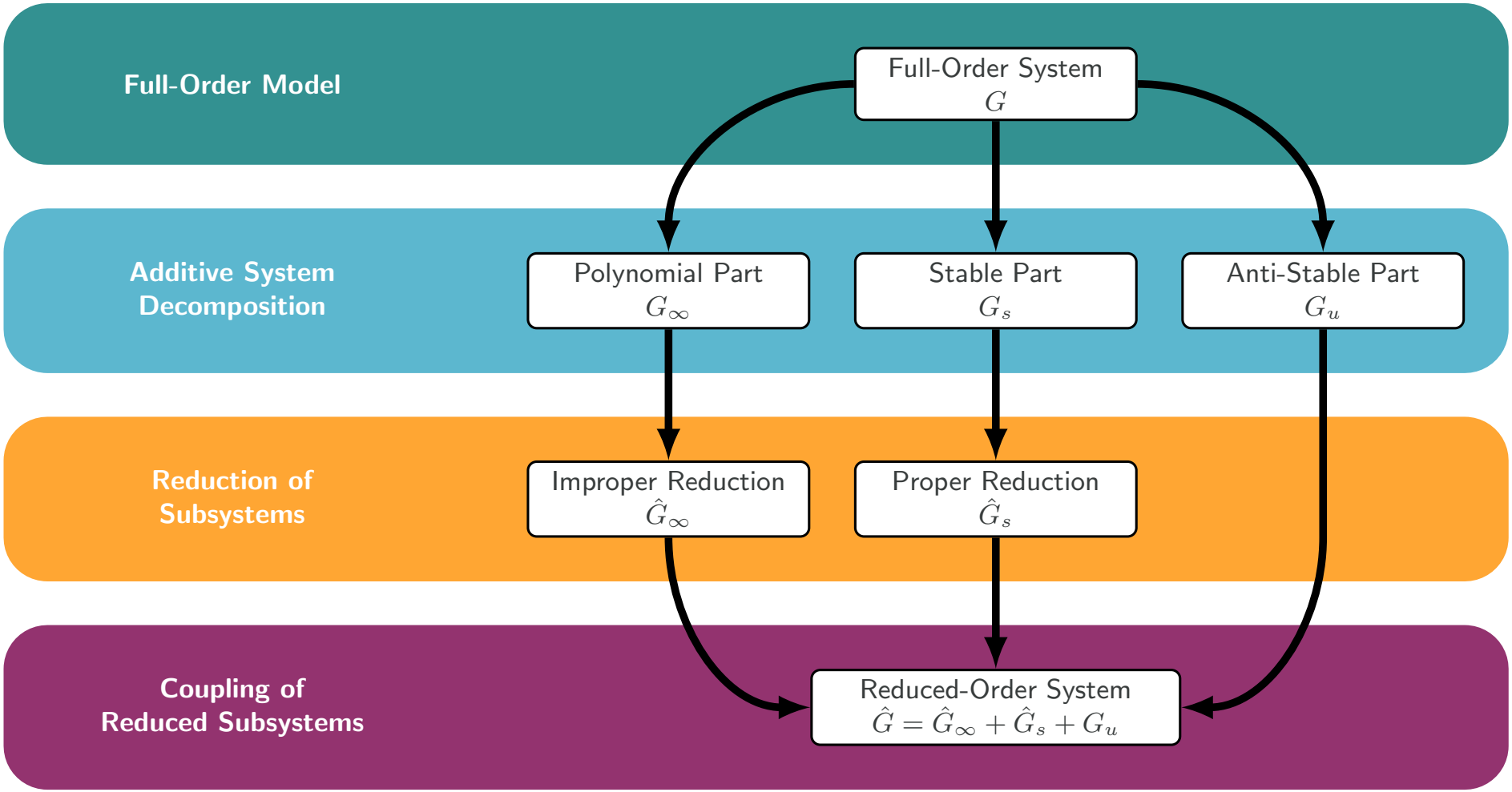
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Reduction of Subsystems
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Proper Reduction
$\hat{G}_s$
Coupling of Reduced Subsystems
Reduced-Order System
$\hat{G} = \hat{G}_\infty + \hat{G}_s + G_u$

Standard Systems **(supported in Control System Toolbox / Simulink)**

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x_{k+1} = Ax_k + Bu_k,$$
$$y(t) = Cx(t) + Du(t). \qquad y_k = Cx_k + Du_k.$$

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Descriptor Systems (partially supported in Control System Toolbox / Simulink)

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Mechanical Systems (not supported in Control System Toolbox / Simulink)

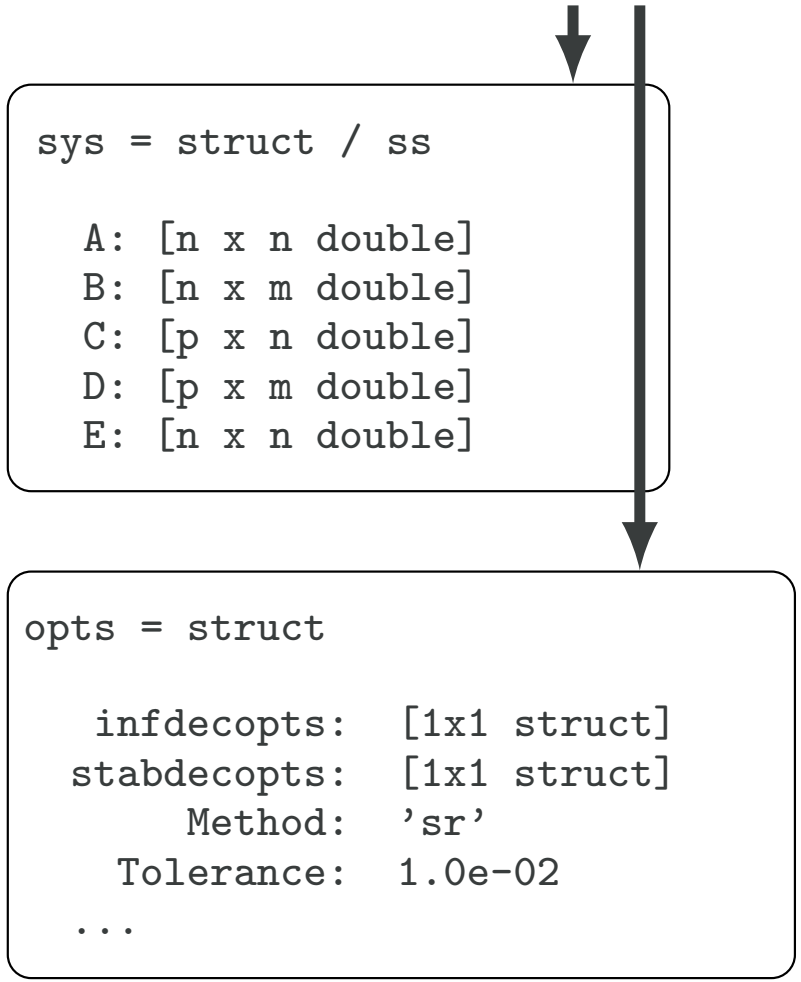
$$M\ddot{x}(t) + E\dot{x}(t) + Kx(t) = B_{\mathrm{u}}u(t),$$
$$y(t) = C_{\mathrm{p}}x(t) + C_{\mathrm{v}}\dot{x}(t) + Du(t).$$

```
[rom, info]  =  ml_ct_dss_bt(sys, opts)
```

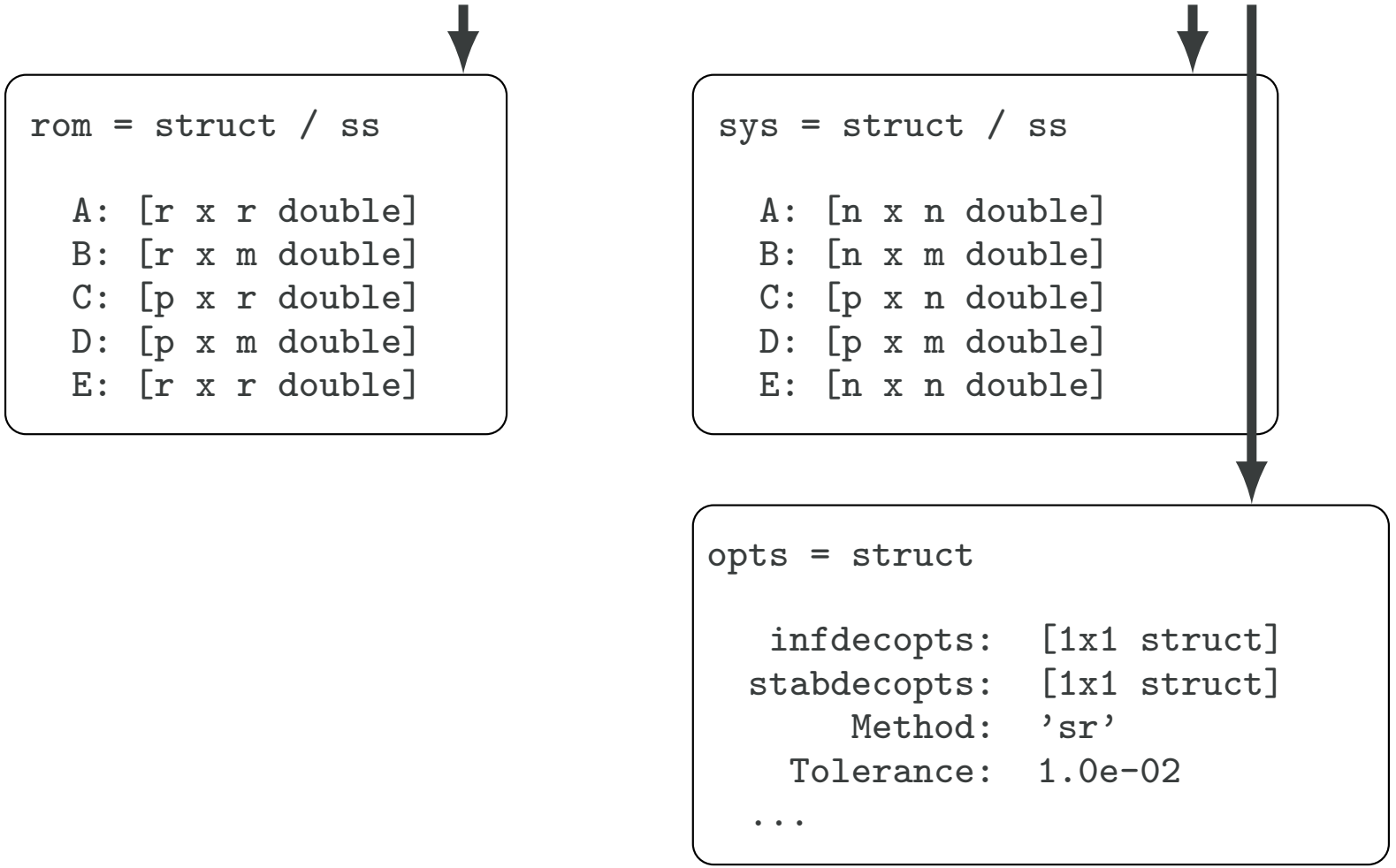
```
[rom, info]  =  ml_ct_dss_bt(sys, opts)
```

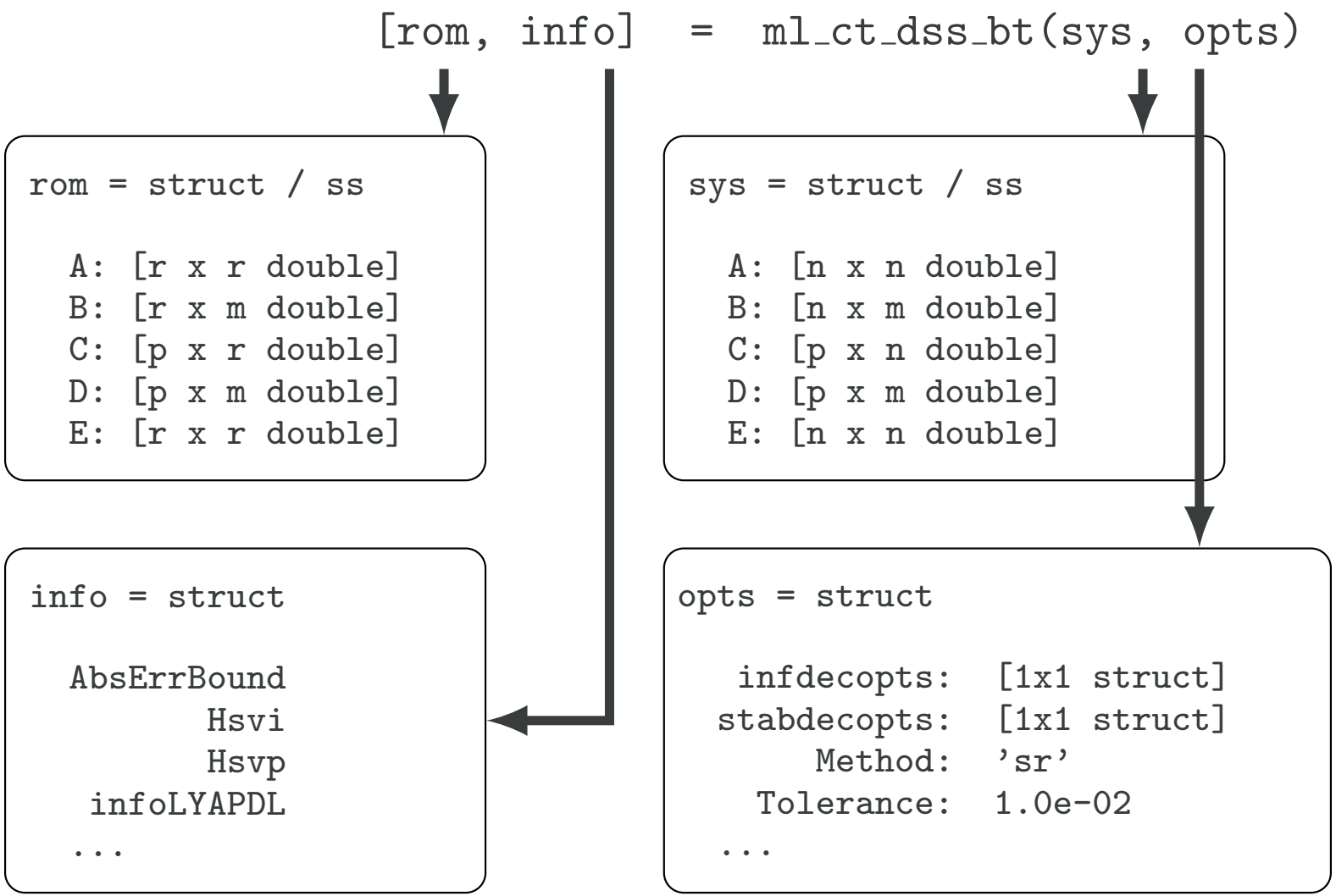
```
sys = struct / ss

  A: [n x n double]
  B: [n x m double]
  C: [p x n double]
  D: [p x m double]
  E: [n x n double]
```

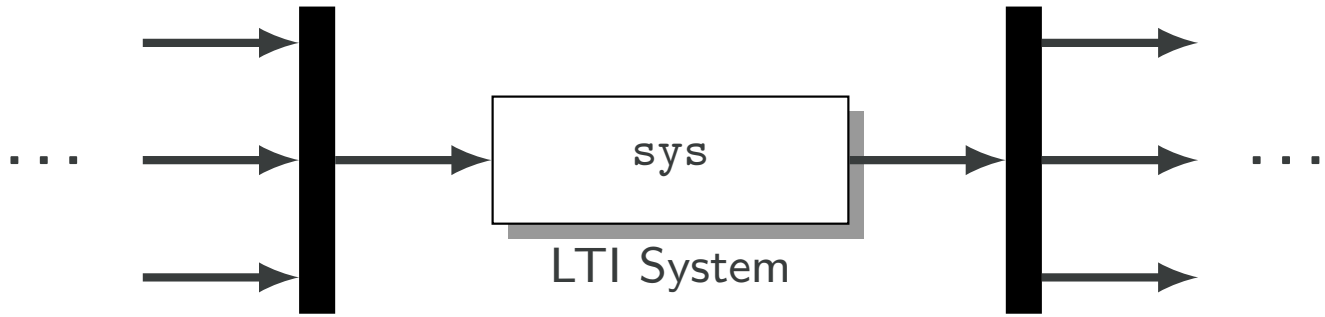
```
[rom, info]  =  ml_ct_dss_bt(sys, opts)
```

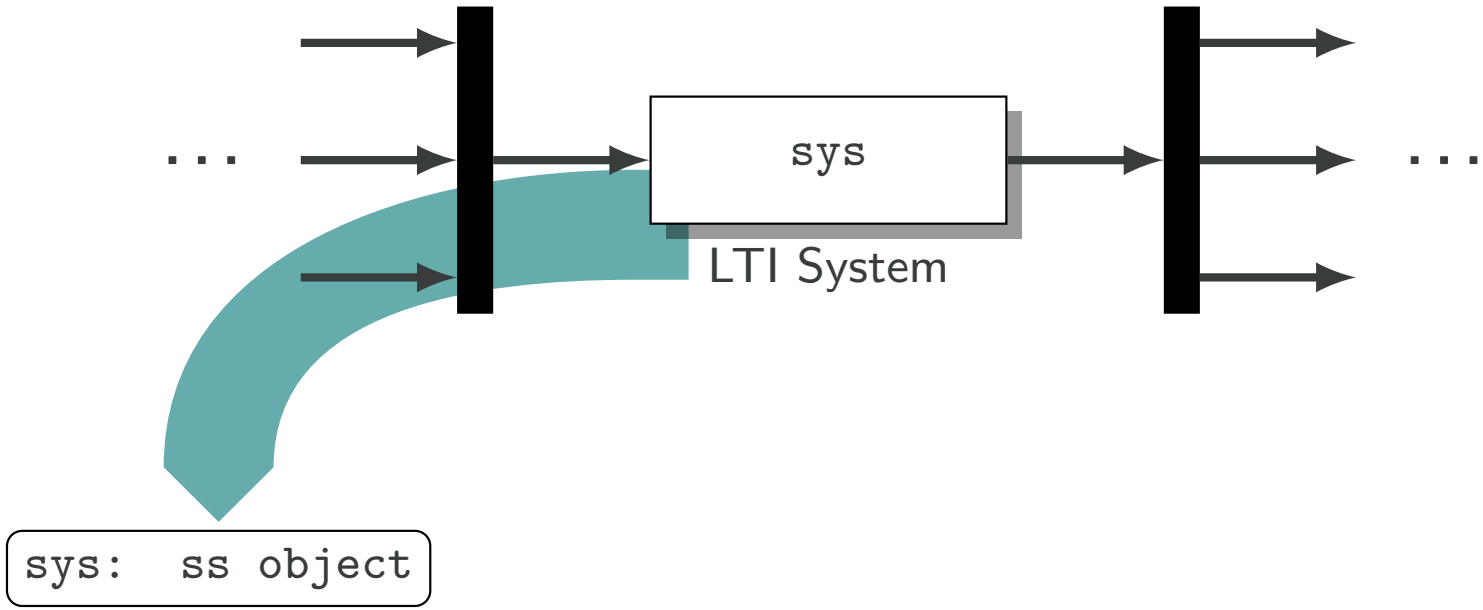


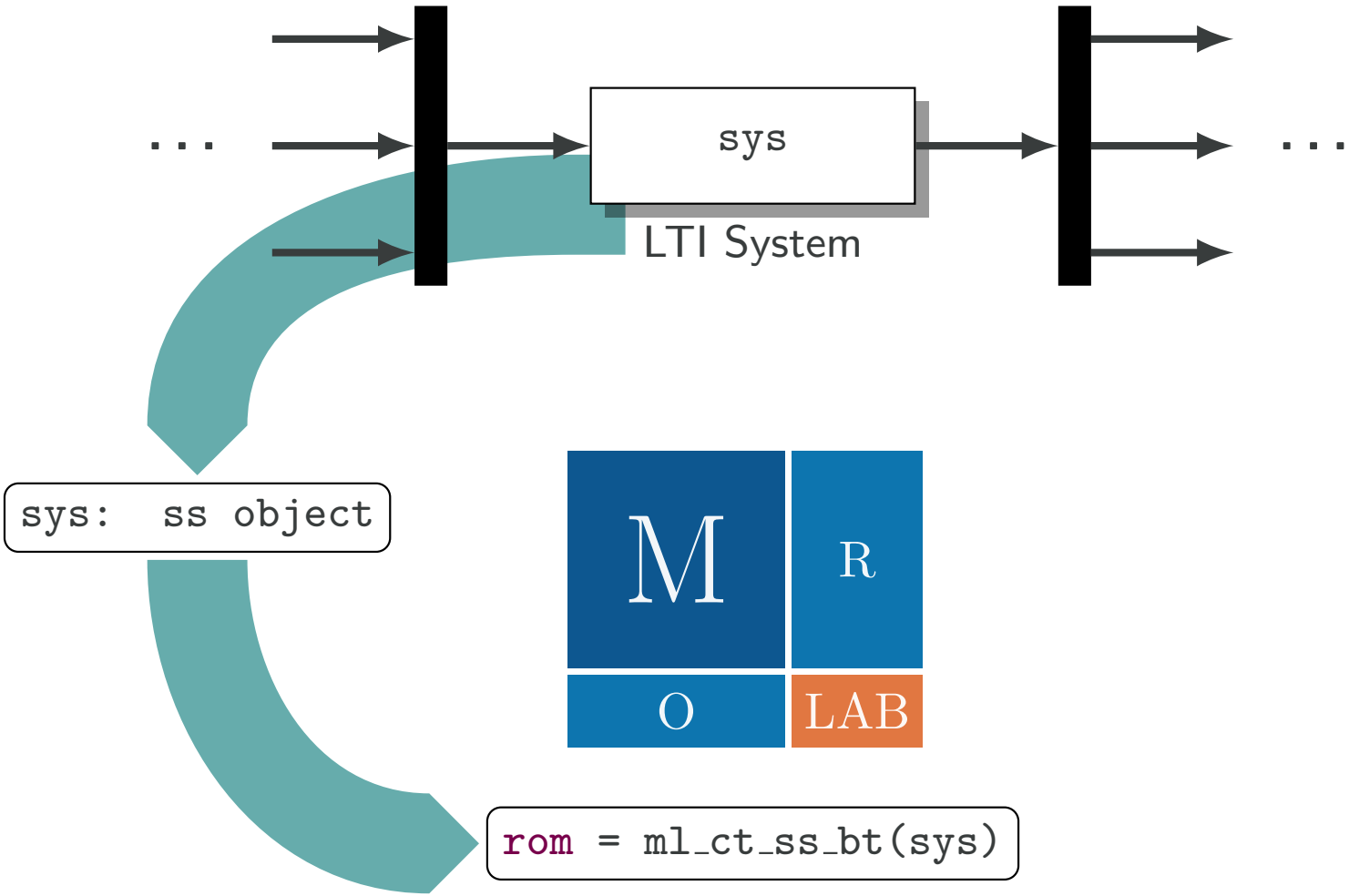
```
[rom, info]  =  ml_ct_dss_bt(sys, opts)
```

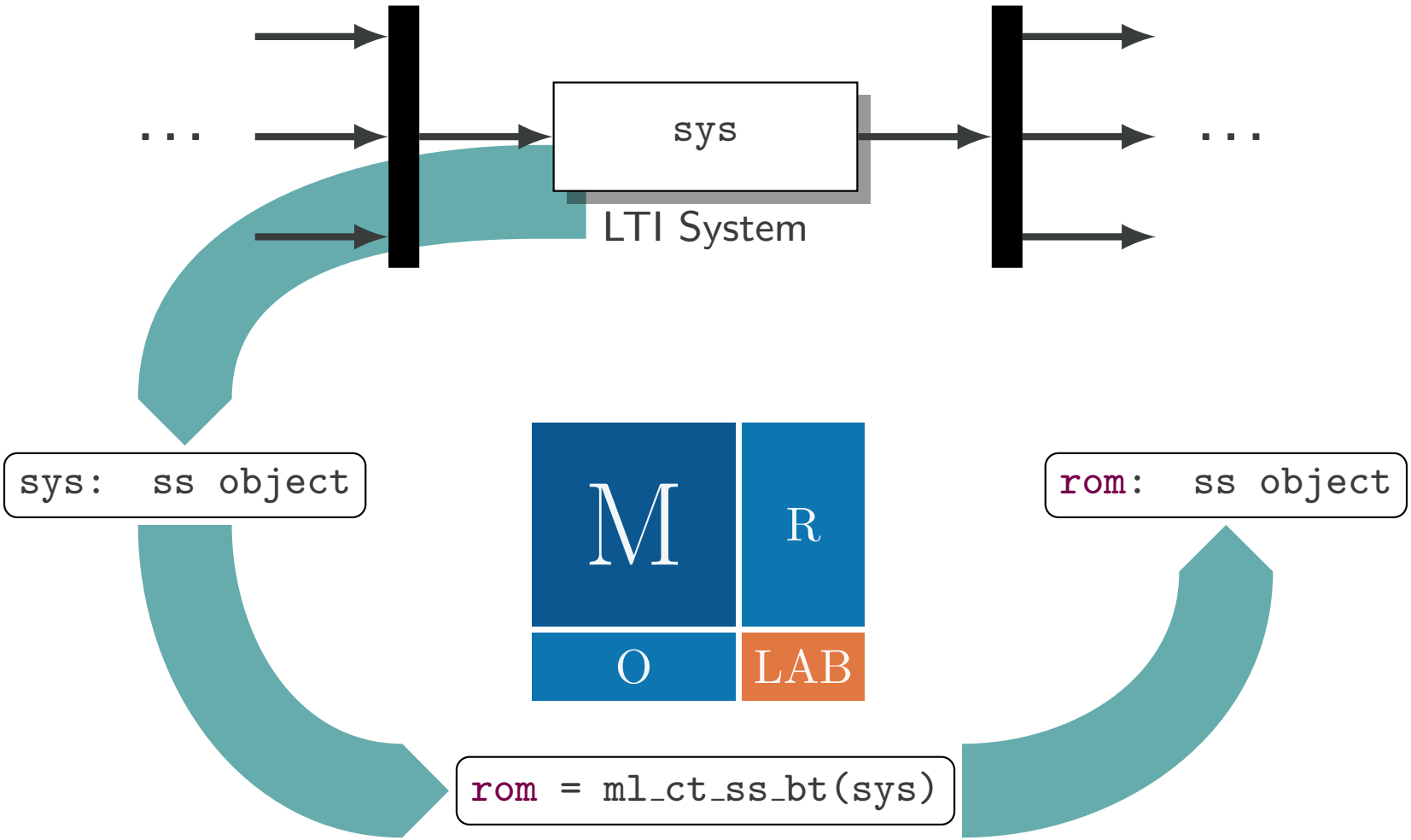



[rom, info] = ml_ct_dss_bt(sys, opts)
rom = struct / ss
A: [r x r double]
B: [r x m double]
C: [p x r double]
D: [p x m double]
E: [r x r double]
sys = struct / ss
A: [n x n double]
B: [n x m double]
C: [p x n double]
D: [p x m double]
E: [n x n double]
info = struct
AbsErrBound
Hsvi
Hsvp
infoLYAPDL
...
opts = struct
infdecopts: [1x1 struct]
stabdecopts: [1x1 struct]
Method: 'sr'
Tolerance: 1.0e-02
...


sys
LTI System


sys
LTI System
sys: ss object


sys
LTI System
sys: ss object
M
R
O
LAB
rom = ml_ct_ss_bt(sys)


...
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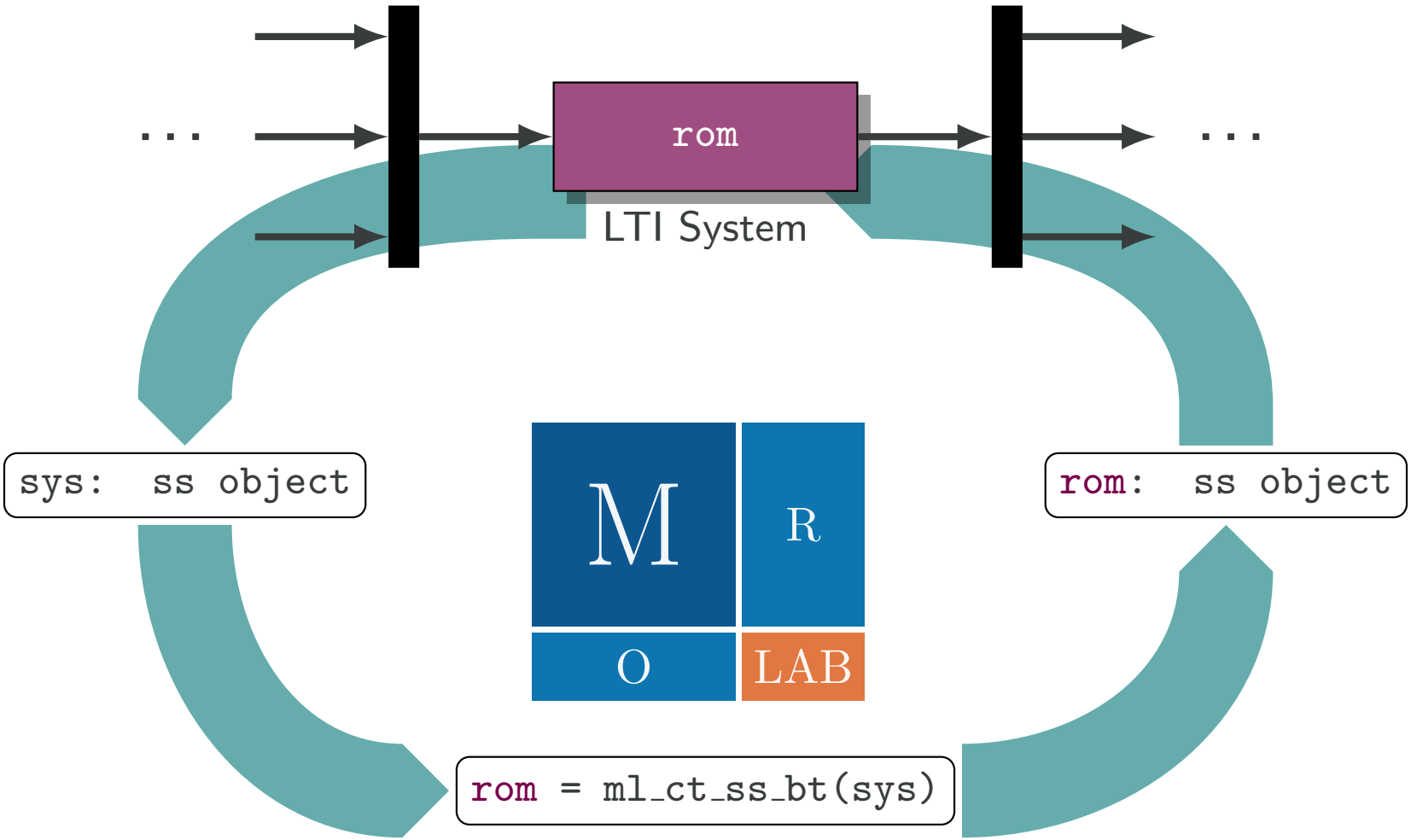
rom
LTI System
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Numerical Examples
Optimal Cooling of Steel Profiles

- Mathematical model: boundary control for linearized 2D heat equation.

$$
\begin{aligned}
c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, \quad \xi \in \Omega \\
\lambda \frac{\partial}{\partial n} x &= \kappa(u_k - x), \quad \xi \in \Gamma_k,\ 1 \le k \le 7, \\
\frac{\partial}{\partial n} x &= 0, \qquad \xi \in \Gamma_7.
\end{aligned}
$$

$\Longrightarrow m = 7, p = 6.$

- FEM Discretization with FENICS (https://zenodo.org/record/5113560), different models for initial mesh ($n = 371$), $1, 2, 3, \ldots$ steps of mesh refinement $\Rightarrow$ $n = 1357; 5177; 20,209; 79,841; \ldots; 1,265,537.$

Source: Physical model: courtesy of Mannesmann/Demag.
Math. model: [Tröltzsch/Unger 1999/2001, Penzl 1999, Saak 2003].
Implementation: [B./Saak 2005, Saak/Behr 2021].

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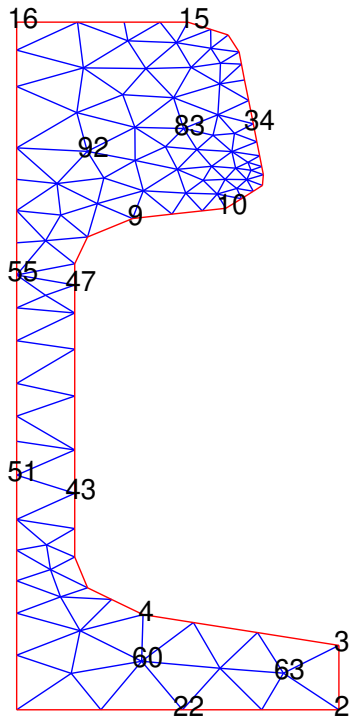

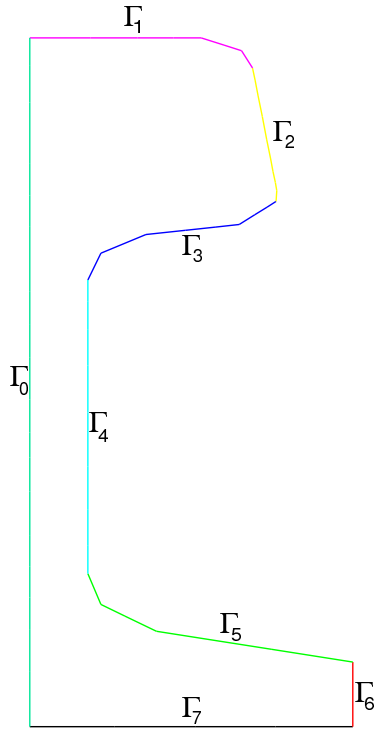

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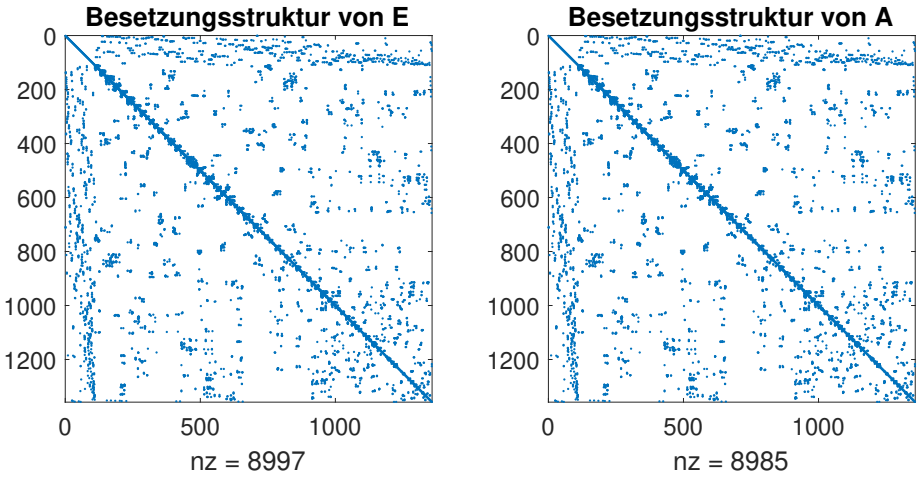
Sparsity patterns of mass/stiffness



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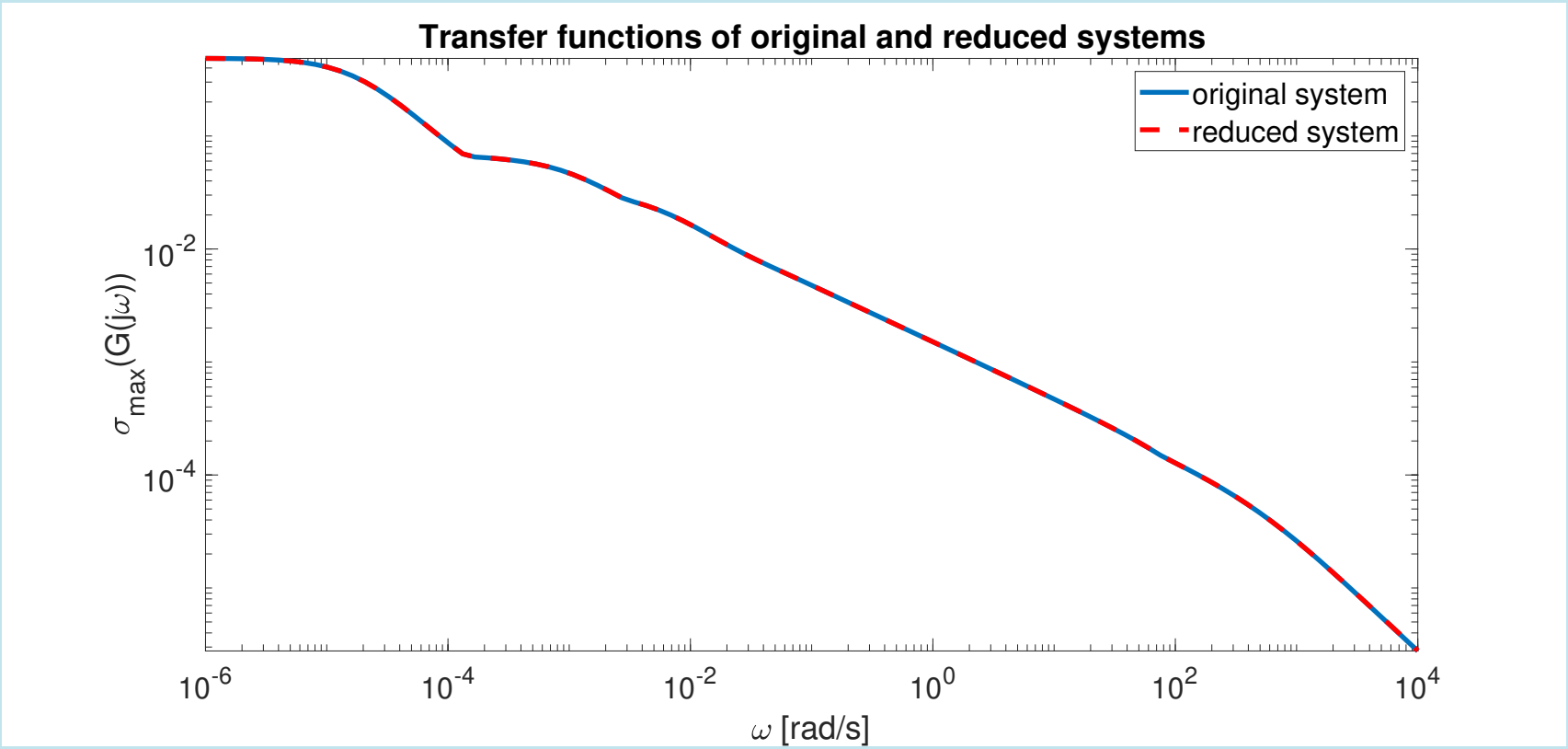
Sigma-Plot ($n = 1,265,537$ vs. $r = 77$)



Computation by **Jens Saak** with M-M.E.S.S. 3.0 in MATLAB R2023a under Linux (Ubuntu 22.04) on a Lenovo P14s Laptop with 12GB RAM and AMD Ryzen 7 PRO 5850U (8 cores@3.55 GHz with 16MB L3 Cache).

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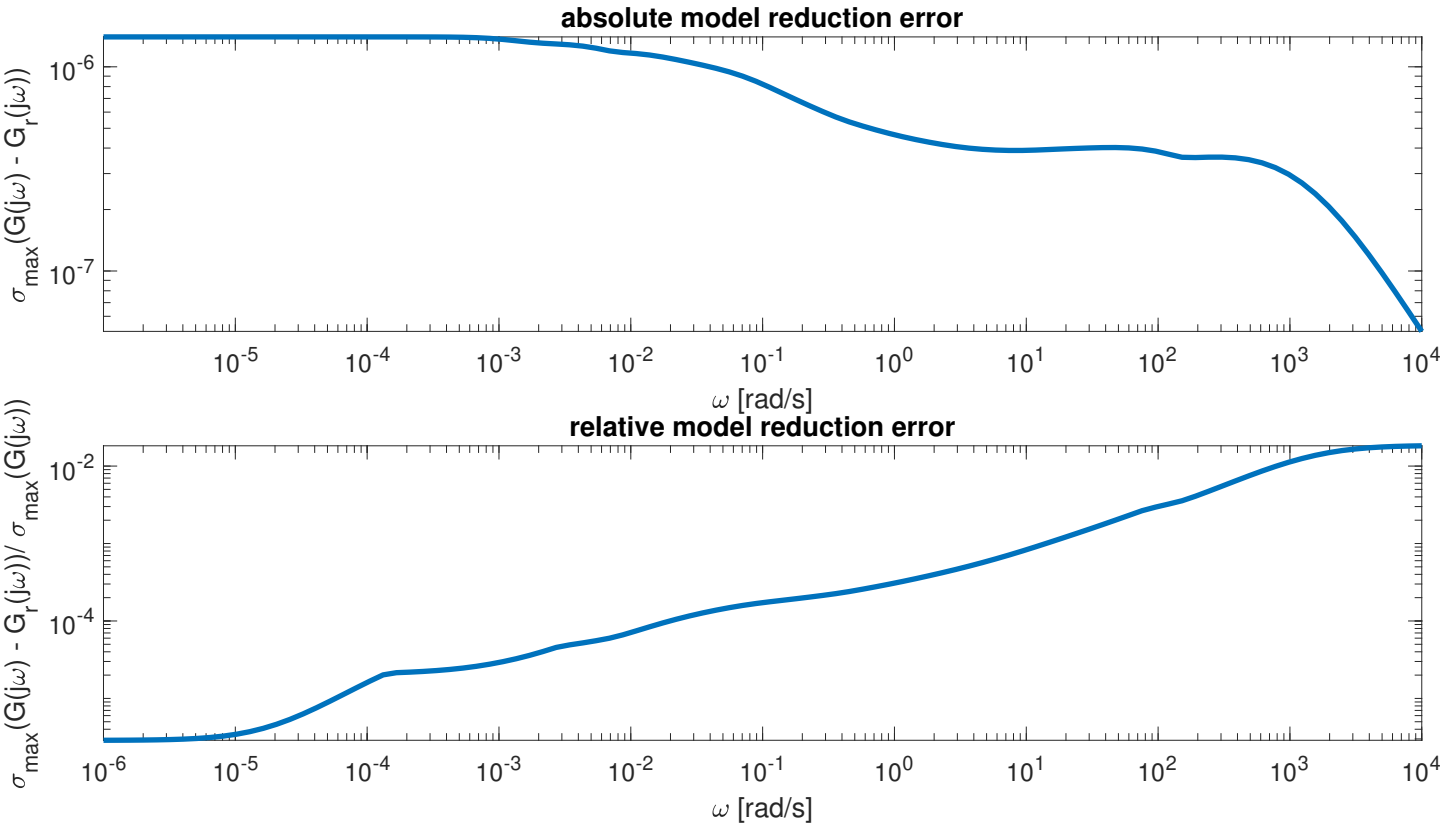

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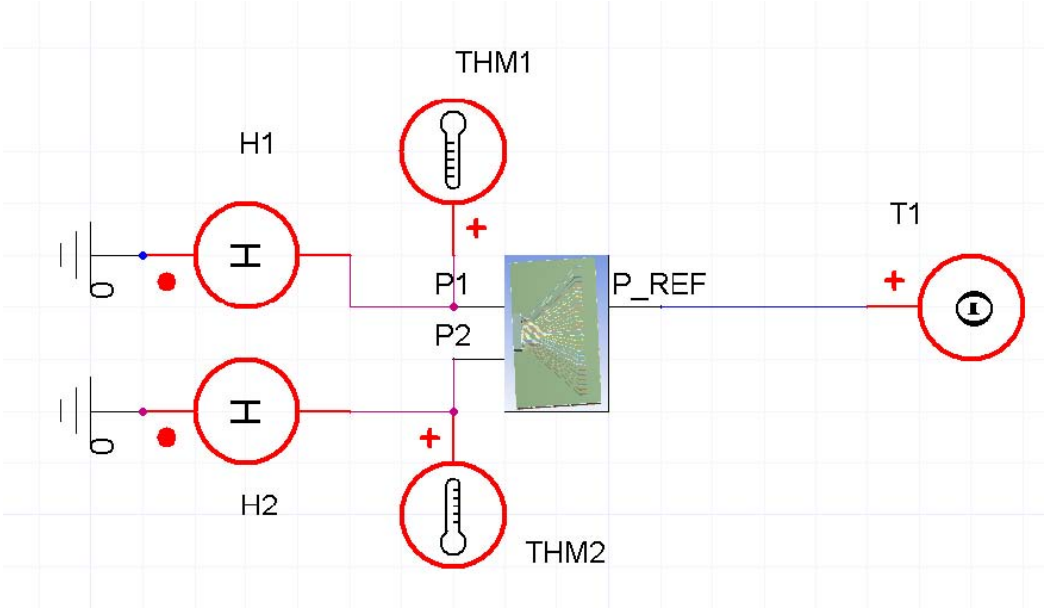
- SIMPLORER® test circuit with 2 transistors.



- Conservative thermic sub-system in SIMPLORER:
  voltage $\rightsquigarrow$ temperature, current $\rightsquigarrow$ heat flow.
- Original model: $n = 270.593$, $m = q = 2 \Rightarrow$
  Computing times (on Intel Xeon dualcore 3GHz, 1 Thread):
  - Main computational cost for set-up data $\approx 22min$.
  - Computation of reduced models from set-up data: $44 - -49sec$ ($r = 20$–70).
  - Bode plot (MATLAB on Intel Core i7, 2,67GHz, 12GB):
    7.5h for original system, $<$ 1min for reduced system.

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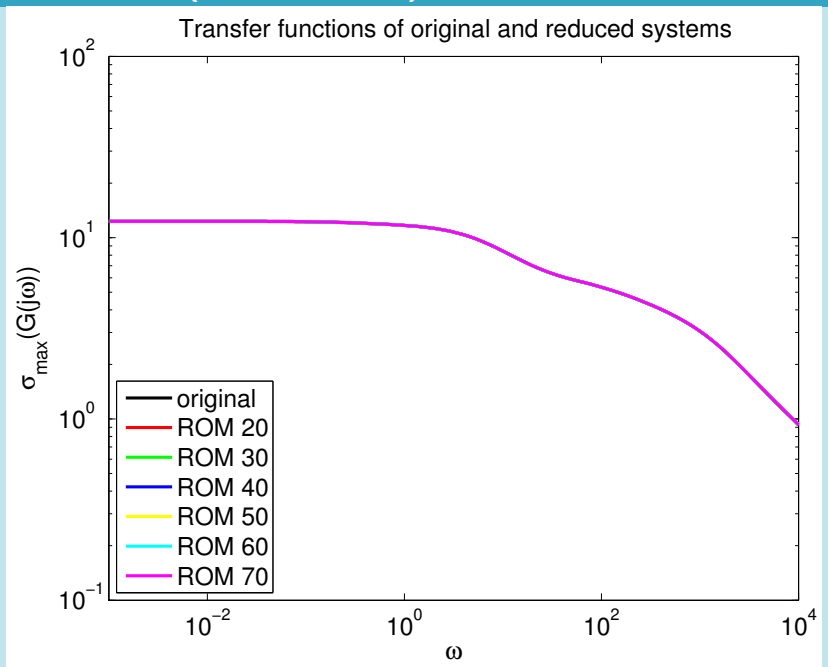
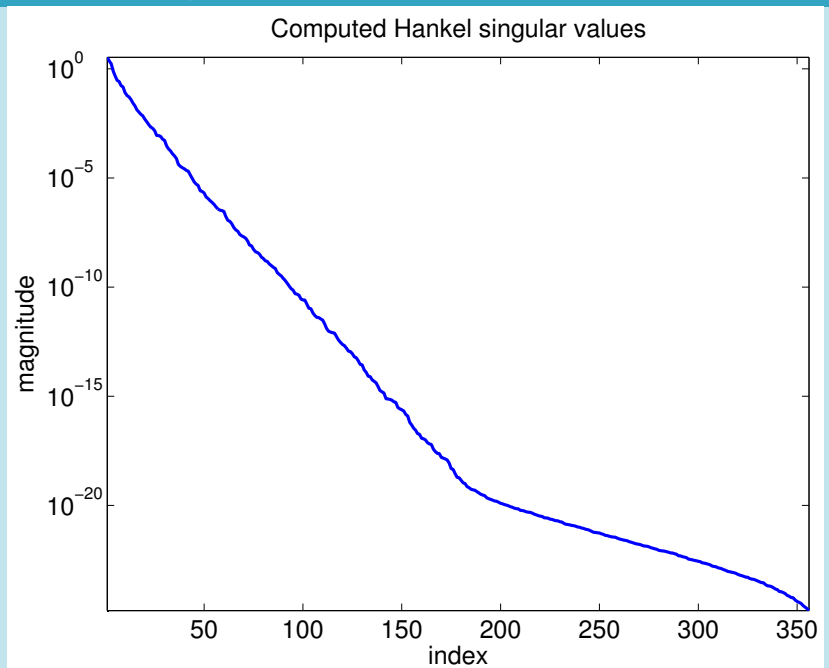
Bode Plot (Amplitude)



Hankel Singular Values



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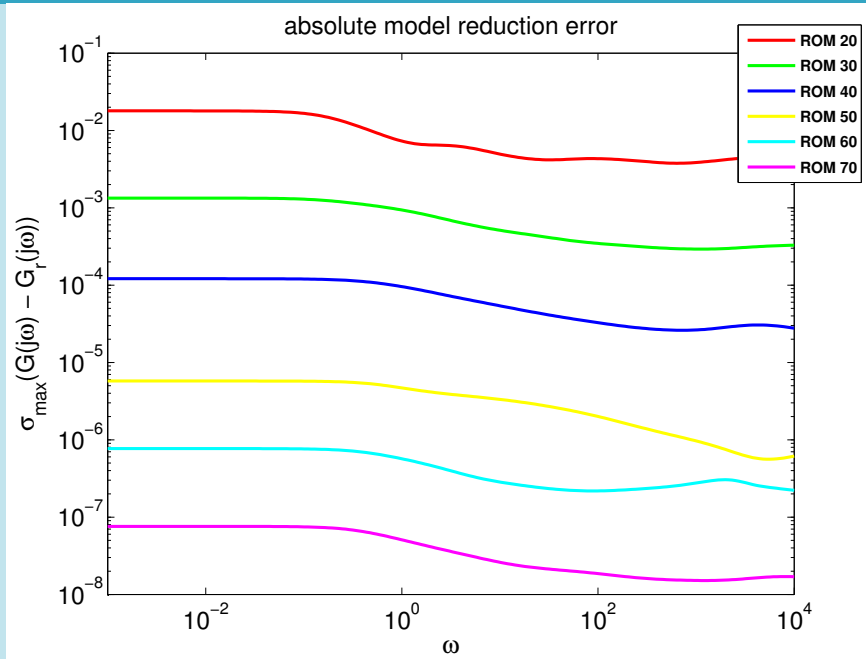
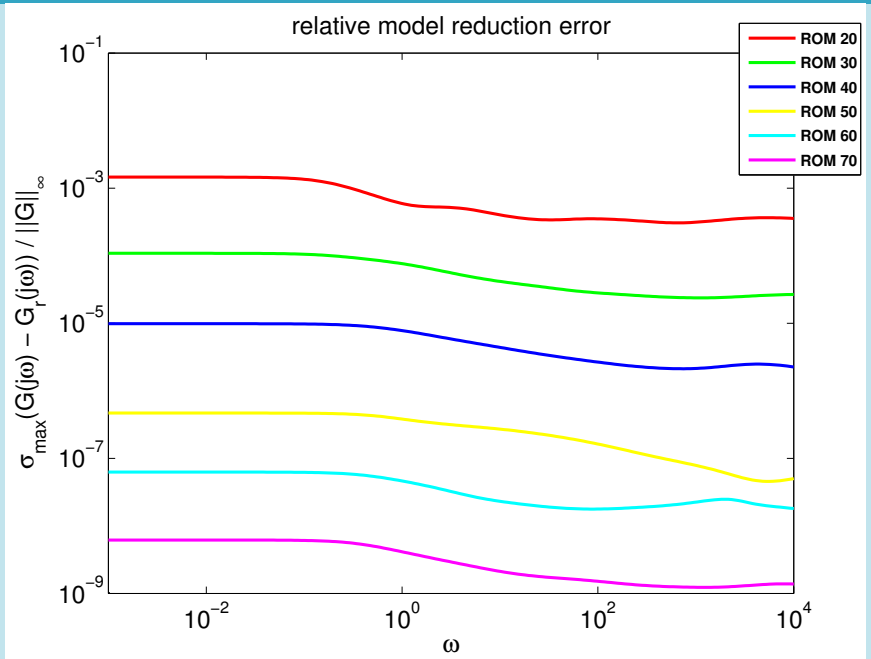
Absolute Error



Relative Error

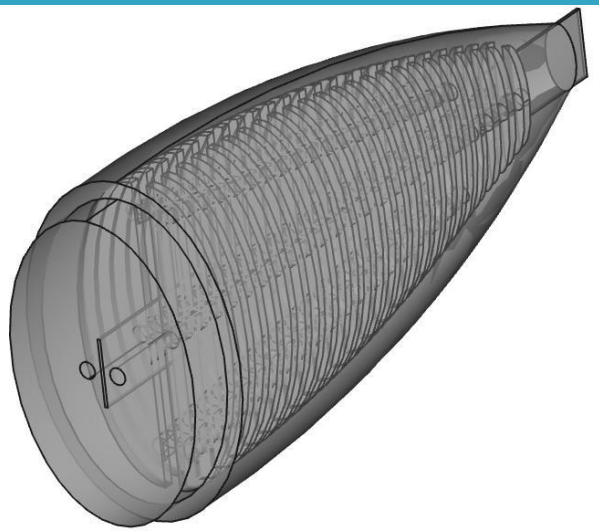


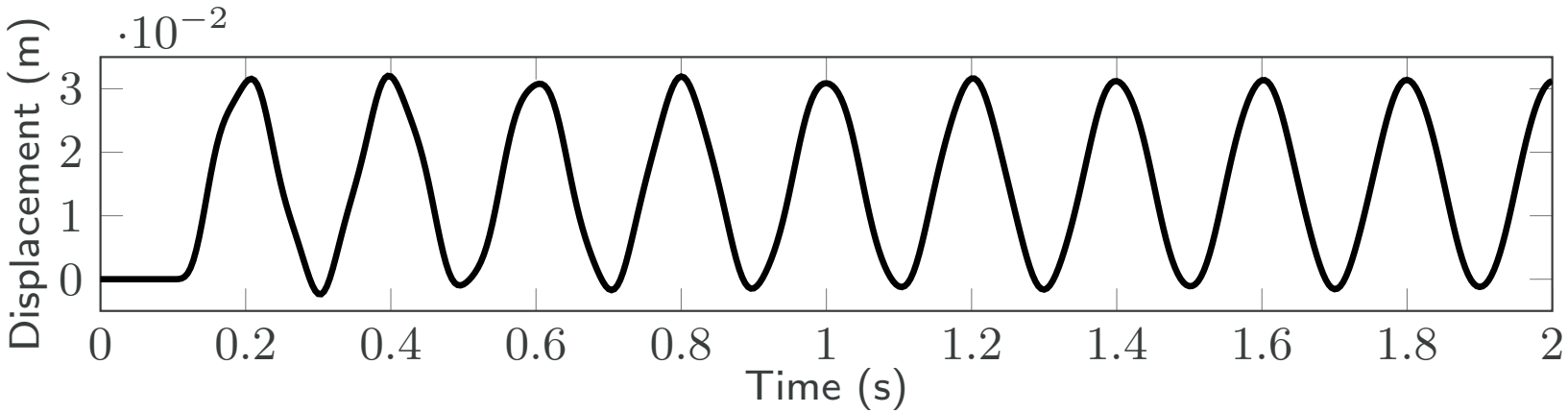
Numerical Examples

Artificial Fishtail Model [Siebelts/Saak/Werner 2019]

Original system:

- mechanical second-order structure
- $n = 779,232$, $m = 1$, $p = 3$
- test simulation time $\approx 4\,\mathrm{h}$





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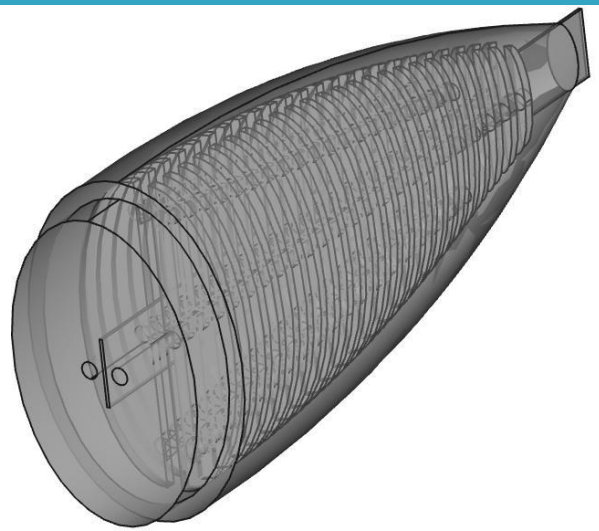

Original system:

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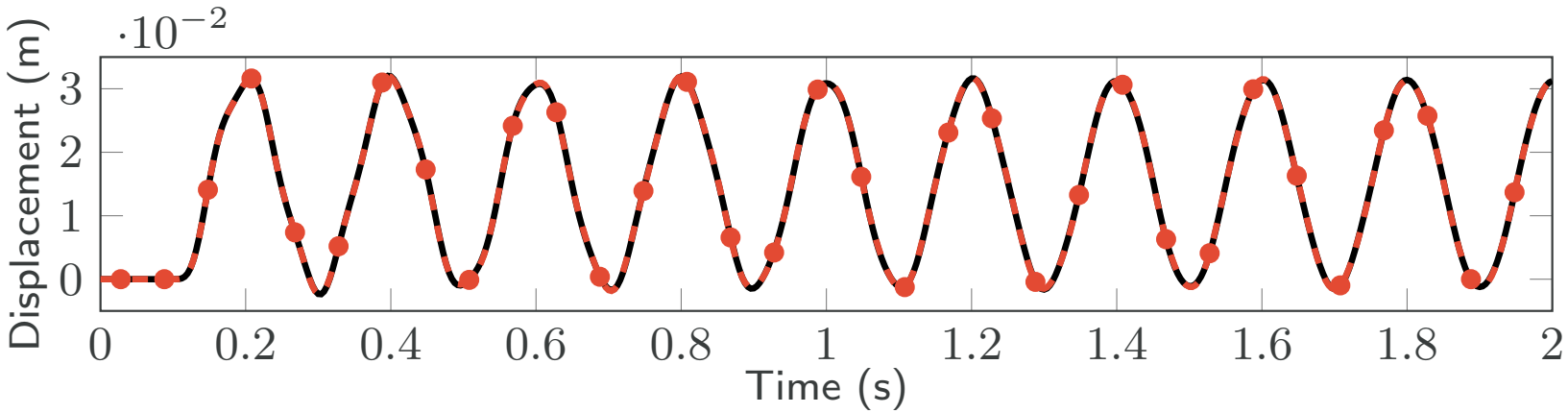
- $n = 779,232$, $m = 1$, $p = 3$
- test simulation time $\approx 4\,\mathrm{h}$

Second-Order Balanced Truncation:

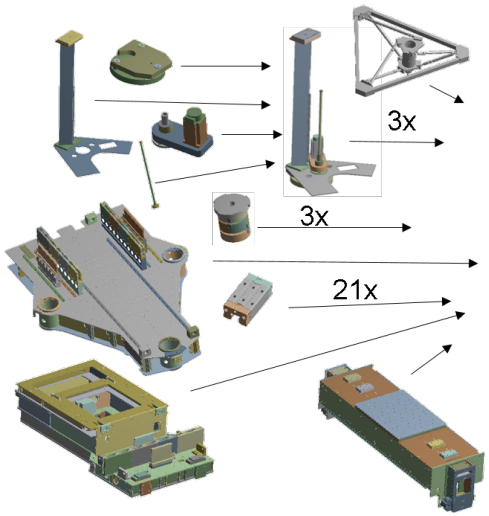
- two-step method with MORLAB backend
- $r = 1$, wall time $\approx 21\,\mathrm{h}$
- test simulation time $\approx 10\,\mathrm{msec}$ (speed up $\approx 1.4 \cdot 10^6$)



Numerical Examples
Thermal Model of Experimental Machine Tool MAX

[Vettermann et al 2021]

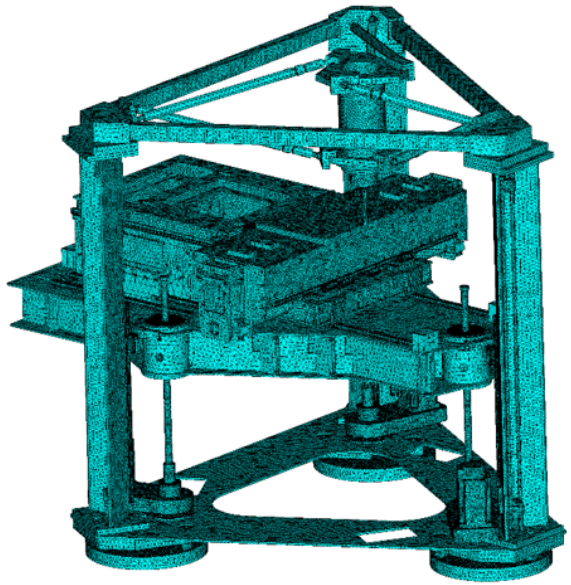
50 subassemblies



CAD model

$\overset{FEM}{\rightsquigarrow}$

FE-Model: **1.2M DOFs**



Industrial challenges for virtual twin:

- **non-homogeneous initial conditions** (IC) — two approaches: augment input with IC ("BTX0") or use superposition ("2phase"),
- subsystem reduction ("output coupled") vs. holistic reduction ("FE-coupled").

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Numerical Examples

MAX: Results considering an inhomogeneous initial condition $x_0 \neq 0$

FE-coupled

method	red. order tol 10^{-3}	t_{red}
2phase	196	6.5h
BTX0	174	4.5h

output-coupled

method	red. order tol 10^{-3}	t_{red}
2phase	3,005	2h
BTX0	2,515	1.8h

Numerical Examples

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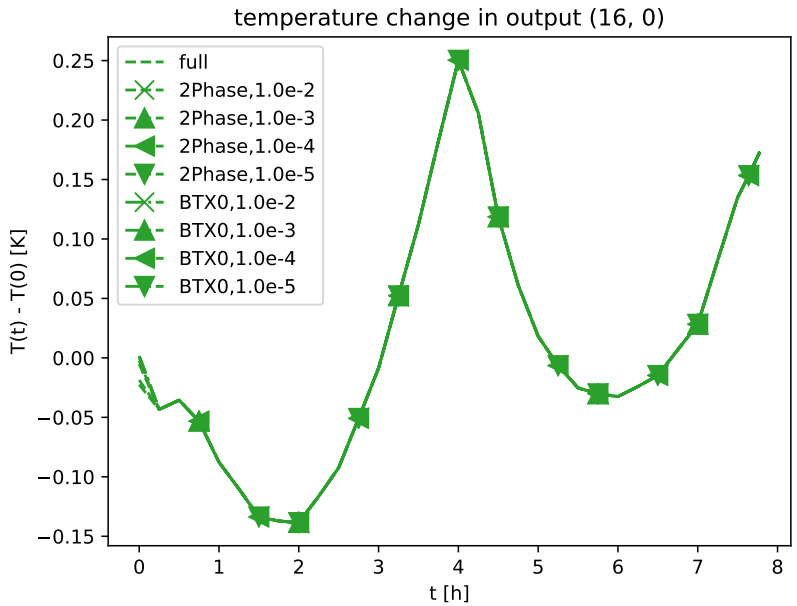
method	red. order tol 10^{-3}	t_{red}
2phase	3,005	2h
BTX0	2,515	1.8h

$\rightarrow$ Required storage for reduced matrices just 1MB!



Vettermann, J., Sauerzapf, S., Naumann, A., Beitelschmidt, M., Herzog, R., Benner, P., Saak, J. (2021): Model order reduction methods for coupled machine tool models. MM Science Journal 2021:4652-4659.

Vortex shedding

- Often encountered in industrial applications: system of the form

$$E\dot{x}(t) = Ax(t) + Bu(t)$$

with singular E $\rightsquigarrow$ linear differential-algebraic (descriptor) system

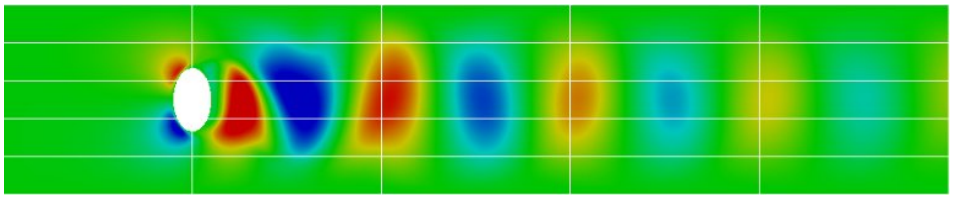
- Here: linearized Navier-Stokes equations for control design: index $= 2$, $n = 22,385$, $m = 2$, $p = 7$.

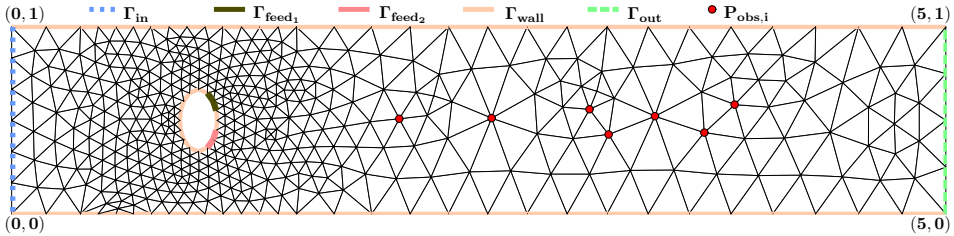
Balanced truncation approximation

- $r = 153$ for tolerance $\tau = 10^{-5}$
- hardware: Intel® Core™ i7-6700 with 16GB RAM
- software: M-M.E.S.S.-2.0.1 in MATLAB® R2018a
- wall time $\approx$ 2min.

M.E.S.S. demo example:

`bt_mor_DAE2('NSE',3,500)`





Model by Heiko K. Weichelt

Numerical Examples
von Kármán Vortex Shedding

[B./Saak/Uddin 2016]

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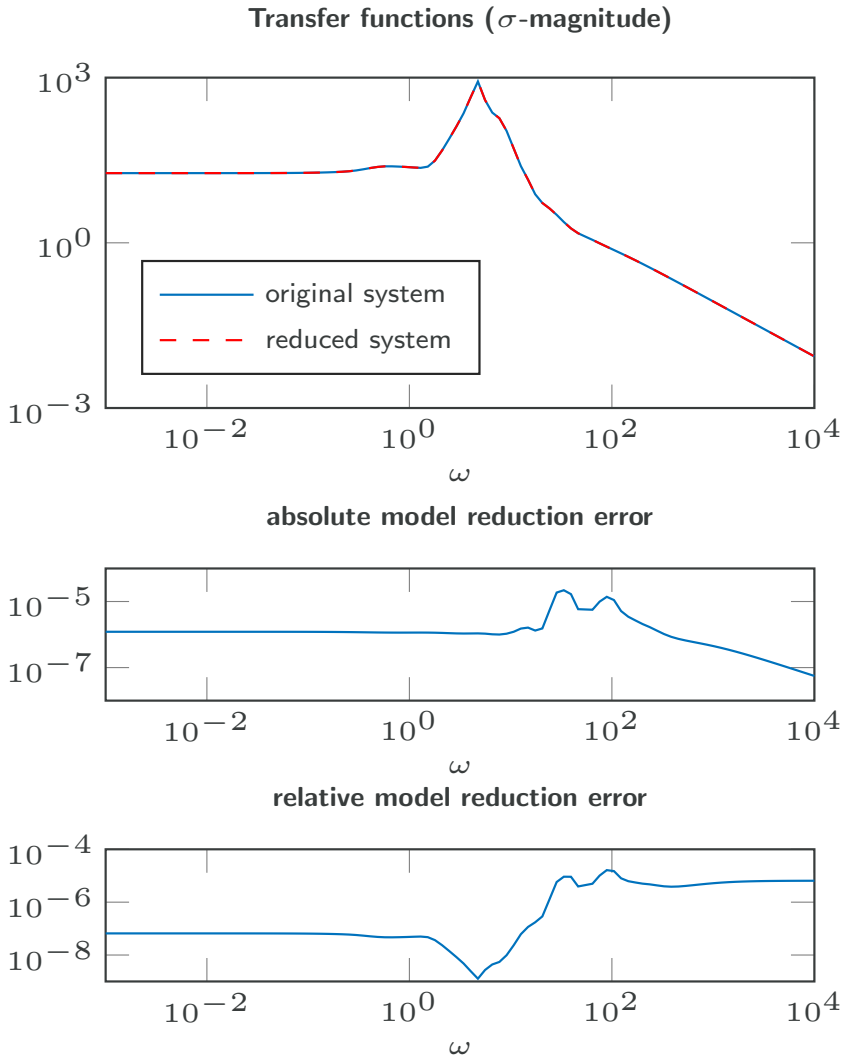

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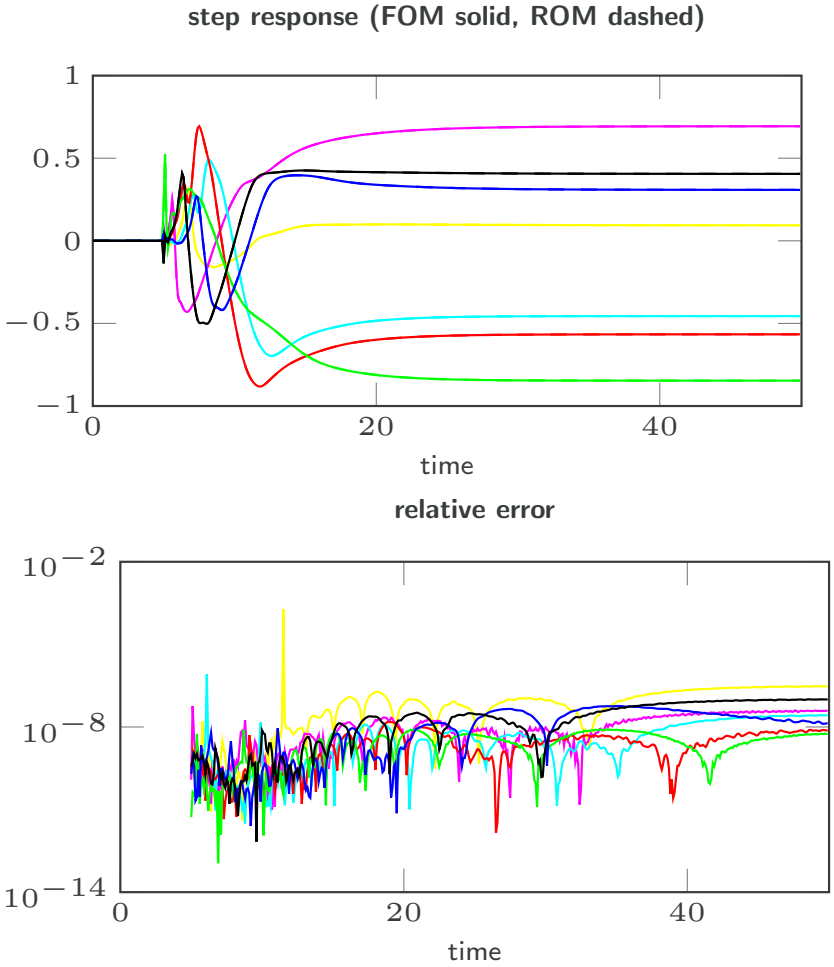

Outline

1. Introduction

2. Balanced Truncation for Linear Systems

3. Balanced Truncation for Polynomial Systems
   - Balanced Truncation for Nonlinear Systems
   - Polynomial Control Systems
   - Gramians for PC Systems
   - Truncated Gramians
   - Numerical Example

4. Conclusions

Balanced Truncation for Nonlinear Systems

Approaches

- Nonlinear balancing based on energy functionals [Scherpen 1993, Gray/Mesko 1996].

Definition [Scherpen 1993, Gray/Mesko 1996]

The reachability energy functional $L_c(x_0)$, and observability energy functional $L_o(x_0)$ of a system are given as:

$$L_c(x_0) = \inf_{\substack{u \in L_2(-\infty,0] \\ x(-\infty)=0,\ x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 dt, \qquad L_o(x_0) = \frac{1}{2}\int_{0}^{\infty} \|y(t)\|^2 dt.$$

Disadvantage: energy functionals are the solutions of nonlinear Hamilton-Jacobi equations which are hardly solvable for large-scale systems.

Note: For linear (LTI) systems,

$$L_c(x_0) = \frac{1}{2} x_0^T P^{-1} x_0, \qquad L_o(x_0) = \frac{1}{2} x_0^T Q x_0,$$

where P, Q are the controllability and observability Gramians, respectively!

Balanced Truncation for Nonlinear Systems

Approaches

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  Disadvantage: energy functionals are the solutions of nonlinear Hamilton-Jacobi equations which are hardly solvable for large-scale systems.
- Empirical Gramians/frequency-domain POD [Lall et al 1999, Willcox/Peraire 2002].

Example: controllability Gramian from time domain data (snapshots)

1. Define reachability Gramian of the system
$$P = \int_0^\infty x(t)x(t)^T dt, \quad \text{where } x(t) \text{ solves } \dot{x} = f(x, \delta),\ x(0) = x_0.$$
2. Use time-domain integrator to produce snapshots $x_k \approx x(t_k)$, $k = 1, \ldots, K$.
3. Approximate $P \approx \sum_{k=0}^{K} w_k x_k x_k^T$ with positive weights w_k.
4. Analogously for observability Gramian.
5. Compute balancing transformation and apply it to nonlinear system.

Disadvantage: Depends on chosen training input (e.g., $\delta(t_0)$) like other POD approaches. For recent developments on empirical Gramians, see [Himpe 2018].

Balanced Truncation for Nonlinear Systems
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  Disadvantage: Depends on chosen training input (e.g., $\delta(t_0)$) like other POD approaches.

  For recent developments on empirical Gramians, see [Himpe 2018].
- $\rightsquigarrow$ **Goal:** computationally efficient and input-independent method!

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- For bilinear systems, such local bounds were derived in [B./Damm 2011] using the solutions to the Lyapunov-plus-positive equations:

$$AP + PA^T + \sum_{i=1}^m A_i P A_i^T + BB^T = 0,$$
$$A^T Q + QA^T + \sum_{i=1}^m A_i^T Q A_i + C^T C = 0.$$

  (Note: if their solutions exist, they define reachability and observability Gramians of BIBO stable bilinear system.)

- **Here, we aim at determining algebraic Gramians for polynomial systems, which**
  - **provide bounds for the energy functionals of polynomial systems,**
  - **generalize the Gramians of linear and bilinear systems, and**
  - **allow us to find the states that are hard to reach as well as hard to observe in an efficient and reliable way.**

Now, consider the class of **polynomial control (PC) Systems**:

$$\dot{x}(t) = Ax(t) + \sum_{j=2}^{n_p} H_j \left(\otimes^j x(t)\right) + \sum_{j=2}^{n_p}\sum_{k=1}^{m} N_j^k \left(\otimes^j x(t)\right) u_k(t) + Bu(t),$$
$$y(t) = Cx(t), \quad x(0) = 0,$$

where

- n_p is the degree of the polynomial part of the system,
- $x(t) \in \mathbb{R}^n$, $\otimes^j x(t) = \underbrace{x(t) \otimes \cdots \otimes x(t)}_{j\text{-times}}$,
- $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$, $n \gg m, p$.
- $A \in \mathbb{R}^{n\times n}$, $H_j, N_j^k \in \mathbb{R}^{n\times n^j}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$.
- Assumption: A is supposed to be Hurwitz $\Rightarrow$ local stability.

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Examples: FitzHugh-Nagumo and Chafee-Infante equations lead to cubic control systems; cubic-quintic Allen-Cahn equation to quintic control system.

- Consider input $\to$ state map of (for simplicity) quadratic-bilinear system ($n_P = 2$, $m = 1,\ N \equiv A_1$):

$$\dot{x}(t) = Ax(t) + Hx(t) \otimes x(t) + Nx(t)u(t) + Bu(t), \qquad x(0) = 0.$$

- Integration yields

$$x(t) = \int_0^t e^{A\sigma_1} Bu(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Nx(t-\sigma_1)u(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Hx(t-\sigma_1) \otimes x(t-\sigma_1)d\sigma_1$$

[Rugh 1981]

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[Rugh 1981]

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- By iteratively inserting expressions for $x(t-\bullet)$, we obtain the **Volterra series expansion** for quadratic-bilinear (and, more general, polynomial) systems. [Rugh 1981]

Expanding the response of the PC system into a Volterra series representation and using the idea of iterated linear systems, we define the reachability Gramian as

$$P = \sum_{k=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \bar{P}_k(t_1, \ldots, t_k) \bar{P}_k(t_1, \ldots, t_k)^T dt_1 \ldots dt_k,$$

where

$$\bar{P}_1(t_1) = e^{At_1} B, \qquad \bar{P}_2(t_1, t_2) = \sum_{k=1}^{m} e^{At_1} N_1^k e^{At_2} B,$$

$$\bar{P}_3(t_1, t_2, t_3) = e^{At_1} H_2 e^{At_2} B \otimes e^{At_3} B, \quad \ldots$$

are the kernels of the Volterra series expansion of the system output.

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are the kernels of the Volterra series expansion of the system output.

Theorem [B./Goyal/Pontes Duff 2018]

The reachability Gramian P of a PC system solves the polynomial Lyapunov equation

$$AP + PA^T + BB^T + \sum_{j=2}^{n_p} H_j \left(\otimes^j P\right) H_j^T + \sum_{j=2}^{n_p} \sum_{k=1}^{m} N_j^k \left(\otimes^j P\right) \left(N_j^k\right)^T = 0.$$

The Observability Gramian is defined as follows:

- First, we write the adjoint system as [Fujimoto et al. 2002]

$$\dot{x}(t) = Ax(t) + \sum_{j=2}^{n_p} H_j x_j^{\otimes}(t) + \sum_{j=1}^{n_p} \sum_{k=1}^{m} N_j^k x_j^{\otimes}(t) u_k(t) + Bu(t),$$

$$\dot{x}_d(t) = -A^T x_d(t) - \sum_{j=2}^{n_p} H_j^{(2)} x_{d,j}^{\otimes}(t) - \sum_{j=1}^{n_p} \sum_{k=1}^{m} \left(N_j^{k,(2)}\right) x_{d,j}^{\otimes}(t) u_{d,k}(t) - C^T u_d(t), \quad x_d(\infty) = 0,$$

$$y_d(t) = B^T x_d(t).$$

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$$y_d(t) = B^T x_d(t).$$

- Then, by taking the kernel of Volterra series, one has

Theorem [B./Goyal/Pontes Duff 2018]

Let P be the reachability Gramian. Then, the observability Gramian Q of the PC system solves the polynomial Lyapunov equation

$$A^T Q + QA + C^T C + \sum_{j=2}^{n_p} H_j^{(2)} \left(\otimes^{j-1} P \otimes Q\right) \left(H_j^{(2)}\right)^T + \sum_{j=2}^{n_p}\sum_{k=1}^{m} N_j^{k,(2)} \left(\otimes^{j-1} P \otimes Q\right) \left(N_j^{k,(2)}\right)^T = 0.$$

Truncated Gramians

- Polynomial Lyapunov equations are very expensive to solve, efficient algorithms have not yet been developed.

- We thus propose truncated Gramians that only involve a finite number of kernels and can be computed using the methods in MORLAB or M-M.E.S.S.:

$$P_{\mathcal{T}} = \sum_{k=1}^{n_p+1} \int_0^\infty \cdots \int_0^\infty \bar{P}_k(t_1,\ldots,t_k)\bar{P}_k(t_1,\ldots,t_k)^T dt_1 \ldots dt_k.$$

Truncated Gramians

The truncated reachability Gramian solves

$$AP_{\mathcal{T}} + P_{\mathcal{T}}A^T + BB^T + \sum_{j=2}^{n_p} H_j \otimes^j P_l H_j^T + \sum_{j=2}^{n_p}\sum_{k=1}^{m} N_j^k \otimes^j P_l \left(N_j^k\right)^T = 0.$$

where $AP_l + P_l A^T + BB^T = 0$

- Advantage: Only need to solve a finite number of (linear) Lyapunov equations.

Numerical Example
The FitzHugh-Nagumo Model, Revisited

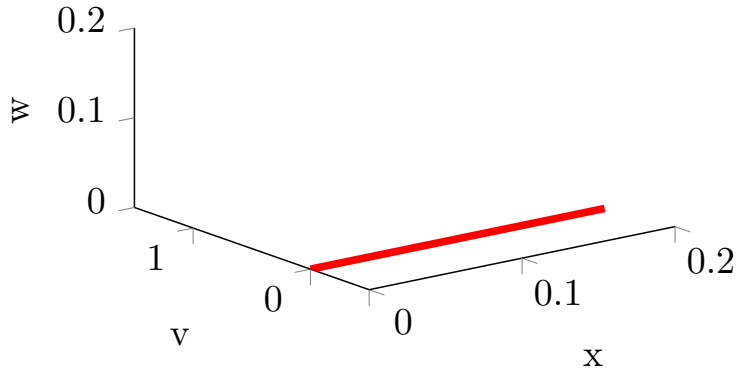
$$\epsilon v_t(x,t) = \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + q,$$
$$w_t(x,t) = hv(x,t) - \gamma w(x,t) + q,$$

with a nonlinear function

$$f(v(x,t)) = v(v-0.1)(1-v),$$

$\epsilon = 0.015$, $h = 0.5$, $\gamma = 2$, $q = 0.05$, $L = 0.2$,
and boundary conditions:

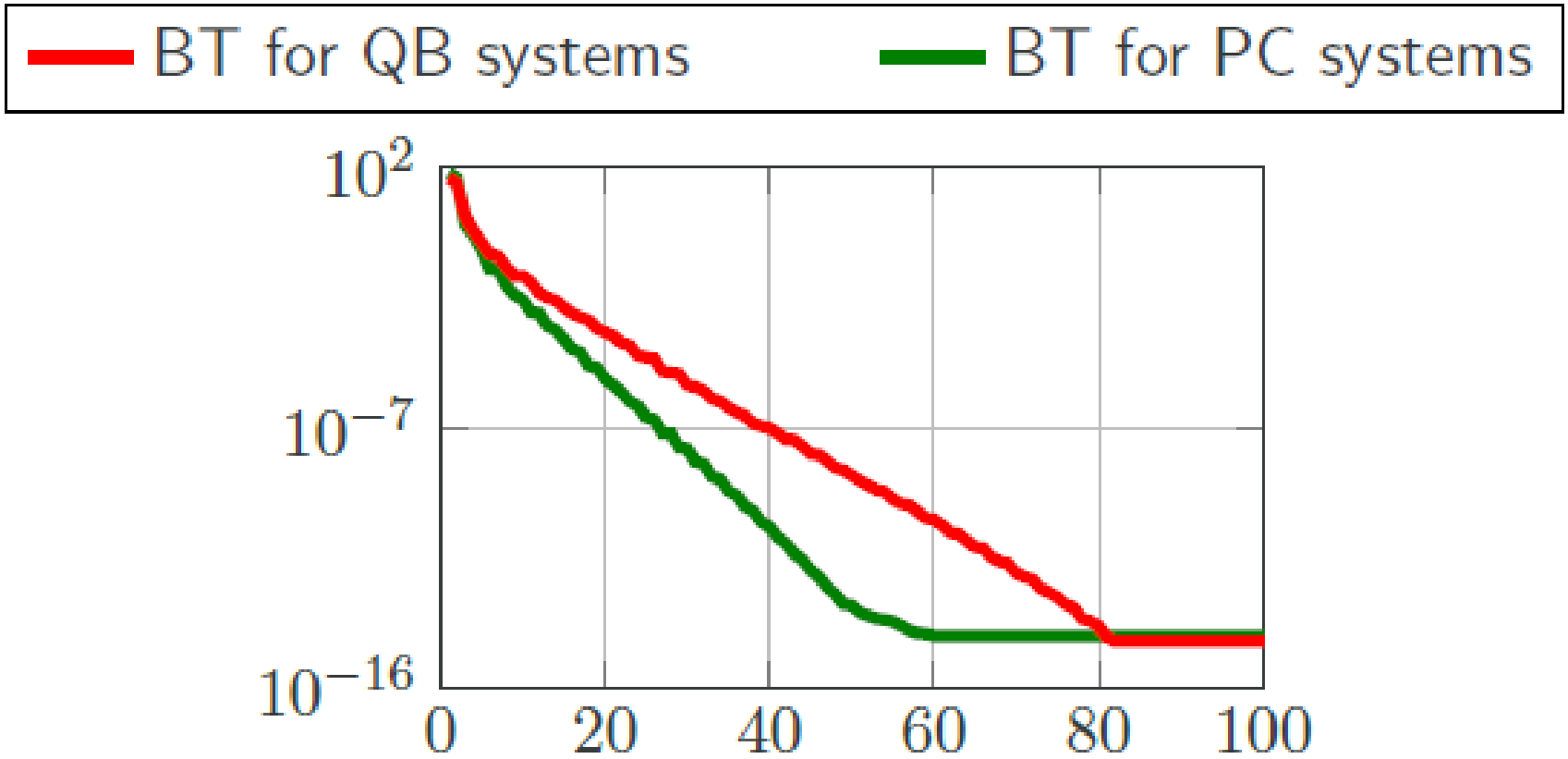
$$v_x(0,t) = i_0(t), \quad v_x(L,t) = 0, \quad t \geq 0,$$



- Spatial discretization leads to PC system with cubic nonlinearity of order $n_{pc} = 600$.
- Lifting: $z := v^2 \;\Rightarrow\; f(v,z) = -vz + 1.1z - 0.1v, \quad z_t = 2vv_t = \ldots \;\rightsquigarrow$
- lifted quadratic-bilinear (QB) system of order $n_{qb} = 900$.
- Outputs of interest $v(0,t), w(0,t)$ are the responses at the left boundary.
- We compare balanced truncation for PC and QB systems.

Numerical Example

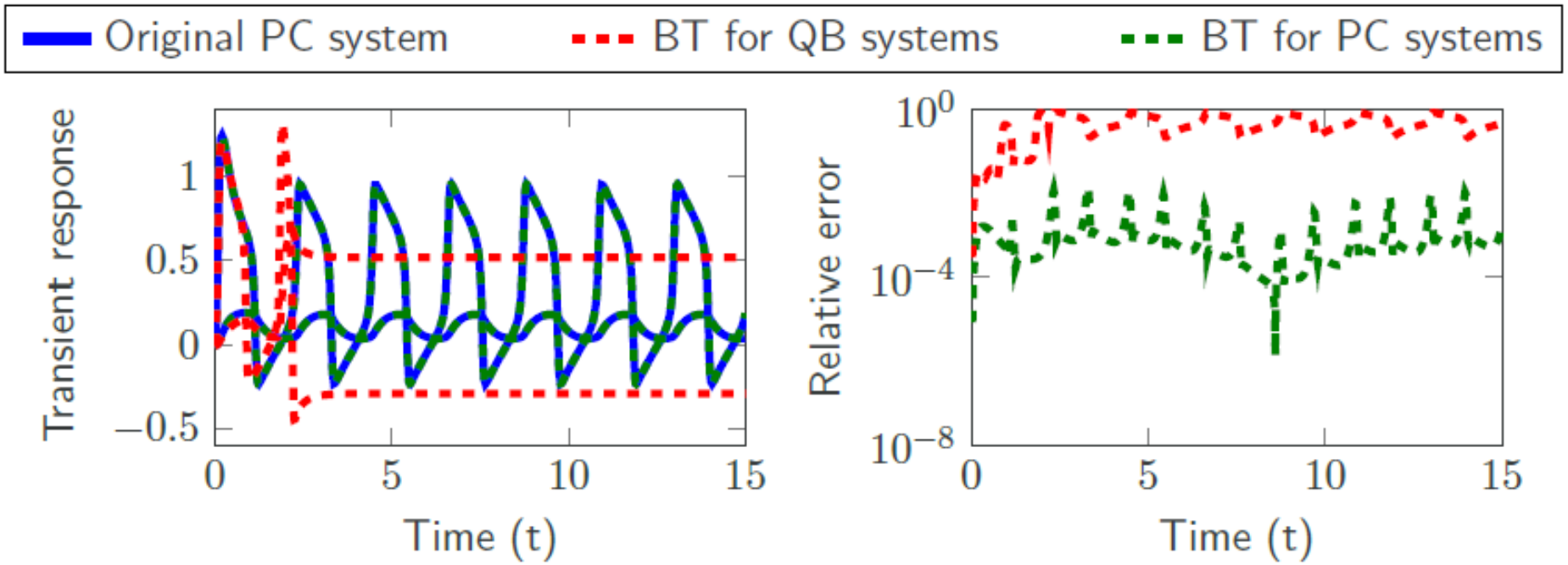
The FitzHugh-Nagumo Model: Singular Value Decay



- Decay of singular values for PC systems is faster $\Rightarrow$ smaller reduced-order model!

Numerical Example

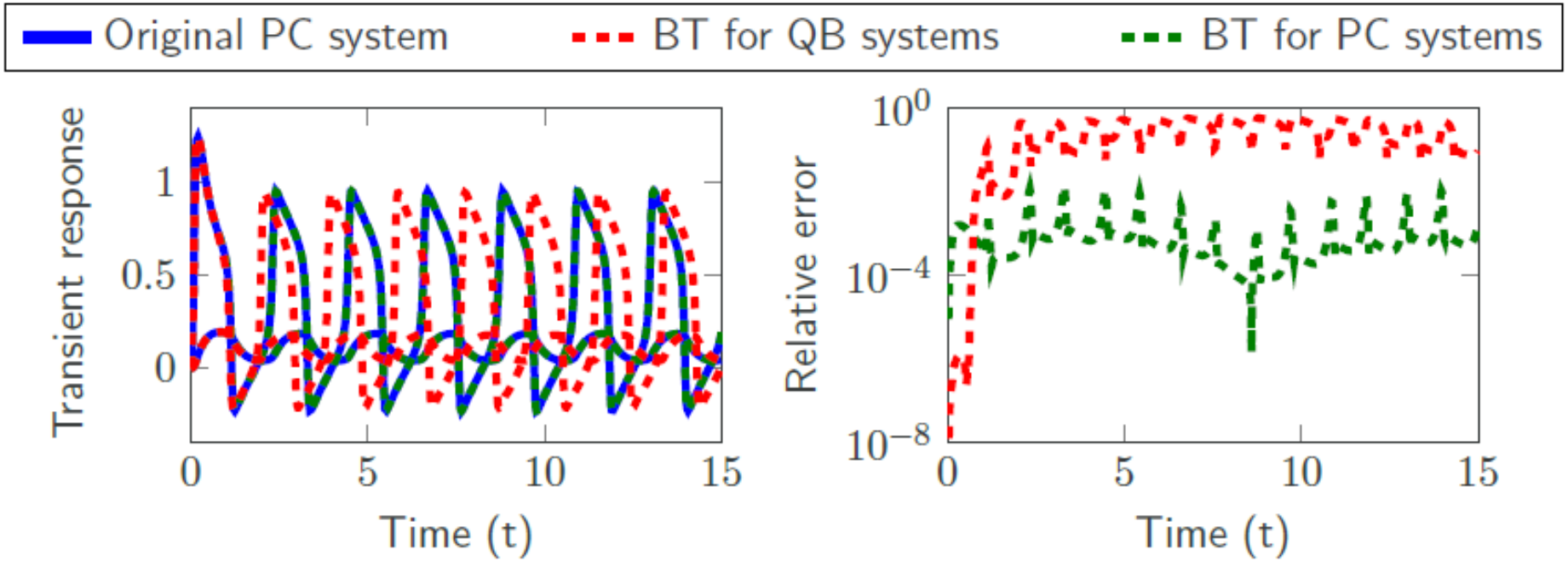
The FitzHugh-Nagumo Model: Time-domain Simulation



- Original PC system of order 600. Original QB system of order 900.
- Reduced PC system of order 10. Reduced QB system of order 10.

Numerical Example

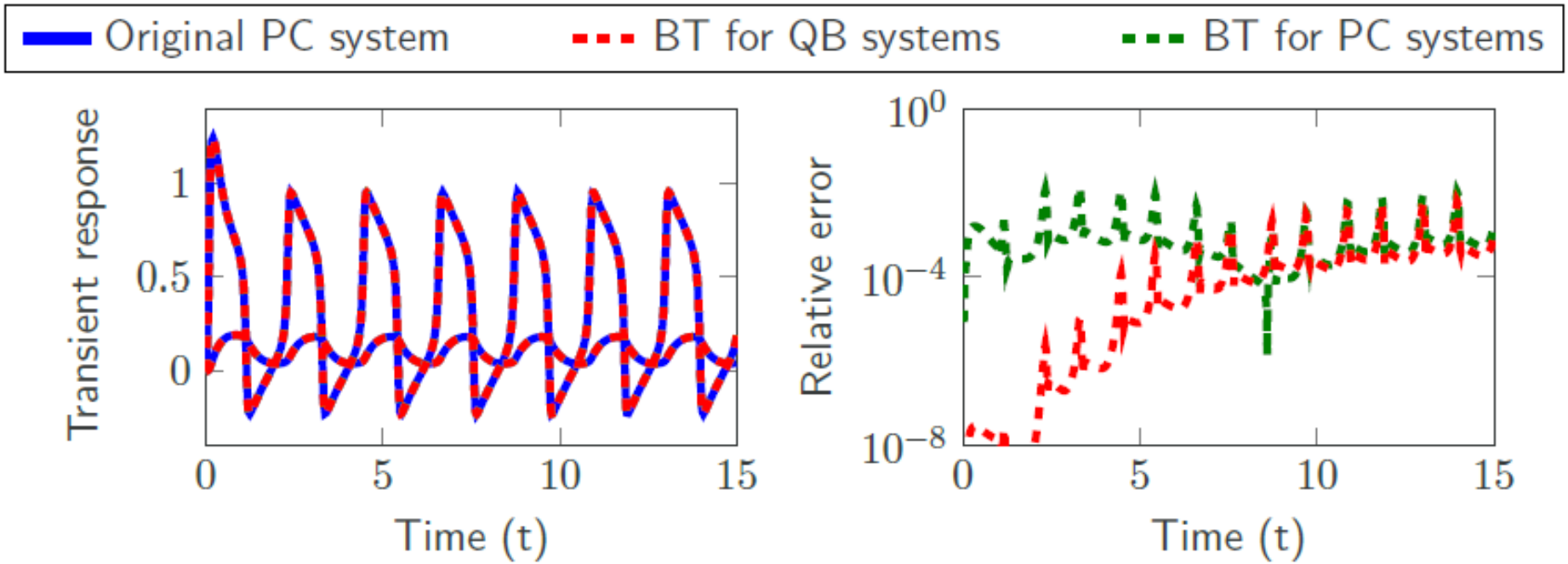
The FitzHugh-Nagumo Model: Time-domain Simulation



- Original PC system of order 600. Original QB system of order 900.
- Reduced PC system of order 10. Reduced QB system of order 30.

Numerical Example

The FitzHugh-Nagumo Model: Time-domain Simulation



- Original PC system of order 600. Original QB system of order 900.
- Reduced PC system of order 10. Reduced QB system of order 43.

Conclusions

BT for linear systems:

- Method of choice for model order reduction in optimal and feedback control.
- Computational efficiency enhanced through advanced techniques from numerical linear algebra so that problems of industrial scale can be reduced.
- Numerous technical details necessary for use in digital and virtual twins.
- Robust and efficient implementations available in numerous software packages like SLICOT, M.E.S.S., MORLAB, pyMOR, ...

Conclusions

BT for linear systems:

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BT for nonlinear systems:

- BT extended to bilinear, quadratic-bilinear, and polynomial systems.
- Not in this talk: local Lyapunov stability is preserved.
- As of yet, only weak motivation by local bounds of energy functionals.
- No error bounds in terms of "Hankel" singular values.
- Computationally efficient (as compared to nonlinear balancing), and input independent.
- **To do:**
  - improve efficiency of Lyapunov solvers with many right-hand sides further;
  - error bound;
  - conditions for existence of new PC Gramians;
  - extension to descriptor systems.

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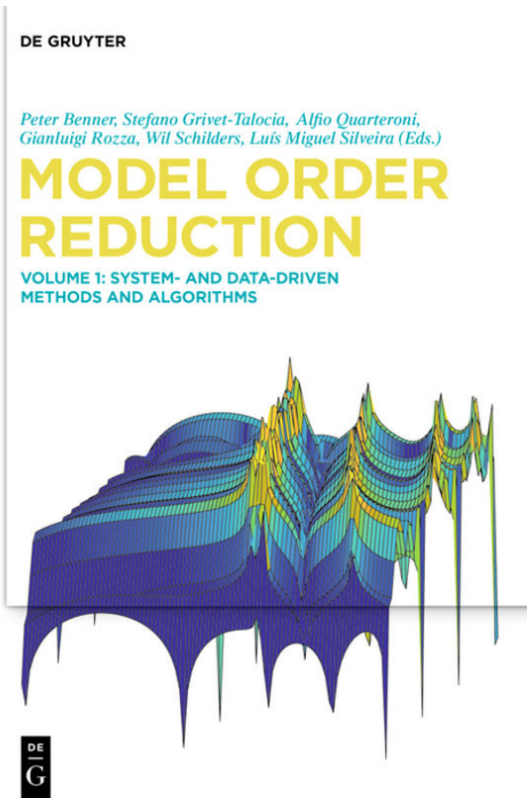
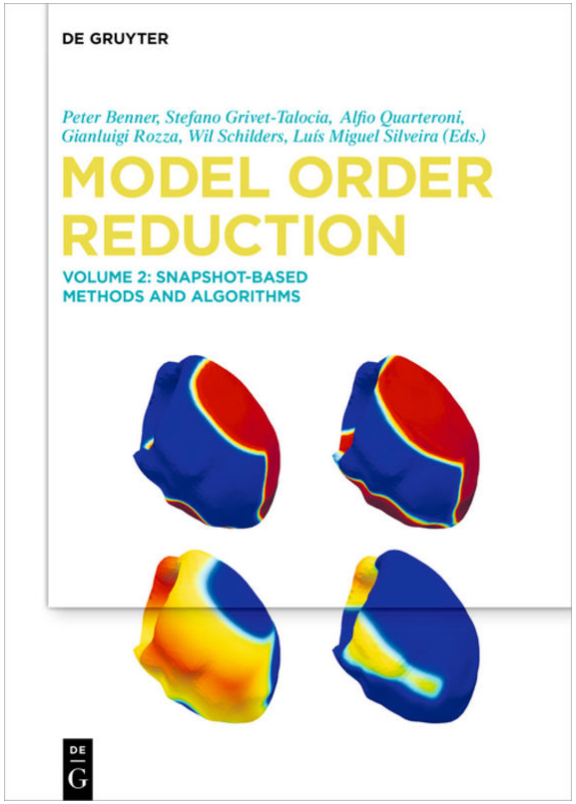
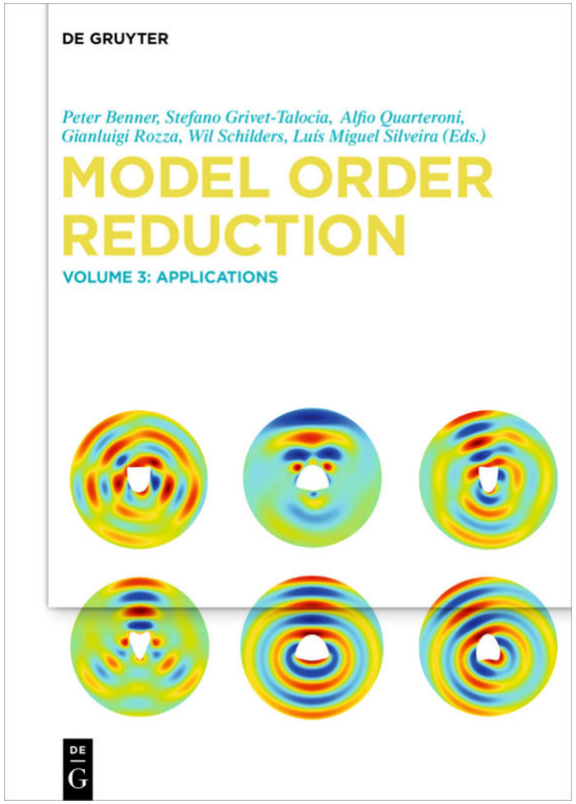
Commercial: 3-Volume Handbook "Model Order Reduction"







- Edited by Peter Benner, Stefano Grivet-Talocia, Alfio Quarteroni, Gianluigi Rozza, Wil Schilders, and Luís Miguel Silveira,
- contains 30 tutorial chapters on modern model reduction techniques, methods, applications, and software,
- published by DeGruyter in 2021, ebook is fully Open Access!