MAX PLANCK INSTITUTE FOR DYNAMICS OF COMPLEX TECHNICAL SYSTEMS MAGDEBURG

CSC

COMPUTATIONAL METHODS IN SYSTEMS AND CONTROL THEORY

Pencil Arithmetic with Applications

Peter Benner

based partially on joint work with Ralph Byers(†)

and recent results of Paul Van Dooren

PVD75 — Proper Value Decomposition 75

A Workshop in honor of Paul Van Dooren

Hotel Sierra Silvana, Selva di Fasano (Br), Italy

July 7–12, 2025

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Early Encounters

- 1995: The boom box talk at ICIAM 1995 in Hamburg.
  ($\rightsquigarrow$ the CD player model reduction benchmark example, used till today!)

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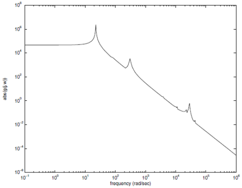

Fig. 24.7. Frequency response (C-DISC)

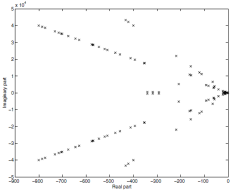

Fig. 24.8. Eigenvalues of A (C-DISC)

[Chahlaoui/Van Dooren 2002/2005]

- Paul's work [Bojanczyk/Golub/Van Dooren 1992, Sreedhar/Van Dooren 1993/1999] (also: [Hench/Laub 1994]) on periodic/product QR and QZ algorithms inspired work on inverse-free spectral projection methods for collapsing products of matrices and matrix pencils and solving periodic Riccati equations $\rightsquigarrow$ joint work with Ralph Byers [B./Byers 2001].
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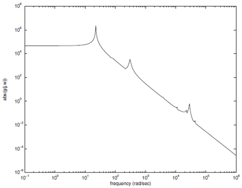

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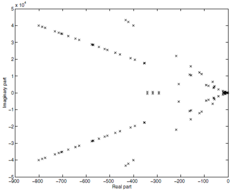

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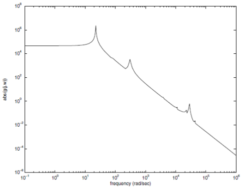

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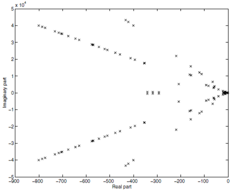

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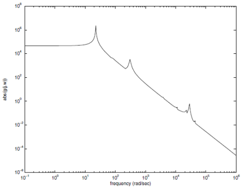

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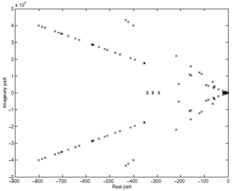

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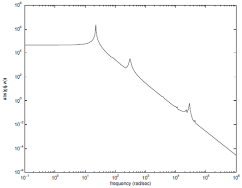

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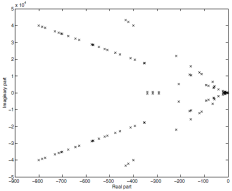

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The NICONET team at the Annual Meeting 2002 in Oxford, organized by Sven Hammarling.

Basic Question(s)

Given two matrix pencils $A - \lambda B, C - \lambda D \in \mathbb{C}^{n \times m}(\lambda)$, can we define meaningful arithmetic operations for them?

E.g., for addition and multiplication, assuming $n = m$ and B, D nonsingular,

$$B^{-1}A + D^{-1}C \quad \text{and} \quad B^{-1}A \cdot D^{-1}C$$

are well-defined via standard matrix addition and multiplication. How about singular cases?

$\rightsquigarrow$ pencil arithmetic [B./Byers 1997 – 2006].

Later question: What does it mean for a pair of matrix pencils to commute? And what are conditions for that?

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Also: relations to 2-point boundary value problems for linear descriptor systems.

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Ralph working on pencil arithmetic in Chemnitz, April 2004.

(Left-handed) matrix relation on $\mathbb{C}^n$

For matrix pencil $A - \lambda E \in \mathbb{C}^{m \times n}$:

$$(E \backslash A) = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \mid Ey = Ax\} \subset \mathbb{C}^n \times \mathbb{C}^n.$$

Also called pullback of E and A (in category theory).

Definition (Action of $(E \backslash A)$ on $x \in \mathbb{C}^n$)

For $x \in \mathbb{C}^n$, the x-section of $(E \backslash A)$ is the set $(E \backslash A)x \equiv \{y \in \mathbb{C}^n \mid (x, y) \in (E \backslash A)\}$.

Applications [3]

Linear descriptor difference equation:

$$E_k x_{k+1} = A_k x_k \iff (x_k, x_{k+1}) \in (E_k \backslash A_k).$$

Linear differential algebraic equation (DAE):

$$E(t)\dot{x}(t) = A(t)x(t) \iff (x, \dot{x}) \in (E(t) \backslash A(t)).$$

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For $E_1,\ A_1 \in \mathbb{C}^{m\times n}$ and $E_2,\ A_2 \in \mathbb{C}^{p\times n}$, the composite or product matrix relation of $(E_2\backslash A_2)$ with $(E_1\backslash A_1)$ is

$$(E_2\backslash A_2)(E_1\backslash A_1) = \left\{(x,z)\in\mathbb{C}^n\times\mathbb{C}^n \;\middle|\; \begin{array}{l}\text{There exists } y\in\mathbb{C}^n \text{ such that}\\ y\in(E_1\backslash A_1)x \text{ and } z\in(E_2\backslash A_2)y.\end{array}\right\}$$

$$= \left\{(x,z)\in\mathbb{C}^n\times\mathbb{C}^n \;\middle|\; \begin{array}{l}\text{There exists } y\in\mathbb{R}^n \text{ such that}\\ \begin{bmatrix} A_1 & -E_1 & 0\\ 0 & A_2 & -E_2\end{bmatrix}\begin{bmatrix} x\\ y\\ z\end{bmatrix} = 0.\end{array}\right\}$$

Remarks

a) Product matrix relation is associative with multiplicative identity $(I\backslash I)$.

b) $(E\backslash A)^{-1} = (A\backslash E)$, but $(E\backslash A)^{-1}(E\backslash A) = (I\backslash I)$ iff E^+A nonsingular.

c) Product relation may or may not have matrix representation with the same number of rows as the factors ($([1]\backslash[0])([0]\backslash[1]) = \{(0,0)\}$).

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Recall product-of-fractions formula for scalars a_1, a_2, e_1, e_2:

$$(a_1/e_1)(a_2/e_2) = (a_1 a_2)/(e_1 e_2)$$

Theorem [2]

Consider relations $(E_1 \backslash A_1)$ and $(E_2 \backslash A_2)$ where $E_1,\ A_1 \in \mathbb{C}^{m\times n}$ and $E_2,\ A_2 \in \mathbb{C}^{p\times n}$. If $\tilde{A}_2 \in \mathbb{C}^{q\times m}$ and $\tilde{E}_1 \in \mathbb{C}^{q\times p}$ satisfy

$$\mathrm{null}[\tilde{A}_2,\ \tilde{E}_1] = \mathrm{range}\begin{bmatrix} -E_1 \\ A_2 \end{bmatrix},$$

then

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Computable, e.g., via QR decomposition

$$\begin{bmatrix} -E_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} Q_{11} & \tilde{A}_2^H \\ Q_{21} & \tilde{E}_1^H \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}.$$

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$$(E_1\backslash A_1)+(E_2\backslash A_2)=\left\{(x,z)\in\mathbb{C}^n\times\mathbb{C}^n \;\middle|\; \begin{array}{l}\exists\, y_1,\, y_2\in\mathbb{C}^n,\text{ such that}\\ y_1\in(E_1\backslash A_1)x,\; y_2\in(E_2\backslash A_2)x,\\ \text{and } z=y_1+y_2.\end{array}\right\}$$

Equivalently,

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$$a_1/e_1 + a_2/e_2 = (e_2 a_1 + e_1 a_2)/(e_1 e_2)$$

Theorem [3]

Consider matrix relations $(E_1 \backslash A_1)$ and $(E_2 \backslash A_2)$ with $E_1,\ A_1 \in \mathbb{C}^{m \times n}$ and $E_2,\ A_2 \in \mathbb{C}^{p \times n}$. If $\tilde{E}_2 \in \mathbb{C}^{q \times m}$ and $\tilde{E}_1 \in \mathbb{C}^{q \times p}$ satisfy

$$\mathrm{null}[\tilde{E}_2,\ \tilde{E}_1] = \mathrm{range}\begin{bmatrix} -E_1 \\ E_2 \end{bmatrix},$$

then

$$\begin{aligned} (E_2 \backslash A_2) + (E_1 \backslash A_1) &= ((\tilde{E}_1 E_2) \backslash (\tilde{E}_2 A_1 + \tilde{E}_1 A_2)) \\ &= \left\{ (x, z) \;\middle|\; \tilde{E}_1 E_2 z = (\tilde{E}_2 A_1 + \tilde{E}_1 A_2) x \right\}. \end{aligned}$$

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Computable, e.g., via QR decomposition

$$\begin{bmatrix} -E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} Q_{11} & \tilde{E}_2^H \\ Q_{21} & \tilde{E}_1^H \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}.$$

Recall sum-of-fractions formula for scalars a_1, a_2, e_1, e_2:

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Remark

If $E_1 = E_2 = I$, then $\tilde{E}_1 = \tilde{E}_2 = I$ is a possibility. $\rightsquigarrow$ Matrix addition

$$(I \backslash A_1) + (I \backslash A_2) = (I \backslash (A_1 + A_2)).$$

Theorem [6] — products respect mutual deflating subspace

Let $(E\backslash A) = (E_2\backslash A_2)(E_1\backslash A_1)$. Suppose that $X \in \mathbb{C}^{n\times k}$, and S_1, T_1, S_2, $T_2 \in \mathbb{C}^{k\times p}$ satisfy

$$E_1 X S_1 = A_1 X T_1, \qquad E_2 X S_2 = A_2 X T_2.$$

If $\tilde{S}_1$, $\tilde{T}_2$ satisfy

$$\mathrm{null}[S_1,\, T_2] = \mathrm{range}\begin{bmatrix} -\tilde{T}_2 \\ \tilde{S}_1 \end{bmatrix},$$

then

$$EX(S_2\tilde{S}_1) = AX(T_1\tilde{T}_2).$$

Moreover, if $p = k$ and $\mathrm{rank}\left(\begin{bmatrix} -\tilde{T}_2 \\ \tilde{S}_1 \end{bmatrix}\right) = k$, then $\lambda(S_2\tilde{S}_1) - (T_1\tilde{T}_2)$ is regular and $\mathrm{range}(X)$ is a right deflating subspace of $\lambda E - A$ if and only if $[S_1,\, T_2]$ has full row rank and

$$\begin{bmatrix} T_1 & 0 \\ -S_1 & T_2 \\ 0 & -S_2 \end{bmatrix} \quad \text{has full column rank.}$$

Theorem [6] — sums respect mutual deflating subspace

Let

$$(E\backslash A) = (E_1\backslash A_1) + (E_2\backslash A_2).$$

Suppose that $X \in \mathbb{C}^{n\times k}$ and S_1, T_1, S_2, $T_2 \in \mathbb{C}^{k\times p}$ satisfy

$$E_1XS_1 = A_1XT_1, \qquad E_2XS_2 = A_2XT_2.$$

If $\tilde{T}_1$ and $\tilde{T}_2$ satisfy $\mathrm{null}[-T_1, T_2] = \mathrm{range}\begin{bmatrix}\tilde{T}_2\\ \tilde{T}_1\end{bmatrix}$, then

$$EX(S_1\tilde{T}_2 + S_2\tilde{T}_1) = AX(T_1\tilde{T}_2).$$

Moreover, if $\lambda(T_1\tilde{T}_2) - (S_1\tilde{T}_2 + S_2\tilde{T}_1)$ is regular, then $\mathrm{range}(X)$ is a right deflating subspace of $\lambda E - A$.

Consider linear, time invariant, DAE

$$E\dot{x} = Ax, \qquad \text{where } E, A \in \mathbb{C}^{m\times n} \tag{1}$$

and $x = x(t) : \mathbb{R} \to \mathbb{C}^n$ is a classical, smooth solution.

Well-known: if $\mathrm{range}(A) \subset \mathrm{range}(E)$ and E has full column rank n, then $\forall\, t_0, t_1 \in \mathbb{R}^n$:

$$x(t_1) = \exp(E^+ A(t_1 - t_0))x(t_0).$$

This can be extended easily to the cases

- $\mathrm{range}(A) \not\subset \mathrm{range}(E)$,
- E rank-deficient

using pencil arithmetic!

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Definition [3]

The exponential relation is defined by

$$\exp(E\backslash(A(t_1 - t_0))) = \sum_{k=0}^{\infty} \frac{(t_1 - t_0)^k}{k!} (E\backslash A)^k.$$

Consider linear, time invariant, DAE

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Theorem [3]

a) If $\lambda E - A$ is regular, then $x(t)$ is a classical solution of (1) iff $\forall\, t_0,\, t_1 \in \mathbb{R}$,

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b) $\exists$ classical solution $x(t)$ of (1) with $x(t_0) = x_0$ and $x(t_1) = x_1$ iff

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Extension to singular pencils possible.
($\exp(E\backslash(A(t_1 - t_0)))$ does not capture this case completely.)

2D Systems

Consider the following system of linear relations

$$\begin{aligned} Bx_{k+1,\ell} &= Ax_{k,\ell}, & Dx_{k+1,\ell+1} &= Cx_{k+1,\ell}, \\ Dx_{k,\ell+1} &= Cx_{k,\ell}, & Bx_{k+1,\ell+1} &= Ax_{k,\ell+1}, \end{aligned} \tag{2}$$

on an infinite two-dimensional grid with "basic cell"

$$\begin{array}{ccc} x_{k+1,\ell} & \overset{C}{\underset{D}{\rightleftarrows}} & x_{k+1,\ell+1} \\ A \uparrow \downarrow B & & A \uparrow \downarrow B \\ x_{k,\ell} & \overset{C}{\underset{D}{\rightleftarrows}} & x_{k,\ell+1} \end{array}$$

Question: are the two paths $x_{k,\ell} \to x_{k+1,\ell+1}$ compatible?

Trivial answer if B, D invertible:

$$x_{k+1,\ell+1} = D^{-1}CB^{-1}Ax_{k,\ell} = B^{-1}AD^{-1}Cx_{k,\ell}, \quad \forall x_{k,\ell}, \quad \text{i.e., iff } (B\backslash A) \text{ and } (D\backslash C) \text{ commute!}$$

Here: consider *descriptor systems*, i.e., B, D singular.

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Some Applications

Periodic 2D Decriptor Systems [B./Van Dooren 2023]

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Consider the following two point boundary value problems

$$\begin{bmatrix} -A & B & 0 \\ 0 & -C & D \\ W_{k,\ell} & 0 & W_{k+1,\ell+1} \end{bmatrix} \begin{bmatrix} x_{k,\ell} \\ x_{k+1,\ell} \\ x_{k+1,\ell+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}, \tag{3}$$

and

$$\begin{bmatrix} -C & D & 0 \\ 0 & -A & B \\ W_{k,\ell} & 0 & W_{k+1,\ell+1} \end{bmatrix} \begin{bmatrix} x_{k,\ell} \\ x_{k,\ell+1} \\ x_{k+1,\ell+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}, \tag{4}$$

where w is an arbitrary n-vector and $W_{k,\ell}$ and $W_{k+1,\ell+1}$ are $n \times n$ matrices that make the systems (3) and (4) have a unique solution.

Definition

The 2D periodic descriptor system (3) is *conditionable* if

$$\left.\begin{bmatrix} D & & & & \\ -A & B & & & \\ & -C & D & & \\ & & \ddots & \ddots & \\ & & & -A & B \\ & & & & -C \end{bmatrix}\right\} m \text{ block rows}$$

has full column rank for all m.

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Theorem

Let the 2D periodic system given in (2) be conditionable. Then any two trajectories

$$x_{k,\ell} \to x_{k+1,\ell} \to x_{k+1,\ell+1}$$

and

$$x_{k,\ell} \to x_{k,\ell+1} \to x_{k+1,\ell+1}$$

corresponding to the two-point BVPs (3) *and* (4) *have the same end-points* $x_{k,\ell}$ *and* $x_{k+1,\ell+1}$ *for all conditionable end-point conditions* $W_{k,\ell}$, $W_{k+1,\ell+1}$ *and* w, *if and only if the orthogonal complements* $\begin{bmatrix} -C_+ & B_+ \end{bmatrix} \in \mathbb{C}^{n\times 2n}$ *and* $\begin{bmatrix} -A_+ & D_+ \end{bmatrix} \in \mathbb{C}^{n\times 2n}$ *defined from*

$$\begin{bmatrix} -C_+ & B_+ \end{bmatrix} \begin{bmatrix} B \\ -C \end{bmatrix} = 0, \quad \begin{bmatrix} -C_+ & B_+ \end{bmatrix} \begin{bmatrix} -C_+^* \\ B_+^* \end{bmatrix} = I_n,$$

$$\begin{bmatrix} -A_+ & D_+ \end{bmatrix} \begin{bmatrix} D \\ -A \end{bmatrix} = 0, \quad \begin{bmatrix} -A_+ & D_+ \end{bmatrix} \begin{bmatrix} -A_+^* \\ D_+^* \end{bmatrix} = I_n$$

satisfy

$$\operatorname{rank} \begin{bmatrix} C_+A & B_+D \\ A_+C & D_+B \end{bmatrix} = n. \tag{3}$$

Definition

Two matrices $A, B \in \mathbb{C}^{n\times n}$ *commute* if

$$AB = BA,$$

i.e., if their *commuatator* or *Lie bracket* is zero:

$$[A, B] := AB - BA = 0.$$

(Trivial) example

If $A, B \in \mathbb{C}^{n\times n}$ are both diagonal, then they commute.

Theorem (Horn/Johnson, "Matrix Analysis", CUP)

If two matrices $A, B \in \mathbb{C}^{n\times n}$ are simultaneously diagonalizable, i.e., $\exists P \in \mathbb{C}^{n\times n}$ nonsingular such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal, then they commute.

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Mass and stiffness matrices of a finite element discretization are simultaneously diagonalizable and thus commute, which is the basis of Modal Analysis.

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Sets of Commuting Matrices

Definition

A set of matrices $\mathcal{M} := \{A_j, j = 1, \ldots, m\}$ is said to be *commuting* if every pair from $\mathcal{M}$ commutes, i.e.

$$A_i A_j = A_j A_i \quad \forall\, i, j \in \{1, \ldots, m\}.$$

Theorem (e.g., Gantmacher 1959)

If a matrix $M \in \mathcal{M}$ from a commuting set has a block-diagonal decomposition

$$T^{-1} M T = \begin{bmatrix} M_{1,1} & 0 \\ 0 & M_{2,2} \end{bmatrix}, \quad M_{1,1} \in \mathbb{C}^{n_1 \times n_1}, M_{2,2} \in \mathbb{C}^{n_2 \times n_2},$$

where the spectra of $M_{1,1}$ and $M_{2,2}$ are disjoint, then every other matrix in that set has a similar block-diagonal decomposition, using the same similarity transformation T.
When partitioned as

$$T = \begin{bmatrix} T_1 & T_2 \end{bmatrix},$$

the column spaces of T_1 and T_2 span the spaces $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. These spaces are also invariant for every matrix in the set $\mathcal{M}$.

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Theorem (B./Van Dooren 2023)

Given two $n \times n$ pencils $zB_1 - A_1$ and $zB_2 - A_2$ so that $\begin{bmatrix} -A_1 & zB_1 \\ zB_2 & -A_2 \end{bmatrix}$ is regular. Under this condition, the pencils $zB_1 - A_1$ and $zB_2 - A_2$ commute if and only if

$$\operatorname{rank} \begin{bmatrix} -A_{+2}A_1 & B_{+1}B_2 \\ -A_{+1}A_2 & B_{+2}B_1 \end{bmatrix} = n, \tag{4}$$

where A_{+1}, A_{+2}, B_{+1} and B_{+2} are defined via

$$\ker \begin{bmatrix} A_{+2} & B_{+1} \end{bmatrix} = \operatorname{range} \begin{bmatrix} B_1 \\ -A_2 \end{bmatrix}, \quad \ker \begin{bmatrix} A_{+1} & B_{+2} \end{bmatrix} = \operatorname{range} \begin{bmatrix} B_2 \\ -A_1 \end{bmatrix}.$$

Moreover, the pencils $zB_1 - A_1$ and $zB_2 - A_2$ will then also be regular.

Note: if B_1 and B_2 are invertible, then the rank condition (4) is equivalent to the commutativity $B_1^{-1}A_1$ and $B_2^{-1}A_2$, meaning

$$B_2^{-1}A_2B_1^{-1}A_1 = B_1^{-1}A_1B_2^{-1}A_2.$$

The rank condition (4) can be computationally tested with $\mathcal{O}(n^3)$ flops [B./Byers 2006].
Not so for a set of $m > 2$ matrix pencils $\rightsquigarrow$ $m(m-1)/2$ such tests.

Theorem (B./Van Dooren 2023)

Given two $n \times n$ pencils $zB_1 - A_1$ and $zB_2 - A_2$ so that $\begin{bmatrix} -A_1 & zB_1 \\ zB_2 & -A_2 \end{bmatrix}$ is regular. Under this condition, the pencils $zB_1 - A_1$ and $zB_2 - A_2$ commute if and only if

$$\operatorname{rank} \begin{bmatrix} -A_{+2}A_1 & B_{+1}B_2 \\ -A_{+1}A_2 & B_{+2}B_1 \end{bmatrix} = n, \tag{4}$$

where A_{+1}, A_{+2}, B_{+1} and B_{+2} are defined via

$$\ker \begin{bmatrix} A_{+2} & B_{+1} \end{bmatrix} = \operatorname{range} \begin{bmatrix} B_1 \\ -A_2 \end{bmatrix}, \quad \ker \begin{bmatrix} A_{+1} & B_{+2} \end{bmatrix} = \operatorname{range} \begin{bmatrix} B_2 \\ -A_1 \end{bmatrix}.$$

Moreover, the pencils $zB_1 - A_1$ and $zB_2 - A_2$ will then also be regular.

Note: if B_1 and B_2 are invertible, then the rank condition (4) is equivalent to the commutativity $B_1^{-1}A_1$ and $B_2^{-1}A_2$, meaning

$$B_2^{-1}A_2B_1^{-1}A_1 = B_1^{-1}A_1B_2^{-1}A_2.$$

The rank condition (4) can be computationally tested with $\mathcal{O}(n^3)$ flops [B./Byers 2006].
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Definition

A set of pencils $\mathcal{P} := \{zB_j - A_j, j = 1, \ldots, m\}$ with B_j invertible is *commuting* if every pair of left quotients $(B_i^{-1}A_i, B_j^{-1}A_j)$ commutes.

(Generalizes to singular B_j using commutativity of all $(B_i \backslash A_i)$ and $(B_j \backslash A_j)$.)

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Theorem

If a pencil $(zB - A) \in \mathcal{P}$ from a commuting set has a block-diagonal decomposition

$$S(zB - A)T = \operatorname{diag}\left(zB_{1,1} - A_{1,1}, \ldots, zB_{k,k} - A_{k,k}\right), \quad B_{i,i}, A_{i,i} \in \mathbb{C}^{n_i \times n_i},$$

where the spectra of the diagonal pencils $zB_{i,i} - A_{i,i}$ are disjoint, then every pencil in $\mathcal{P}$ has such a decomposition, using equivalence transformations (S_j, T) with the same right transformation T.

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A set of diagonalizable pencils $\mathcal{P}$ is a commuting set if and only if the pencils are simultaneously diagonalizable using equivalence transformations (S_j, T) with the same right transformation T.

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1. This allows computational $\mathcal{O}(n^3)$ algorithm to test commutativity based on Jacobi algorithm
2. Case of singular B_i can be treated via Moebius transformations.

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All the best, Paul!

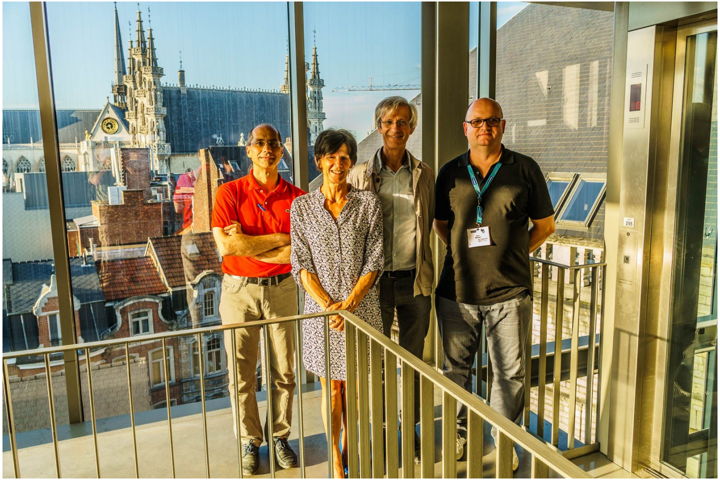

Photo taken by Frank Uhlig during ILAS conference, July 2016, in Leuven.