MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Model reduction of descriptor systems with quadratic output functionals

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Problem setting

DAE with quadratic output

$$E\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = x(t)^{\mathrm{T}} M x(t)$$

with $E,\ A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $M \in \mathbb{R}^{n\times n}$, where E is singular and $M = M^{\mathrm{T}}$

- Aim: find surrogate model that
  - approximates the input-to-output behavior,
  - has the same structure,
  - is of smaller dimension.
- Method: apply balanced truncation.

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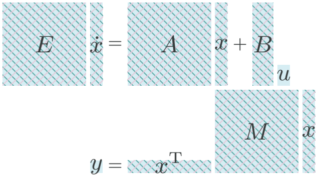

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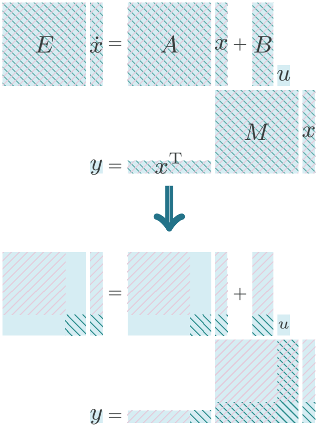

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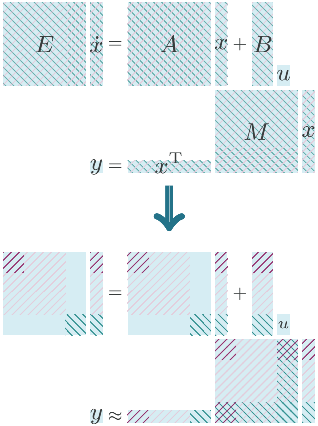

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Table of contents

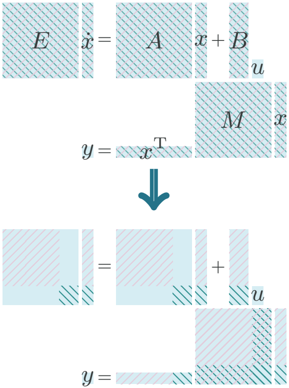
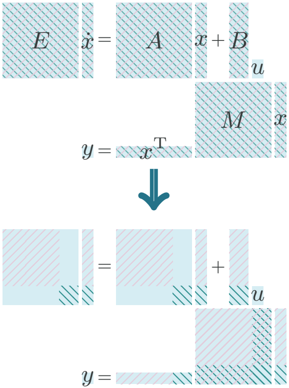


Table of contents



Weierstraß canonical form

DAE with quadratic output

$$E\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = x(t)^{\mathrm{T}} M x(t).$$

- There exist W, T nonsingular[1] such that

$$E = W \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} T, \quad A = W \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} T, \quad Tx(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

with

  - n_f number of finite eigenvalues of (E, A),
  - n_∞ number of infinite eigenvalues of (E, A),
  - J nonsingular, N nilpotent with nilpotency index ν.

[1] P. Kunkel; V. Mehrmann, Differential-Algebraic Equations: Analysis and Numerical Solution, EMS Publishing House, 2006.

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Transformed system

$$\begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + W^{-1}Bu(t),$$

$$y(t) = \begin{bmatrix} x_1(t)^{\mathrm{T}} & x_2(t)^{\mathrm{T}} \end{bmatrix} T^{-\mathrm{T}} M T^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Weierstraß canonical form

Transformed state equation

$$E\dot{x}(t) = Ax(t) + Bu(t) \quad \Rightarrow \quad \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + W^{-1}Bu(t)$$

- Decompose state $x(t) = x_p(t) + x_i(t) = T^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.
- Proper state

$$x_p(t) = T^{-1} \begin{bmatrix} x_1(t) \\ 0 \end{bmatrix} = \int_0^t T^{-1} \begin{bmatrix} e^{J(t-\tau)} & 0 \\ 0 & 0 \end{bmatrix} W^{-1}Bu(\tau)\mathrm{d}\tau$$

- Improper state

$$x_i(t) = T^{-1} \begin{bmatrix} 0 \\ x_2(t) \end{bmatrix} = \sum_{k=0}^{\nu-1} T^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -N^k \end{bmatrix} W^{-1}Bu^{(k)}(t)$$

Weierstraß canonical form

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$$E\dot{x}(t) = Ax(t) + Bu(t) \qquad \Rightarrow \qquad \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + W^{-1}Bu(t)$$

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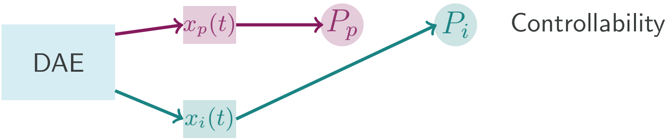
Weierstraß canonical form

DAE with quadratic output

$$E\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = x(t)^{\mathrm{T}} M x(t).$$

- Consider proper and improper state $x(t) = x_p(t) + x_i(t)$.



Weierstraß canonical form

DAE with quadratic output

$$E\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = x(t)^{\mathrm{T}} M x(t).$$

- Consider proper and improper state $x(t) = x_p(t) + x_i(t)$.
- Output $y(t) = x_p(t)^{\mathrm{T}} M x_p(t) + x_p(t)^{\mathrm{T}} M x_i(t) + x_i(t)^{\mathrm{T}} M x_p(t) + x_i(t)^{\mathrm{T}} M x_i(t)$.

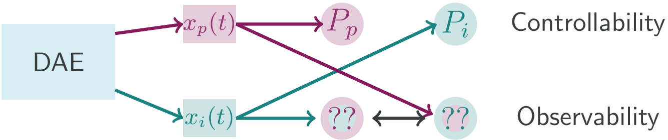
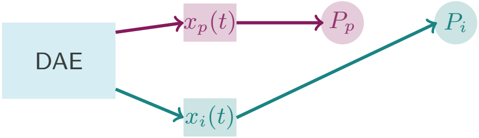


Table of contents



Controllability

Proper state

- Consider the proper state

$$x_p(t) = \int_0^t F_J(t-\tau)Bu(\tau)\mathrm{d}\tau.$$

Improper state

- Consider the improper state

$$x_i(t) = \sum_{k=0}^{\nu-1} F_N(k)Bu^{(k)}(t).$$

Controllability

Proper state

- Consider the proper state

$$x_p(t) = \int_0^t F_J(t-\tau)Bu(\tau)\mathrm{d}\tau.$$

- Define proper controllability mapping

$$\mathcal{C}_p(t) = F_J(t)B.$$

Improper state

- Consider the improper state

$$x_i(t) = \sum_{k=0}^{\nu-1} F_N(k)Bu^{(k)}(t).$$

- Define improper controllability mapping

$$\mathcal{C}_i(k) = F_N(k)B.$$

Controllability

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$$x_p(t) = \int_0^t F_J(t-\tau)Bu(\tau)\mathrm{d}\tau.$$

- Define proper controllability mapping

$$\mathcal{C}_p(t) = F_J(t)B.$$

- Define corresponding Gramian

$$P_p = \int_0^\infty \mathcal{C}_p(t)\mathcal{C}_p(t)^{\mathrm{T}}\mathrm{d}t$$
$$= \int_0^\infty F_J(t)BB^{\mathrm{T}}F_J(t)^{\mathrm{T}}\mathrm{d}t.$$

Improper state

- Consider the improper state

$$x_i(t) = \sum_{k=0}^{\nu-1} F_N(k)Bu^{(k)}(t).$$

- Define improper controllability mapping

$$\mathcal{C}_i(k) = F_N(k)B.$$

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Controllability

Controllability Gramians:

$$P_p = \int_0^\infty F_J(t) BB^{\mathrm{T}} F_J(t)^{\mathrm{T}} \mathrm{d}t, \qquad P_i = \sum_{k=0}^{\nu-1} F_N(k) BB^{\mathrm{T}} F_N(k)^{\mathrm{T}}$$

Theorem [2]

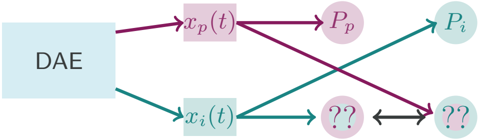
The controllability Gramians P_p and P_i solve the projected Lyapunov equations

$$\begin{aligned} EP_pA^{\mathrm{T}} + AP_pE^{\mathrm{T}} &= -P_l BB^{\mathrm{T}} P_l^{\mathrm{T}}, & P_p &= P_r P_p P_r^{\mathrm{T}}, \\ AP_iA^{\mathrm{T}} - EP_iE^{\mathrm{T}} &= (I - P_l) BB^{\mathrm{T}} (I - P_l)^{\mathrm{T}}, & 0 &= P_r P_p P_r^{\mathrm{T}} \end{aligned}$$

where $P_l = W \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} W^{-1}$, $P_r = T^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} T$ are projections.

[2] T. Stykel, Gramian-based model reduction for descriptor systems, Math. Control Signals Systems, 16(4):297-319, 2004.

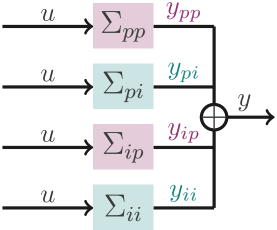
Table of contents



Superposition

Original system

Decomposed system



- Original and decomposed system result in the same output:

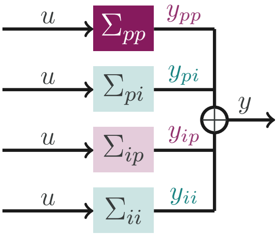
$$y(t) = \underbrace{x_p(t)^{\mathrm{T}} M x_p(t)}_{=: y_{pp}(t)} + \underbrace{x_p(t)^{\mathrm{T}} M x_i(t)}_{=: y_{pi}(t)} + \underbrace{x_i(t)^{\mathrm{T}} M x_p(t)}_{=: y_{ip}(t)} + \underbrace{x_i(t)^{\mathrm{T}} M x_i(t)}_{=: y_{ii}(t)}.$$

- We consider the observability of the right states under consideration of the left state.

Superposition

Original system

Decomposed system



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- We consider the observability of the right states under consideration of the left state.

Proper-proper Observability

DAE system with proper-proper output

$$x_p(t) = \int_0^t F_J(t) B u(\tau) \mathrm{d}\tau, \qquad y_{pp}(t) = x_p(t)^{\mathrm{T}} M x_p(t)$$

- Investigate the output

$$y_{pp}(t) = x_p(t)^{\mathrm{T}} M x_p(t) = \int_0^t \int_0^t u(\tau_1)^{\mathrm{T}} B^{\mathrm{T}} F_J(t-\tau_1)^{\mathrm{T}} M F_J(t-\tau_2) B u(\tau_2) \mathrm{d}\tau_1 \mathrm{d}\tau_2$$

$$= \int_0^t \int_0^t \operatorname{vec}\left(B^{\mathrm{T}} F_J(t-\tau_1)^{\mathrm{T}} M F_J(t-\tau_2) B\right) (u(\tau_2) \otimes u(\tau_1)) \mathrm{d}\tau_1 \mathrm{d}\tau_2$$

- We recognize $\mathcal{C}_p(t-\tau_2) = F_J(t-\tau_2)B$.
- The remaining observability mapping is defined as

$$\mathcal{O}_{pp}(t_1, t_2) := B^{\mathrm{T}} F_J(t_1)^{\mathrm{T}} M F_J(t_2).$$

Proper-proper Observability

- The observability mapping is defined as

$$\mathcal{O}_{pp}(t_1,t_2) = B^{\mathrm{T}} F_J(t_1)^{\mathrm{T}} M F_J(t_2).$$

- Define the corresponding observability Gramian[3]

$$\begin{aligned} Q_{pp} &:= \int_0^\infty \int_0^\infty \mathcal{O}_{pp}(t_1,t_2)^{\mathrm{T}} \mathcal{O}_{pp}(t_1,t_2) \mathrm{d}t_1 \mathrm{d}t_2 \\ &= \int_0^\infty \int_0^\infty F_J(t_2)^{\mathrm{T}} M F_J(t_1) B B^{\mathrm{T}} F_J(t_1)^{\mathrm{T}} M F_J(t_2) \mathrm{d}t_1 \mathrm{d}t_2 \end{aligned}$$

$$\boxed{Q_{pp} = \int_0^\infty F_J(t_2)^{\mathrm{T}} M P_p M F_J(t_2) \mathrm{d}t_2.}$$

[3] P. Benner, P. Goyal, and I. Pontes Duff, Gramians, energy functionals and balanced truncation for linear dynamical systems with quadratic outputs, IEEE Trans. Autom. Control, 67(2):886-893, 2021.

Summary — Gramians

Observability Gramians:

$$Q_{pp} = \int_0^\infty F_J(t)^{\mathrm{T}} M P_p M F_J(t) \mathrm{d}t, \qquad Q_{ip} = \int_0^\infty F_J(t)^{\mathrm{T}} M P_i M F_J(t) \mathrm{d}t \quad \Rightarrow \quad Q_p = Q_{pp} + Q_{ip},$$

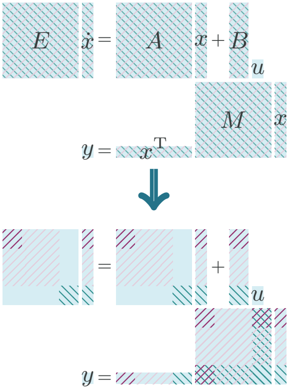
$$Q_{pi} = \sum_{k=0}^{\nu-1} F_N(k)^{\mathrm{T}} M P_p M F_N(k), \qquad Q_{ii} = \sum_{k=0}^{\nu-1} F_N(k)^{\mathrm{T}} M P_i M F_N(k) \quad \Rightarrow \quad Q_i = Q_{pi} + Q_{ii}$$

Theorem

The observability Gramians Q_{pp}, Q_{pi}, Q_{ip} and Q_{ii} solve the projected Lyapunov equations

$$\begin{aligned}
E^{\mathrm{T}} Q_{pp} A + A^{\mathrm{T}} Q_{pp} E &= -P_r^{\mathrm{T}} M P_p M P_r, & P_p &= P_l^{\mathrm{T}} Q_{pp} P_l, \\
A^{\mathrm{T}} Q_{pi} A - E^{\mathrm{T}} Q_{pi} E &= (I - P_r)^{\mathrm{T}} M P_p M (I - P_r), & 0 &= P_l^{\mathrm{T}} Q_{pi} P_l, \\
E^{\mathrm{T}} Q_{ip} A + A^{\mathrm{T}} Q_{ip} E &= -P_r^{\mathrm{T}} M P_i M P_r, & P_p &= P_l^{\mathrm{T}} Q_{ip} P_l, \\
A^{\mathrm{T}} Q_{ii} A - E^{\mathrm{T}} Q_{ii} E &= (I - P_r)^{\mathrm{T}} M P_i M (I - P_r), & 0 &= P_l^{\mathrm{T}} Q_{ii} P_l.
\end{aligned}$$

Table of contents



Balanced truncation

- **Idea:** Truncate states that are hard to reach and to observe.
- Evaluate energy norms to detect most dominant subspaces.

Input energy:

- Energy norm of the proper input-to-state mapping $\mathcal{C}_p(t)$ and $\mathcal{C}_i(t)$:

$$\|\mathcal{C}_p\| = \int_0^\infty \operatorname{tr}\left(\mathcal{C}_p(t)\mathcal{C}_p(t)\right) \mathrm{d}t = \operatorname{tr}\left(P_p\right) = \sigma_1 + \cdots + \sigma_{n_f},$$

$$\|\mathcal{C}_i\| = \sum_0^{\nu-1} \operatorname{tr}\left(\mathcal{C}_i(k)\mathcal{C}_i(k)\right) = \operatorname{tr}\left(P_i\right) = \theta_1 + \cdots + \theta_{n_\infty},$$

where $\sigma_1, \ldots, \sigma_{n_f}$ are the nonzero singular values of P_p and $\theta_1, \ldots, \theta_{n_\infty}$ those from P_i.

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$$\|\mathcal{C}_i\| = \sum_0^{\nu-1} \operatorname{tr}\left(\mathcal{C}_i(k)\mathcal{C}_i(k)\right) = \operatorname{tr}\left(P_i\right) = \theta_1 + \cdots + \theta_{n_\infty},$$

where $\sigma_1, \ldots, \sigma_{n_f}$ are the nonzero singular values of P_p and $\theta_1, \ldots, \theta_{n_\infty}$ those from P_i.

$\Rightarrow$ Small singular values and respective states have little effect on system dynamics.

$\Rightarrow$ Truncate the respective subspaces.

Balanced truncation

Output energy:

- Evaluate the energy norms of the proper and improper state-to-output mappings $\mathcal{O}_{pp}(t_1, t_2)$, $\mathcal{O}_{ip}$, $\mathcal{O}_{pi}$, and $\mathcal{O}_{ii}$, which yields:

$$\|\mathcal{O}_{pp}\| = \operatorname{tr}(Q_{pp}), \quad \|\mathcal{O}_{ip}\| = \operatorname{tr}(Q_{ip}), \quad \|\mathcal{O}_{pi}\| = \operatorname{tr}(Q_{pi}), \quad \|\mathcal{O}_{ii}\| = \operatorname{tr}(Q_{ii}).$$

Proper output energy corresponding to a differential right state:

$$E_{y_p} = \|\mathcal{O}_{pp}\| + \|\mathcal{O}_{ip}\| = \operatorname{tr}(Q_{pp} + Q_{ip}) = \operatorname{tr}(Q_p)$$

Improper output energy corresponding to an algebraic right state:

$$E_{y_i} = \|\mathcal{O}_{pi}\| + \|\mathcal{O}_{ii}\| = \operatorname{tr}(Q_{pi} + Q_{ii}) = \operatorname{tr}(Q_i)$$

Balanced truncation

Output energy:

- Evaluate the energy norms of the proper and improper state-to-output mappings $\mathcal{O}_{pp}(t_1, t_2)$, $\mathcal{O}_{ip}$, $\mathcal{O}_{pi}$, and $\mathcal{O}_{ii}$, which yields:

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Proper output energy corresponding to a differential right state:

$$\begin{aligned} E_{y_p} &= \|\mathcal{O}_{pp}\| + \|\mathcal{O}_{ip}\| = \operatorname{tr}(Q_{pp} + Q_{ip}) \\ &= \operatorname{tr}(Q_p) \end{aligned}$$

Improper output energy corresponding to an algebraic right state:

$$\begin{aligned} E_{y_i} &= \|\mathcal{O}_{pi}\| + \|\mathcal{O}_{ii}\| = \operatorname{tr}(Q_{pi} + Q_{ii}) \\ &= \operatorname{tr}(Q_i) \end{aligned}$$

$\Rightarrow$ Truncate states corresponding to small eigenvalues of P_p and Q_p.

Balanced truncation

- Use low-rank factors $P_p = R_p R_p^{\mathrm{T}}$, $P_i = R_i R_i^{\mathrm{T}}$, $Q_p = S_p S_p^{\mathrm{T}}$, $Q_i = S_i S_i^{\mathrm{T}}$.
- Compute the SVDs: $S_p^{\mathrm{T}} E R_p = U_p \Sigma V_p^{\mathrm{T}} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^{\mathrm{T}} \\ V_2^{\mathrm{T}} \end{bmatrix}$, $\quad S_i^{\mathrm{T}} A R_i = U_i \Theta V_i^{\mathrm{T}}$.
- Balancing and truncating projection matrices:

$$W_{\mathrm{r}} = \begin{bmatrix} S_p^{\mathrm{T}} U_1 \Sigma_1^{-\frac{1}{2}} & S_i U_i \Theta^{-\frac{1}{2}} \end{bmatrix}, \qquad T_{\mathrm{r}} = \begin{bmatrix} R_p^{\mathrm{T}} V_1 \Sigma_1^{-\frac{1}{2}} & R_i V_i \Theta^{-\frac{1}{2}} \end{bmatrix}$$

Reduced system

$$W_{\mathrm{r}}^{\mathrm{T}} E T_{\mathrm{r}} \dot{\widehat{x}}(t) = W_{\mathrm{r}}^{\mathrm{T}} A T_{\mathrm{r}} \widehat{x}(t) + W_{\mathrm{r}}^{\mathrm{T}} B u(t),$$

$$\widehat{y}(t) = \widehat{x}(t)^{\mathrm{T}} T_{\mathrm{r}}^{\mathrm{T}} M T_{\mathrm{r}} \widehat{x}(t)$$

Balanced truncation

- Use low-rank factors $P_p = R_p R_p^{\mathrm{T}}$, $P_i = R_i R_i^{\mathrm{T}}$, $Q_p = S_p S_p^{\mathrm{T}}$, $Q_i = S_i S_i^{\mathrm{T}}$.
- Compute the SVDs: $S_p^{\mathrm{T}} E R_p = U_p \Sigma V_p^{\mathrm{T}} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^{\mathrm{T}} \\ V_2^{\mathrm{T}} \end{bmatrix}$, $\quad S_i^{\mathrm{T}} A R_i = U_i \Theta V_i^{\mathrm{T}}$.
- Balancing and truncating projection matrices:

$$W_{\mathrm{r}} = \begin{bmatrix} S_p^{\mathrm{T}} U_1 \Sigma_1^{-\frac{1}{2}} & S_i U_i \Theta^{-\frac{1}{2}} \end{bmatrix}, \qquad T_{\mathrm{r}} = \begin{bmatrix} R_p^{\mathrm{T}} V_1 \Sigma_1^{-\frac{1}{2}} & R_i V_i \Theta^{-\frac{1}{2}} \end{bmatrix}$$

Reduced system

$$\begin{bmatrix} I_r & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} \dot{\widehat{x}}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \widehat{A}_1 & 0 \\ 0 & I_{n_\infty} \end{bmatrix} \begin{bmatrix} \widehat{x}_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \end{bmatrix} u(t),$$

$$\widehat{y}(t) = \begin{bmatrix} \widehat{x}_1(t) \\ x_2(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \widehat{M}_{11} & \widehat{M}_{12} \\ \widehat{M}_{12}^{\mathrm{T}} & \widehat{M}_{22} \end{bmatrix} \begin{bmatrix} \widehat{x}_1(t) \\ x_2(t) \end{bmatrix}$$

Error Estimation

Output error

$$\|y(t)-\widehat{y}(t)\|_{L_\infty} \leq \|y_{pp}(t)-\widehat{y}_{pp}(t)\|_{L_\infty} + \|y_{pi}(t)-\widehat{y}_{pi}(t)\|_{L_\infty} + \|y_{ip}(t)-\widehat{y}_{ip}(t)\|_{L_\infty} + \underbrace{\|y_{ii}(t)-\widehat{y}_{ii}(t)\|_{L_\infty}}_{=0}$$

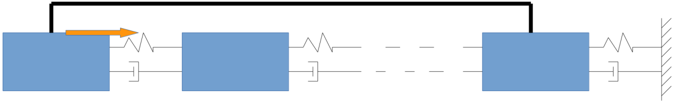
- As example estimate

$$\|y_{ip}(t)-\widehat{y}_{ip}(t)\|_{L_\infty} \leq \left(\operatorname{tr}\left(B^{\mathrm{T}}Q_{ip}B\right) - 2\operatorname{tr}\left(B^{\mathrm{T}}\widetilde{Q}_{ip}\hat{B}\right) + \operatorname{tr}\left(\hat{B}^{\mathrm{T}}\hat{Q}_{ip}\hat{B}\right)\right)^{\frac{1}{2}} \nu^{\frac{1}{2}} \|u\|_{\mathcal{C}^{\nu-1}} \|u\|_{L_2}$$

with

- $\widehat{Q}_{ip}$ Gramian of the reduced system,
- $\widetilde{Q}_{ip}$ mixed Gramian that solves a particular projected Sylvester equation.

A numerical example



Mechanical system described by an index 3 DAE system[4]

- Input $u(t) = \sin(2t)^2 e^{-\frac{t}{2}}$.
- Output matrix $M = I_{n_f+n_\infty}$.
- Original dimensions: $n_f = 1200,\ n_\infty = 1$.

[4] V. Mehrmann and T. Stykel, Balanced truncation model reduction for large-scale systems in descriptor form. In P. Benner, V. Mehrmann, and D. C. Sorensen, Dimension Reduction of Large-Scale Systems, volume 45 of Lect. Notes Comput. Sci. Eng., pages 83-115.Springer-Verlag, Berlin/Heidelberg, Germany, 2005.

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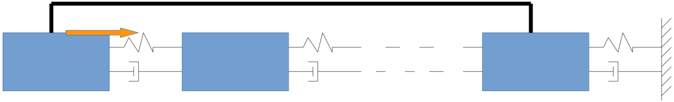

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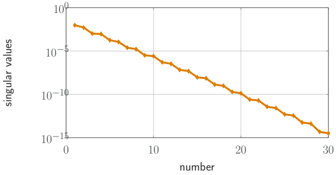
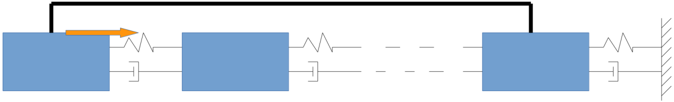
- Reduced dimensions: $\hat{n}_f = 20,\ \hat{n}_\infty = 1$.

Singular value decay in Σ:



A numerical example



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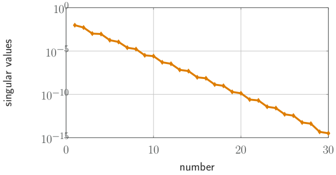
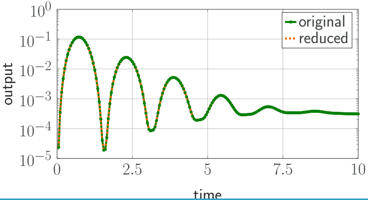
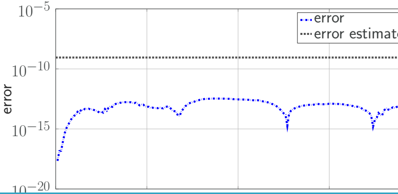


Output and output error:





Summary

- We defined (new) Gramians:

$$
\begin{aligned}
P_p &= \int_0^\infty F_J(t)BB^{\mathrm{T}}F_J(t)^{\mathrm{T}}\mathrm{d}t, & P_i &= \sum_{k=0}^{\nu-1} F_N(k)BB^{\mathrm{T}}F_N(k)^{\mathrm{T}},\\
Q_{pp} &= \int_0^\infty F_J(t)^{\mathrm{T}}MP_pMF_J(t)\mathrm{d}t, & Q_{pi} &= \sum_{k=0}^{\nu-1} F_N(k)^{\mathrm{T}}MP_pMF_N(k),\\
Q_{ip} &= \int_0^\infty F_J(t)^{\mathrm{T}}MP_iMF_J(t)\mathrm{d}t, & Q_{ii} &= \sum_{k=0}^{\nu-1} F_N(k)^{\mathrm{T}}MP_iMF_N(k),\\
Q_p &= Q_{pp} + Q_{ip}, & Q_i &= Q_{pi} + Q_{ii}.
\end{aligned}
$$

- We investigated the energy functionals of the systems

$$
E_u(x_p^*) = (x_p^*)P_p^I x_p^*, \qquad E_{y_p}(x_p^*) \leq (x_p^*)^{\mathrm{T}}E^{\mathrm{T}}Q_pEx_p^*.
$$

- We propose a balanced truncation method for DAE systems with quadratic output.
- We derived an error estimator.

Thank you for your attention!

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