MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Solving Quadratic Lyapunov Equations

Peter Benner   Steffen W. R. Werner (Virginia Tech)
Pawan K. Goyal (appliedAI Initiative)

SIAM LA24
MS84 "Matrix and Tensor Equations in Action:
Simulation, Model Reduction and Scientific Machine Learning"
Paris, May 13–17, 2024

Supported by:

DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Partners:

OTTO VON GUERICKE
UNIVERSITÄT
MAGDEBURG

MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Solving Quadratic Lyapunov Equations
. . . and why

Peter Benner Steffen W. R. Werner (Virginia Tech)
Pawan K. Goyal (appliedAI Initiative)

SIAM LA24
MS84 "Matrix and Tensor Equations in Action:
Simulation, Model Reduction and Scientific Machine Learning"
Paris, May 13–17, 2024

Supported by:
DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Partners:
OTTO VON GUERICKE
UNIVERSITÄT
MAGDEBURG

Overview

Outline

Problem Setting

Quadratic Lyapunov Equation

For $A, A_k \in \mathbb{R}^{n\times n}$, $k = 1, \ldots, m$, $B \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{n\times n^2}$, find symmetric positive semidefinite solution P of

$$\begin{aligned} 0 &= AP + PA^T + \sum_{k=1}^{m} A_k P A_k^T + H(P\otimes P)H^T + BB^T \\ &=: \quad \mathcal{L}(P) \quad + \quad \Pi(P) \quad + \quad \mathcal{K}(P) \quad + BB^T. \end{aligned}$$

We assume A stable throughout, i.e. $\Lambda(A) \subset \mathbb{C}^-$, implying $\Lambda(\mathcal{L}) \subset \mathbb{C}^-$.

Problem Setting

Quadratic Lyapunov Equation

For $A, A_k \in \mathbb{R}^{n\times n}$, $k = 1, \dots, m$, $B \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{n\times n^2}$, find symmetric positive semidefinite solution P of

$$\begin{aligned} 0 &= AP + PA^T + \sum_{k=1}^{m} A_k P A_k^T + H(P \otimes P)H^T + BB^T \\ &=: \quad \mathcal{L}(P) \quad + \quad \Pi(P) \quad + \quad \mathcal{K}(P) \quad + BB^T. \end{aligned}$$

We assume A stable throughout, i.e. $\Lambda(A) \subset \mathbb{C}^-$, implying $\Lambda(\mathcal{L}) \subset \mathbb{C}^-$.

Note:

- $\mathcal{L}$ is the standard Lyapunov operator.
- For $A_k \equiv 0$, $H = 0$, we obtain a standard Lyapunov equation.
- $\Pi, \mathcal{K}$ are nonnegative operators, i.e., they map spsd matrices to spsd matrices.
- $\mathcal{L} + \Pi$ is linear ("Lyapunov-plus-positive"/"bilinear Lyapunov"); $\mathcal{K}$ is nonlinear.

Nonlinear Control Systems

$$\Sigma : \begin{cases} E\dot{x}(t) &= f(t, x(t), u(t)), \quad Ex(t_0) = Ex_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

with

- (generalized) states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^q$.

If E singular $\rightsquigarrow$ descriptor system. Here, $E = I_n$ for simplicity.

Original System ($E = I_n$)

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), \\ y(t) = g(t, x(t), u(t)). \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^q$.

Reduced-Order Model (ROM)

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t) \in \mathbb{R}^q$.

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Model Reduction for Control Systems

Original System ($E = I_n$)

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), \\ y(t) = g(t, x(t), u(t)). \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^q$.

Reduced-Order Model (ROM)

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t) \in \mathbb{R}^q$.

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

System Classes

Control-Affine (Autonomous) Systems

$$\begin{aligned}\dot{x}(t) &= f(t,x,u) &= \mathcal{A}(x(t)) + \mathcal{B}(x(t))u(t), \quad \mathcal{A}:\mathbb{R}^n \to \mathbb{R}^n,\ \mathcal{B}:\mathbb{R}^n \to \mathbb{R}^{n\times m},\\ y(t) &= g(t,x,u) &= \mathcal{C}(x(t)) + \mathcal{D}(x(t))u(t), \quad \mathcal{C}:\mathbb{R}^n \to \mathbb{R}^q,\ \mathcal{D}:\mathbb{R}^n \to \mathbb{R}^{q\times m}.\end{aligned}$$

System Classes

Control-Affine (Autonomous) Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = \mathcal{A}(x(t)) + \mathcal{B}(x(t))u(t), \quad \mathcal{A}:\mathbb{R}^n \to \mathbb{R}^n,\ \mathcal{B}:\mathbb{R}^n \to \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) = \mathcal{C}(x(t)) + \mathcal{D}(x(t))u(t), \quad \mathcal{C}:\mathbb{R}^n \to \mathbb{R}^q,\ \mathcal{D}:\mathbb{R}^n \to \mathbb{R}^{q\times m}. \end{aligned}$$

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = Ax(t) + Bu(t), \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}. \end{aligned}$$

System Classes

Control-Affine (Autonomous) Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = \mathcal{A}(x(t)) + \mathcal{B}(x(t))u(t), \quad \mathcal{A}:\mathbb{R}^n \to \mathbb{R}^n,\ \mathcal{B}:\mathbb{R}^n \to \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) = \mathcal{C}(x(t)) + \mathcal{D}(x(t))u(t), \quad \mathcal{C}:\mathbb{R}^n \to \mathbb{R}^q,\ \mathcal{D}:\mathbb{R}^n \to \mathbb{R}^{q\times m}. \end{aligned}$$

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = Ax(t) + Bu(t), \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}. \end{aligned}$$

Bilinear Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = Ax(t) + \sum_{i=1}^{m} u_i(t)A_i x(t) + Bu(t), \quad A, A_i \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \quad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}. \end{aligned}$$

System Classes

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) &= Ax(t) + Bu(t), \qquad & A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) &= Cx(t) + Du(t), \qquad & C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}. \end{aligned}$$

Bilinear Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = Ax(t) + \sum_{i=1}^{m} u_i(t) A_i x(t) + Bu(t), \quad & A, A_i \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) = Cx(t) + Du(t), & C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}. \end{aligned}$$

Quadratic-Bilinear (QB) Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = Ax(t) + H\left(x(t) \otimes x(t)\right) + \sum_{i=1}^{m} u_i(t) A_i x(t) + Bu(t), \\ & \qquad A, A_i \in \mathbb{R}^{n\times n},\ H \in \mathbb{R}^{n\times n^2},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}. \end{aligned}$$

QB systems can be obtained as approximation (by truncating Taylor expansion) to weakly nonlinear systems [Phillips '03].

C. Gu. QLMOR: A Projection-Based Nonlinear Model Order Reduction Approach Using Quadratic-Linear Representation of Nonlinear Systems. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 30(9):1307–1320, 2011.

L. Feng, X. Zeng, C. Chiang, D. Zhou, and Q. Fang. Direct nonlinear order reduction with variational analysis. In: Proceedings of DATE 2004, pp. 1316–1321.

J. R. Phillips. Projection-based approaches for model reduction of weakly nonlinear time-varying systems. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 22(2):171–187, 2003.

Quadratic-Bilinearization

QB systems can be obtained as approximation (by truncating Taylor expansion) to weakly nonlinear systems [Phillips '03].

But exact representation of smooth nonlinear systems possible:

Theorem [Gu '09/'11]

Assume that the state equation of a nonlinear system is given by

$$\dot{x} = a_0 x + a_1 g_1(x) + \ldots + a_k g_k(x) + Bu,$$

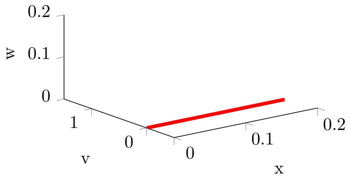
where $g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$ are compositions of uni-variable rational, exponential, logarithmic, trigonometric or root functions, respectively. Then, by iteratively taking derivatives and adding algebraic equations, respectively, the nonlinear system can be transformed into a QB(DAE) system.

C. Gu. QLMOR: A Projection-Based Nonlinear Model Order Reduction Approach Using Quadratic-Linear Representation of Nonlinear Systems. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 30(9):1307–1320, 2011.

L. Feng, X. Zeng, C. Chiang, D. Zhou, and Q. Fang. Direct nonlinear order reduction with variational analysis. In: Proceedings of DATE 2004, pp. 1316–1321.

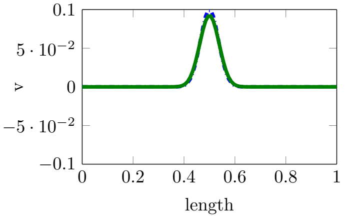
J. R. Phillips. Projection-based approaches for model reduction of weakly nonlinear time-varying systems. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 22(2):171–187, 2003.

FitzHugh-Nagumo model



- Model describes activation and de-activation of neurons.
- Contains a cubic nonlinearity, which can be transformed to QB form.

Sine-Gordon equation



- Applications in biomedical studies, mechanical transmission lines, etc.
- Contains $\sin$ function, which can also be rewritten into QB form.

Basic concept

- System $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is **balanced**, if **system Gramians**, i.e., solutions P, Q of the **Lyapunov equations**

  $$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

Basic concept

- System $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions P, Q of the **Lyapunov equations**

$$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

  satisfy: $P = Q = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.
- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

Basic concept

- System $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions P, Q of the **Lyapunov equations**

  $$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.
- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization (needs P, Q!) of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C) \quad &\mapsto \quad (TAT^{-1}, TB, CT^{-1}) \\ &= \quad \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right). \end{aligned}$$

Balanced Truncation for Linear Systems

Basic concept

- System $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions P, Q of the **Lyapunov equations**

  $$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.
- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization (needs P, Q!) of the system via state-space transformation

  $$\begin{aligned} \mathcal{T} : (A, B, C) \quad &\mapsto \quad (TAT^{-1}, TB, CT^{-1}) \\ &= \quad \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right). \end{aligned}$$

- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}) = (A_{11}, B_1, C_1)$.

Properties

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.

Properties

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \|G - \hat{G}\|_{\mathcal{H}_\infty} \|u\|_2 \leq \left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right) \|u\|_2 .$$

Balanced Truncation for Linear Systems

Properties

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \|G - \hat{G}\|_{\mathcal{H}_\infty} \|u\|_2 \leq \left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right) \|u\|_2 .$$

Practical implementation

- Rather than solving Lyapunov equations for P, Q (n^2 unknowns!), find $S, R \in \mathbb{R}^{n \times s}$ with $s \ll n$ such that $P \approx SS^T$, $Q \approx RR^T$.

  Many algorithms: [Antoulas, Baur, B., Chu, Druskin, Hammarling, Fassbender, Freitag, Grasedyck, Gugercin, Jaimoukha, Knizherman, Köhler, Kressner, Kürschner, Li, Palitta, Penzl, Quintana-Ortí, Saad, Saak, Simoncini, Sorensen, Stykel, White, ...]

Properties

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \|G - \hat{G}\|_{\mathcal{H}_\infty} \|u\|_2 \leq \left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right) \|u\|_2 .$$

Practical implementation

- Rather than solving Lyapunov equations for P, Q (n^2 unknowns!), find $S, R \in \mathbb{R}^{n \times s}$ with $s \ll n$ such that $P \approx SS^T$, $Q \approx RR^T$.
  Many algorithms: [Antoulas, Baur, B., Chu, Druskin, Hammarling, Fassbender, Freitag, Grasedyck, Gugercin, Jaimoukha, Knizherman, Köhler, Kressner, Kürschner, Li, Palitta, Penzl, Quintana-Ortí, Saad, Saak, Simoncini, Sorensen, Stykel, White, ...]
- Reduced-order model directly obtained via small-scale ($s \times s$) SVD of $R^T S$!
- No $\mathcal{O}(n^3)$ or $\mathcal{O}(n^2)$ computations necessary!

Outline

- Nonlinear balancing based on energy functionals [Scherpen '93, Gray/Mesko '96].

Definition [Scherpen '93, Gray/Mesko '96]

The reachability energy functional, $L_c(x_0)$, and observability energy functional, $L_o(x_0)$ of a system are given as:

$$L_c(x_0) = \inf_{\substack{u \in L_2(-\infty,0] \\ x(-\infty)=0,\ x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 dt, \qquad L_o(x_0) = \frac{1}{2}\int_0^{\infty} \|y(t)\|^2 dt.$$

Disadvantage: energy functionals are the solutions of nonlinear Hamilton-Jacobi equations which are hardly solvable for large-scale systems.

- Nonlinear balancing based on energy functionals [Scherpen '93, Gray/Mesko '96].

  Disadvantage: energy functionals are the solutions of nonlinear Hamilton-Jacobi equations which are hardly solvable for large-scale systems.

- Empirical Gramians/frequency-domain POD [Lall et al '99, Willcox/Peraire '02].

Example: controllability Gramian from time domain data (snapshots)

1. Define reachability Gramian of the system

$$P = \int_0^\infty x(t)x(t)^T dt, \quad \text{where } x(t) \text{ solves } \dot{x} = f(x, \delta),\ x(0) = x_0.$$

2. Use time-domain integrator to produce snapshots $x_k \approx x(t_k)$, $k = 1, \ldots, K$.
3. Approximate $P \approx \sum_{k=0}^{K} w_k x_k x_k^T$ with positive weights w_k.
4. Analogously for observability Gramian.
5. Compute balancing transformation and apply it to nonlinear system.

Disadvantage: Depends on chosen training input (e.g., $\delta(t_0)$) like other POD approaches.

- Nonlinear balancing based on energy functionals [Scherpen '93, Gray/Mesko '96].

  Disadvantage: energy functionals are the solutions of nonlinear Hamilton-Jacobi equations which are hardly solvable for large-scale systems.

- Empirical Gramians/frequency-domain POD [Lall et al '99, Willcox/Peraire '02].

  Disadvantage: Depends on chosen training input (e.g., $\delta(t_0)$) like other POD approaches.

- $\rightsquigarrow$ **Goal:** computationally efficient and input-independent method!

W. S. Gray and J. P. Mesko. Controllability and observability functions for model reduction of nonlinear systems. In Proc. of the Conf. on Information Sci. and Sys., pp. 1244–1249, 1996.

S. Lall, J. Marsden, and S. Glavaški. A subspace approach to balanced truncation for model reduction of nonlinear control systems. International Journal of Robust and Nonlinear Control, 12:519–535, 2002.

J. M. A. Scherpen. Balancing for nonlinear systems. Systems & Control Letters, 21:143–153, 1993.

K. Willcox and J. Peraire, Balanced model reduction via the proper orthogonal decomposition. AIAA Journal, 40:2323–2330, 2002.

New "Gramians"

- A possible solution is to obtain bounds for the energy functionals, instead of computing them exactly.

New "Gramians"

- A possible solution is to obtain bounds for the energy functionals, instead of computing them exactly.
- For example, (locally) $L_c(x_0) \geq \frac{1}{2} x_0^T \tilde{P}^{-1} x_0$, where $\tilde{P} = \tilde{P}^T > 0$ [Gray/Mesko '96].
- For bilinear systems, such local bounds were derived in [B./Damm '11] using the solutions to the Lyapunov-plus-positive equations:

$$AP + PA^T + \sum_{i=1}^m A_i P A_i^T + BB^T = 0,$$
$$A^T Q + QA^T + \sum_{i=1}^m A_i^T Q A_i + C^T C = 0.$$

(If their solutions exist, they define reachability and observability Gramians of BIBO stable bilinear system.)

- Efficient solution methods for Lyapunov-plus-positive equations are derived in [B./Breiten '13, Shank/Simoncini/Szyld '16, Kürschner '17, ...].

New "Gramians"

- A possible solution is to obtain bounds for the energy functionals, instead of computing them exactly.
- For example, (locally) $L_c(x_0) \geq \frac{1}{2} x_0^T \tilde{P}^{-1} x_0$, where $\tilde{P} = \tilde{P}^T > 0$ [Gray/Mesko '96].
- For bilinear systems, such local bounds were derived in [B./Damm '11] using the solutions to the Lyapunov-plus-positive equations:

$$AP + PA^T + \sum_{i=1}^{m} A_i P A_i^T + BB^T = 0,$$
$$A^T Q + QA^T + \sum_{i=1}^{m} A_i^T Q A_i + C^T C = 0.$$

  (If their solutions exist, they define reachability and observability Gramians of BIBO stable bilinear system.)
- Efficient solution methods for Lyapunov-plus-positive equations are derived in [B./Breiten '13, Shank/Simoncini/Szyld '16, Kürschner '17, ...].
- **Here we aim at determining algebraic Gramians for QB systems, which**
  - **provide bounds for the energy functionals of QB systems,**
  - **generalize the Gramians of linear and bilinear systems, and**
  - **allow us to find the states that are hard to control as well as hard to observe in an efficient and reliable way.**

- Consider input $\rightarrow$ state map of QB system $(m = 1,\ N \equiv A_1)$:

$$\dot{x}(t) = Ax(t) + Hx(t) \otimes x(t) + Nx(t)u(t) + Bu(t), \qquad x(0) = 0.$$

- Integration yields

$$x(t) = \int_0^t e^{A\sigma_1} Bu(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Nx(t-\sigma_1)u(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Hx(t-\sigma_1) \otimes x(t-\sigma_1)d\sigma_1$$

[Rugh '81]

- Consider input $\rightarrow$ state map of QB system ($m=1,\ N \equiv A_1$):

$$\dot{x}(t) = Ax(t) + Hx(t) \otimes x(t) + Nx(t)u(t) + Bu(t), \qquad x(0) = 0.$$

- Integration yields

$$\begin{aligned}
x(t) &= \int_0^t e^{A\sigma_1} Bu(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Nx(t-\sigma_1)u(t-\sigma_1)d\sigma_1 \\
&\qquad + \int_0^t e^{A\sigma_1} Hx(t-\sigma_1) \otimes x(t-\sigma_1)d\sigma_1 \\
&= \int_0^t e^{A\sigma_1} Bu(t-\sigma_1)d\sigma_1 + \int_0^t \int_0^{t-\sigma_1} e^{A\sigma_1} Ne^{A\sigma_2} Bu(t-\sigma_1)u(t-\sigma_1-\sigma_2)d\sigma_1 d\sigma_2 \\
&\quad + \int_0^t \int_0^{t-\sigma_1} \int_0^{t-\sigma_1} e^{A\sigma_1} H(e^{A\sigma_2}B \otimes e^{A\sigma_3}B)u(t-\sigma_1-\sigma_2)u(t-\sigma_1-\sigma_3)d\sigma_1 d\sigma_2 d\sigma_3 + \ldots
\end{aligned}$$

[Rugh '81]

- Consider input $\rightarrow$ state map of QB system ($m = 1,\ N \equiv A_1$):

$$\dot{x}(t) = Ax(t) + Hx(t) \otimes x(t) + Nx(t)u(t) + Bu(t), \qquad x(0) = 0.$$

- Integration yields

$$\begin{aligned} x(t) &= \int_0^t e^{A\sigma_1} Bu(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Nx(t-\sigma_1)u(t-\sigma_1)d\sigma_1 \\ &\qquad + \int_0^t e^{A\sigma_1} Hx(t-\sigma_1) \otimes x(t-\sigma_1)d\sigma_1 \\ &= \int_0^t e^{A\sigma_1} Bu(t-\sigma_1)d\sigma_1 + \int_0^t \int_0^{t-\sigma_1} e^{A\sigma_1} N e^{A\sigma_2} Bu(t-\sigma_1)u(t-\sigma_1-\sigma_2)d\sigma_1 d\sigma_2 \\ &\quad + \int_0^t \int_0^{t-\sigma_1} \int_0^{t-\sigma_1} e^{A\sigma_1} H(e^{A\sigma_2} B \otimes e^{A\sigma_3} B)u(t-\sigma_1-\sigma_2)u(t-\sigma_1-\sigma_3)d\sigma_1 d\sigma_2 d\sigma_3 + \ldots \end{aligned}$$

- By iteratively inserting expressions for $x(t - \bullet)$, we obtain the **Volterra series expansion** for the QB system. [Rugh '81]

Using the *Volterra kernels*, we can define the controllability mappings

$$\Pi_1(t_1) := e^{At_1}B, \qquad \Pi_2(t_1, t_2) := e^{At_1}N\Pi_1(t_2),$$
$$\Pi_3(t_1, t_2, t_3) := e^{At_1}[H(\Pi_1(t_2) \otimes \Pi_1(t_3)), N\Pi_2(t_1, t_2)], \ldots$$

and a candidate for a new Gramian:

$$P := \sum_{k=1}^{\infty} P_k, \qquad \text{where} \quad P_k = \int_0^{\infty} \cdots \int_0^{\infty} \Pi_k(t_1, \ldots, t_k)\Pi_k(t_1, \ldots, t_k)^T \, dt_1 \ldots dt_k.$$

Using the *Volterra kernels*, we can define the controllability mappings

$$\Pi_1(t_1) := e^{At_1}B, \qquad \Pi_2(t_1,t_2) := e^{At_1}N\Pi_1(t_2),$$
$$\Pi_3(t_1,t_2,t_3) := e^{At_1}[H(\Pi_1(t_2)\otimes\Pi_1(t_3)), N\Pi_2(t_1,t_2)], \ldots$$

and a candidate for a new Gramian:

$$P := \sum_{k=1}^{\infty} P_k, \qquad \text{where} \quad P_k = \int_0^\infty \cdots \int_0^\infty \Pi_k(t_1,\ldots,t_k)\Pi_k(t_1,\ldots,t_k)^T \, dt_1 \ldots dt_k.$$

Theorem [B./Goyal '16]

If it exists, the new controllability Gramian P for QB (MIMO) systems with stable A solves the **quadratic Lyapunov equation**

$$AP + PA^T + \sum_{k=1}^{m} A_k P A_k^T + H(P\otimes P)H^T + BB^T = 0.$$

Note: $H = 0 \rightsquigarrow$ "bilinear reachability Gramian"; if additionally, all $A_k = 0 \rightsquigarrow$ linear one.

Outline

1. Motivation

2. Balanced Truncation for Nonlinear Systems

3. Numerical Methods
   - Truncated Gramians
   - Fix point iterations
   - Numerical Example

- Now, the main obstacle for using the new Gramians is the solution of the (quadratic) Lyapunov-type equations.

- Now, the main obstacle for using the new Gramians is the solution of the (quadratic) Lyapunov-type equations.
- Fix point iteration scheme can be employed $\rightsquigarrow$ next subsection.

- Now, the main obstacle for using the new Gramians is the solution of the (quadratic) Lyapunov-type equations.
- Fix point iteration scheme can be employed $\rightsquigarrow$ next subsection.
- For model order reduction, we proposed truncated Gramians for QB systems.

Definition (Truncated Gramians) [B./Goyal '16, B./Goyal/Redmann '17]

The truncated Gramians $P_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ for QB systems satisfy

$$AP_{\mathcal{T}} + P_{\mathcal{T}}A^T = -BB^T - \sum_{k=1}^{m} A_k P_l A_k^T - H(P_l \otimes P_l)H^T,$$

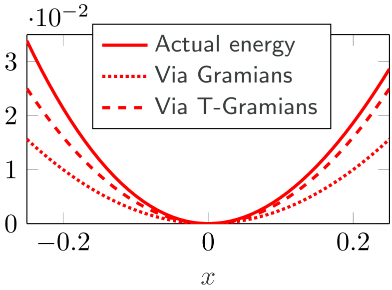
$$A^T Q_{\mathcal{T}} + Q_{\mathcal{T}}A = -C^T C - \sum_{k=1}^{m} A_k^T Q_l A_k - H^{(2)}(P_l \otimes Q_l)(H^{(2)})^T,$$

where

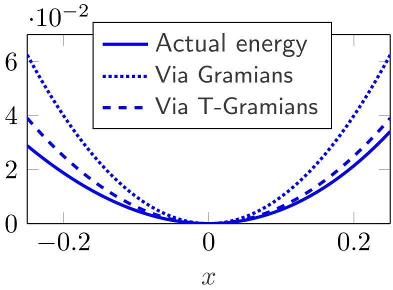
$$AP_l + P_l A^T = -BB^T \quad \text{and} \quad A^T Q_l + Q_l A = -C^T C.$$

- T-Gramians approximate energy functionals better than the actual Gramians.

- T-Gramians approximate energy functionals better than the actual Gramians.



(a) Input energy lower bounds.



(b) Output energy upper bounds.

Figure: Comparison of energy functionals for $-a = b = c = 2, h = 1, n = 0$.

- T-Gramians approximate energy functionals better than the actual Gramians.

- $\sigma_i(P \cdot Q) > \sigma_i(P_{\mathcal{T}} \cdot Q_{\mathcal{T}}) \Rightarrow$ obtain smaller order of reduced system if truncation is done at the same cutoff threshold.

Truncated Gramians

Advantages of truncated Gramians (T-Gramians)

- T-Gramians approximate energy functionals better than the actual Gramians.
- $\sigma_i(P \cdot Q) > \sigma_i(P_{\mathcal{T}} \cdot Q_{\mathcal{T}}) \Rightarrow$ obtain smaller order of reduced system if truncation is done at the same cutoff threshold.
- Most importantly, we need solutions of only four standard Lyapunov equations.

- T-Gramians approximate energy functionals better than the actual Gramians.
- $\sigma_i(P \cdot Q) > \sigma_i(P_{\mathcal{T}} \cdot Q_{\mathcal{T}}) \Rightarrow$ obtain smaller order of reduced system if truncation is done at the same cutoff threshold.
- Most importantly, we need solutions of only four standard Lyapunov equations.
- Interpretation of controllability/observability of the system via T-Gramians:
  - If the system is to be steered from 0 to x_0, where $x_0 \notin \text{range}(P_{\mathcal{T}})$, then $L_c(x_0) = \infty$.
  - If the system is controllable and $x_0 \in \ker(Q_{\mathcal{T}})$, then $L_o(x_0) = 0$.

Quadratic Lyapunov equation

$$AP + PA^T + \sum_{k=1}^{m} A_k P_l A_k^T + H(P_l \otimes P_l)H^T + BB^T.$$

Quadratic Lyapunov equation

$$AP + PA^T + \sum_{k=1}^{m} A_k P_l A_k^T + H(P_l \otimes P_l)H^T + BB^T.$$

Lyapunov iteration:

$$\begin{aligned} P_0 &= 0 \\ \mathcal{L}(P_{j+1}) &= AP_{j+1} + P_{j+1}A^T \\ &= -BB^T - \sum_{k=1}^{m} A_k P_j A_k^T - H(P_j \otimes P_j)H^T, \qquad j = 0, 1, \ldots \end{aligned}$$

Note: P_1 is reachability Gramian of associated linear-time invariant system.

Quadratic Lyapunov equation

$$AP + PA^T + \sum_{k=1}^{m} A_k P_l A_k^T + H(P_l \otimes P_l)H^T + BB^T.$$

Lyapunov iteration:

$$\begin{aligned} P_0 &= 0 \\ \mathcal{L}(P_{j+1}) &= AP_{j+1} + P_{j+1}A^T \\ &= -BB^T - \sum_{k=1}^{m} A_k P_j A_k^T - H(P_j \otimes P_j)H^T, \qquad j = 0, 1, \ldots \end{aligned}$$

Note: P_1 is reachability Gramian of associated linear-time invariant system.

Lyapunov operator remains constant over the iteration; this can be exploited using, e.g., Bartels-Stewart, Hammarling, sign function,. . .

Quadratic Lyapunov equation

$$AP + PA^T + \sum_{k=1}^{m} A_k P_l A_k^T + H(P_l \otimes P_l)H^T + BB^T.$$

Lyapunov iteration:

$$\begin{aligned} P_0 &= 0 \\ \mathcal{L}(P_{j+1}) &= AP_{j+1} + P_{j+1}A^T \\ &= -BB^T - \sum_{k=1}^{m} A_k P_j A_k^T - H(P_j \otimes P_j)H^T, \qquad j = 0, 1, \ldots \end{aligned}$$

Note: P_1 is reachability Gramian of associated linear-time invariant system.

Lyapunov operator remains constant over the iteration; this can be exploited using, e.g., Bartels-Stewart, Hammarling, sign function,. . .

Conjecture: converges to the minimal spsd solution under certain stability assumptions.

Quadratic Lyapunov equation

$$AP + PA^T + \sum_{k=1}^{m} A_k P_l A_k^T + H(P_l \otimes P_l)H^T + BB^T.$$

Lyapunov-plus-positive iteration:

$$\begin{aligned} P_0 &= 0 \\ \big(\mathcal{L} + \Pi\big)(P_{j+1}) &= AP_{j+1} + P_{j+1}A^T + \sum_{k=1}^{m} A_k P_{j+1} A_k^T \\ &= -BB^T - H(P_k \otimes P_j)H^T, \qquad j = 0, 1, \ldots \end{aligned}$$

Note: P_1 is reachability Gramian of associated bilinear system if it exists.

Quadratic Lyapunov equation

$$AP + PA^T + \sum_{k=1}^{m} A_k P_l A_k^T + H(P_l \otimes P_l)H^T + BB^T.$$

Lyapunov-plus-positive iteration:

$$\begin{aligned} P_0 &= 0 \\ (\mathcal{L} + \Pi)(P_{j+1}) &= AP_{j+1} + P_{j+1}A^T + \sum_{k=1}^{m} A_k P_{j+1} A_k^T \\ &= -BB^T - H(P_k \otimes P_j)H^T, \qquad j = 0, 1, \ldots \end{aligned}$$

Note: P_1 is reachability Gramian of associated bilinear system if it exists.

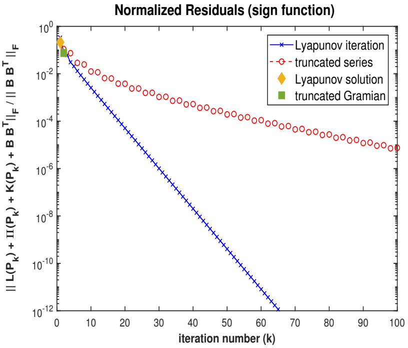
Conjecture: converges to the minimal spsd solution under BIBO stability assumptions.

Toy data (homogeneous bilinear term)

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \; H = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \rightsquigarrow \quad X = \begin{bmatrix} 0.3410 & 0 \\ 0 & 0.7522 \end{bmatrix}.$$

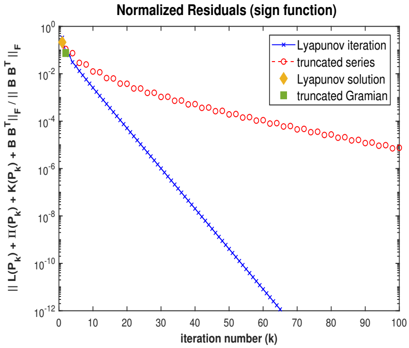
Toy data (homogeneous bilinear term)

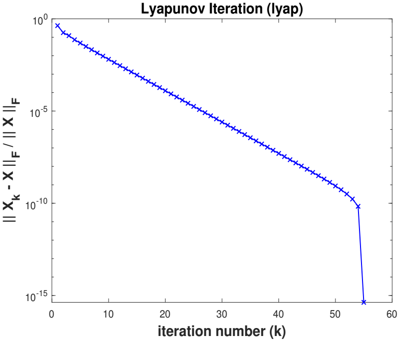
$$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \; H = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \rightsquigarrow \quad X = \begin{bmatrix} 0.3410 & 0 \\ 0 & 0.7522 \end{bmatrix}.$$



Toy data (homogeneous bilinear term)

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \; H = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \rightsquigarrow \quad X = \begin{bmatrix} 0.3410 & 0 \\ 0 & 0.7522 \end{bmatrix}.$$





Conclusions

- Extension of balanced truncation to asymptotically stable quadratic (and polynomial) systems leads to quadratic Lyapunov equations.
- Can be solved by fix point iteration or truncated series formula.
- Each step requires solution of standard Lyapunov equation; via sign function method or other Lyapunov solvers.
- Factorized versions possible $\rightsquigarrow$ large-scale, sparse solvers can be employed, but controlling rank growth might be problematic.
- Code will be available in upcoming MORLAB release, see `https://www.mpi-magdeburg.mpg.de/projects/morlab`.

Conclusions

- Extension of balanced truncation to asymptotically stable quadratic (and polynomial) systems leads to quadratic Lyapunov equations.
- Can be solved by fix point iteration or truncated series formula.
- Each step requires solution of standard Lyapunov equation; via sign function method or other Lyapunov solvers.
- Factorized versions possible $\rightsquigarrow$ large-scale, sparse solvers can be employed, but controlling rank growth might be problematic.
- Code will be available in upcoming MORLAB release, see `https://www.mpi-magdeburg.mpg.de/projects/morlab`.
- Acceleration of fix point iterations via vector extrapolation possible.
- Open problems: Existence of solutions, variety of solutions, convergence of fix point iterations.

Own References

P. Benner and T. Damm.
Lyapunov Equations, Energy Functionals, and Model Order Reduction of Bilinear and Stochastic Systems.
SIAM Journal on Control and Optimiztion, 49(2):686–711, 2011.

P. Benner, P. Goyal, and M. Redmann.
Truncated Gramians for Bilinear Systems and their Advantages in Model Order Reduction.
In P. Benner, M. Ohlberger, T. Patera, G. Rozza, K. Urban (eds.), Model Reduction of Parametrized Systems, pp. 285–300, Springer International Publishing, Cham, 2017.

P. Benner and P. Goyal.
Balanced Truncation Model Order Reduction for Quadratic-Bilinear Control Systems.
arXiv Preprint arXiv:1705.00160, April 2017.

P. Benner and S. W. R. Werner.
MORLAB - The Model Order Reduction LABoratory.
In P. Benner, T. Breiten, H. Faßbender, M. Hinze, T. Stykel, and R. Zimmermann (eds.), Model Reduction of Complex Dynamical Systems, pp. 393–415, Birkhäuser, Cham, 2021.