ADI-based Galerkin-Methods for Algebraic Lyapunov and Riccati Equations

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Overview

Algebraic Riccati equation (ARE) for $A, G = G^T, W = W^T \in \mathbb{R}^{n \times n}$ given and $X \in \mathbb{R}^{n \times n}$ unknown:

$$0 = \mathcal{R}(X) := A^T X + XA - XGX + W.$$

$G = 0 \Longrightarrow$ Lyapunov equation:

$$0 = \mathcal{L}(X) := A^T X + XA + W.$$

Typical situation in model reduction and optimal control problems for semi-discretized PDEs:

- $n = 10^3 - 10^6$ ($\Longrightarrow 10^6 - 10^{12}$ unknowns!),
- A has sparse representation ($A = -M^{-1}S$ for FEM),
- G, W low-rank with $G, W \in \{BB^T, C^T C\}$, where $B \in \mathbb{R}^{n \times m}$, $m \ll n$, $C \in \mathbb{R}^{p \times n}$, $p \ll n$.
- Standard (eigenproblem-based) $\mathcal{O}(n^3)$ methods are not applicable!

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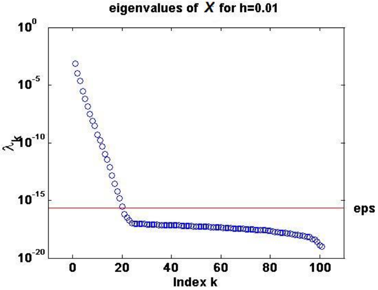
Consider spectrum of ARE solution (analogous for Lyapunov equations).

Example:

- Linear 1D heat equation with point control,
- $\Omega = [0, 1]$,
- FEM discretization using linear B-splines,
- $h = 1/100 \implies n = 101$.



Idea: $X = X^T \geq 0 \implies$

$$X = ZZ^T = \sum_{k=1}^{n} \lambda_k z_k z_k^T \approx Z^{(r)}(Z^{(r)})^T = \sum_{k=1}^{r} \lambda_k z_k z_k^T.$$

$\implies$ Goal: compute $Z^{(r)} \in \mathbb{R}^{n \times r}$ directly w/o ever forming X!

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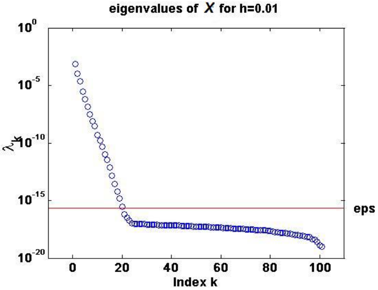

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Numerical solution of linear-quadratic optimal control problem for parabolic PDEs via Galerkin approach, spatial FEM discretization $\rightsquigarrow$

LQR Problem (finite-dimensional)

$$\text{Min } \mathcal{J}(u) = \frac{1}{2}\int_0^\infty \left(y^T Qy + u^T Ru\right) dt \quad \text{for} \quad u \in \mathcal{L}_2(0,\infty;\mathbb{R}^m),$$

subject to $M\dot{x} = -Sx + Bu, \quad x(0) = x_0, \quad y = Cx,$
with stiffness $S \in \mathbb{R}^{n\times n}$, mass $M \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$.

Solution of finite-dimensional LQR problem: feedback control

$$u_*(t) = -B^T X_* x(t) =: -K_* x(t),$$

where $X_* = X_*^T \geq 0$ is the unique stabilizing[1] solution of the ARE

$$0 = \mathcal{R}(X) := \mathcal{C}^T\mathcal{C} + \mathcal{A}^T X + X\mathcal{A} - X\mathcal{B}\mathcal{B}^T X,$$

with $\mathcal{A} := -M^{-1}S$, $\mathcal{B} := M^{-1}BR^{-\frac{1}{2}}$, $\mathcal{C} := CQ^{-\frac{1}{2}}$.

[1] X is stabilizing $\Leftrightarrow \Lambda(\mathcal{A} - \mathcal{B}\mathcal{B}^T X) \subset \mathbb{C}^-$.

Motivation
Linear-quadratic Optimal Control

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Linear, Time-Invariant (LTI) Systems

$$\Sigma : \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax + Bu, \\ y(t) & = & Cx + Du, \end{array} \right. \qquad \begin{array}{ll} A \in \mathbb{R}^{n\times n}, & B \in \mathbb{R}^{n\times m}, \\ C \in \mathbb{R}^{p\times n}, & D \in \mathbb{R}^{p\times m}. \end{array}$$

(A, B, C, D) is a realization of Σ (nonunique).

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Model Reduction Based on Balancing

Given $P, Q \in \mathbb{R}^{n\times n}$ symmetric positive definite (spd), and a contragredient transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$TPT^T = T^{-T}QT^{-1} = \mathrm{diag}(\sigma_1, \ldots, \sigma_n), \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0.$$

Balancing Σ w.r.t. P, Q:

$$\Sigma \equiv (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D) \equiv \Sigma.$$

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For Balanced Truncation: $P/Q =$ controllability/observability Gramian of Σ, i.e., for asymptotically stable systems, P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0.$$

Basic Model Reduction Procedure

1. Given $\Sigma \equiv (A, B, C, D)$ and balancing (w.r.t. given P, Q spd) transformation $T \in \mathbb{R}^{n\times n}$ nonsingular, compute

$$\begin{aligned}(A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D)\\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D\right)\end{aligned}$$

2. Truncation $\rightsquigarrow$ reduced-order model:

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Implementation: SR Method

1. Given Cholesky (square) or (low-rank approximation to) full-rank (maybe rectangular, "thin") factors of P, Q
$$P = S^T S, \quad Q = R^T R.$$
2. Compute SVD
$$SR^T = [\, U_1,\ U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$
3. Set
$$W = R^T V_1 \Sigma_1^{-1/2}, \qquad V = S^T U_1 \Sigma_1^{-1/2}.$$
4. Reduced-order model is
$$(\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (W^T A V, W^T B, CV, D) \ \ (\equiv (A_{11}, B_1, C_1, D).)$$

Recall Peaceman Rachford ADI:
Consider $Au = s$ where $A \in \mathbb{R}^{n \times n}$ spd, $s \in \mathbb{R}^n$. ADI Iteration Idea:
Decompose $A = H + V$ with $H, V \in \mathbb{R}^{n \times n}$ such that

$$(H + pI)v = r$$
$$(V + pI)w = t$$

can be solved easily/efficiently.

ADI Iteration

If H, V spd $\Rightarrow \exists p_k$, $k = 1, 2, \ldots$ such that

$$\begin{aligned} u_0 &= 0 \\ (H + p_k I)u_{k-\frac{1}{2}} &= (p_k I - V)u_{k-1} + s \\ (V + p_k I)u_k &= (p_k I - H)u_{k-\frac{1}{2}} + s \end{aligned}$$

converges to $u \in \mathbb{R}^n$ solving $Au = s$.

ADI Method for Lyapunov Equations
Background

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ADI Method for Lyapunov Equations
Motivation

The Lyapunov operator

$$\mathcal{L}: \quad P \quad \mapsto \quad AX + XA^T$$

can be decomposed into the linear operators

$$\mathcal{L}_H : X \mapsto AX \qquad \mathcal{L}_V : X \mapsto XA^T.$$

In analogy to the standard ADI method we find the

ADI iteration for the Lyapunov equation [Wachspress 1988

$$\begin{aligned} P_0 &= 0 \\ (A + p_k I)X_{k-\frac{1}{2}} &= -W - P_{k-1}(A^T - p_k I) \\ (A + p_k I)X_k^T &= -W - X_{k-\frac{1}{2}}^T (A^T - p_k I) \end{aligned}$$

Low-Rank ADI for Lyapunov equations

- For $A \in \mathbb{R}^{n\times n}$ stable, $B \in \mathbb{R}^{n\times m}$ $(w \ll n)$, consider Lyapunov equation
$$AX + XA^T = -BB^T.$$
- ADI Iteration: [Wachspress 1988]
$$\begin{aligned}(A + p_k I)X_{k-\frac{1}{2}} &= -BB^T - X_{k-1}(A^T - p_k I)\\ (A + \overline{p_k} I){X_k}^T &= -BB^T - X_{k-\frac{1}{2}}(A^T - \overline{p_k} I)\end{aligned}$$
with parameters $p_k \in \mathbb{C}^-$ and $p_{k+1} = \overline{p_k}$ if $p_k \notin \mathbb{R}$.
- For $X_0 = 0$ and proper choice of p_k: $\lim_{k\to\infty} X_k = X$ superlinear.
- Re-formulation using $X_k = Y_k Y_k^T$ yields iteration for Y_k...

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Lyapunov equation $0 = AX + XA^T + BB^T$.

Setting $X_k = Y_k Y_k^T$, some algebraic manipulations $\Longrightarrow$

Algorithm [Penzl '97/'00, Li/White '99/'02, B. 04, B./Li/Penzl '99/'08]

$$V_1 \leftarrow \sqrt{-2\mathrm{Re}\,(p_1)}(A + p_1 I)^{-1}B, \qquad Y_1 \leftarrow V_1$$

FOR $k = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{Re}\,(p_k)}{\mathrm{Re}\,(p_{k-1})}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1} V_{k-1}\right)$$

$$Y_k \leftarrow \left[\begin{array}{cc} Y_{k-1} & V_k \end{array}\right]$$

$$Y_k \leftarrow \mathrm{rrlq}(Y_k, \tau) \qquad \text{\% column compression}$$

At convergence, $Y_{k_{\max}} Y_{k_{\max}}^T \approx X$, where (without column compression)

$$Y_{k_{\max}} = \left[\begin{array}{ccc} V_1 & \ldots & V_{k_{\max}} \end{array}\right], \quad V_k = \left[\;\right] \in \mathbb{C}^{n \times m}.$$

Note: Implementation in real arithmetic possible by combining two steps.

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Numerical Results

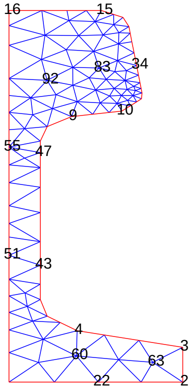
Optimal Cooling of Steel Profiles

- Mathematical model: boundary control for linearized 2D heat equation.

$$
\begin{aligned}
c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, && \xi \in \Omega \\
\lambda \frac{\partial}{\partial n} x &= \kappa(u_k - x), && \xi \in \Gamma_k,\ 1 \leq k \leq 7, \\
\frac{\partial}{\partial n} x &= 0, && \xi \in \Gamma_7.
\end{aligned}
$$

$\Longrightarrow m = 7, p = 6.$

- FEM Discretization, different models for initial mesh ($n = 371$), 1, 2, 3, 4 steps of mesh refinement $\Rightarrow$ $n = 1357, 5177, 20209, 79841.$



Source: Physical model: courtesy of Mannesmann/Demag.

Math. model: Tröltzsch/Unger 1999/2001, Penzl 1999, Saak 2003.

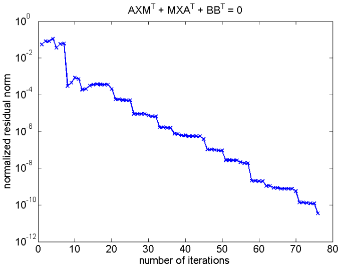
Numerical Results

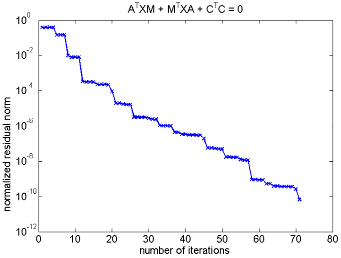
Optimal Cooling of Steel Profiles

- Solve dual Lyapunov equations needed for balanced truncation, i.e.,

$$APM^T + MPA^T + BB^T = 0, \qquad A^TQM + M^TQA + C^TC = 0,$$

  for 79,841. Note: $m = 7, p = 6$.
- 25 shifts chosen by Penzl's heuristic from 50/25 Ritz values of A of largest/smallest magnitude, no column compression performed.
- New version in MESS (Matrix Equations Sparse Solvers) requires no factorization of mass matrix!
- Computations done on Core2Duo at 2.8GHz with 3GB RAM and 32Bit-MATLAB®.





- $A \in \mathbb{R}^{n \times n} \equiv$ FDM matrix for 2D heat equation on $[0,1]^2$ (LYAPACK benchmark `demo_l1`, $m = 1$).
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CPU Times

n	CMESS	LYAPACK	MESS
100	0.023	0.124	0.158
625	0.042	0.104	0.227
2,500	0.159	0.702	0.989
10,000	0.965	6.22	5.644
40,000	11.09	71.48	34.55
90,000	34.67	418.5	90.49
160,000	109.3	out of memory	219.9
250,000	193.7	out of memory	403.8
562,500	930.1	out of memory	1216.7
1,000,000	2220.0	out of memory	2428.6

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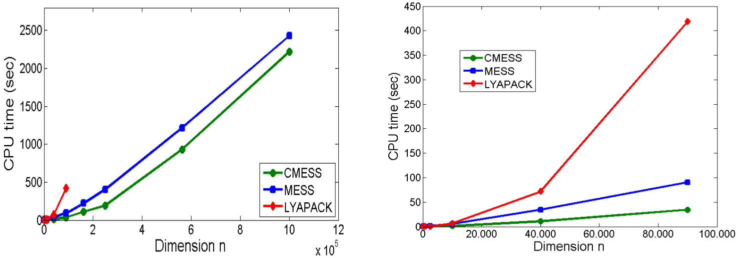

Note: for n=1,000,000, first sparse LU needs $\sim 1,100$ sec., using UMFPACK this reduces to 30 sec.

Factored Galerkin-ADI Iteration

Lyapunov equation $0 = AX + XA^T + BB^T$

Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis $\mathrm{range}(Z)$, $Z \in \mathbb{R}^{n \times r}$, for subspace $\mathcal{Z} \subset \mathbb{R}^n$, $\dim \mathcal{Z} = r$.
2. Set $\hat{A} := Z^T AZ$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A}\hat{X} + \hat{X}\hat{A}^T + \hat{B}\hat{B}^T = 0$.
4. Use $X \approx Z\hat{X}Z^T$.

Examples:

- Krylov subspace methods, i.e., for $m = 1$:

$$\mathcal{Z} = \mathcal{K}(A, B, r) = \mathrm{span}\{B, AB, A^2B, \dots, A^{r-1}B\}$$

  [Saad '90, Jaimoukha/Kasenally '94, Jbilou '02–'08].

- K-PIK [Simoncini '07],

$$\mathcal{Z} = \mathcal{K}(A, B, r) \cup \mathcal{K}(A^{-1}, B, r).$$

Factored Galerkin-ADI Iteration

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Examples:

- ADI subspace [B./R.-C. Li/Truhar '08]:

$$\mathcal{Z} = \mathrm{colspan}\left[\; V_1, \quad \ldots, \quad V_r \;\right].$$

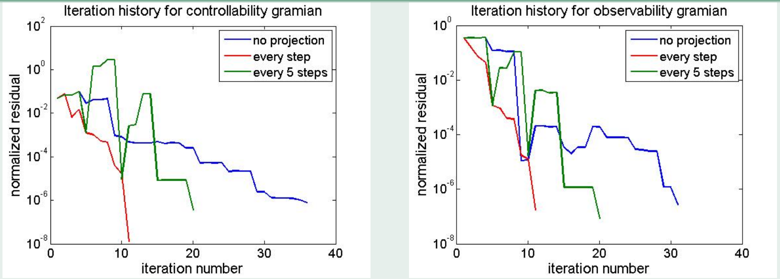
Note:

1. ADI subspace is rational Krylov subspace [J.-R. Li/White '02].
2. Similar approach: ADI-preconditioned global Arnoldi method [Jbilou '08].

FEM semi-discretized control problem for parabolic PDE:

- optimal cooling of rail profiles,
- $n = 20,209, \ m = 7, \ p = 6.$

Good ADI shifts



CPU times: 80s (projection every 5th ADI step) vs. 94s (no projection).

Computations by Jens Saak.

Factored Galerkin-ADI Iteration

Numerical examples

FEM semi-discretized control problem for parabolic PDE:

- optimal cooling of rail profiles,
- $n = 20,209$, $m = 7$, $p = 6$.

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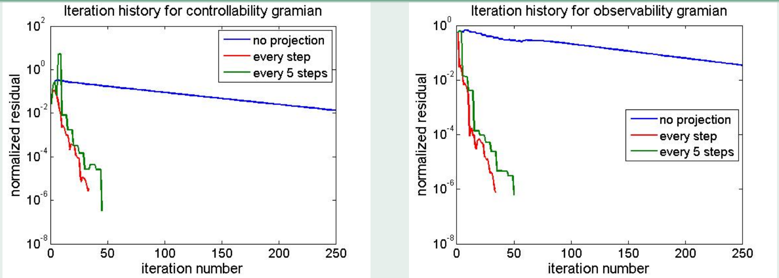


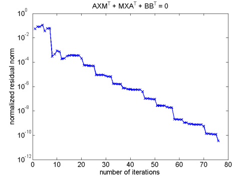
CPU times: 368s (projection every 5th ADI step) vs. 1207s (no projection).

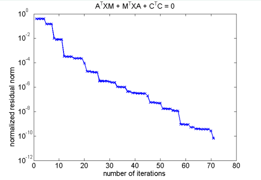
Computations by Jens Saak.

Factored Galerkin-ADI Iteration

Numerical examples: optimal cooling of rail profiles, $n = 79,841$, $m = 7$, $p = 6$.

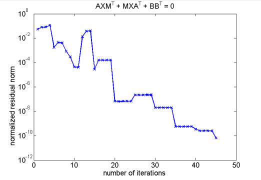
MESS w/o Galerkin projection and column compression

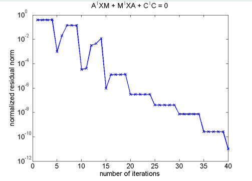




Rank of solution factors: 532 / 426

MESS with Galerkin projection and column compression





Rank of solution factors: 269 / 205

Newton's Method for AREs [Kleinman '68, Mehrmann '91, Lancaster/Rodman '95, B./Byers '94/'98, B. '97, Guo/Laub '99]

- Consider $0 = \mathcal{R}(X) = C^TC + A^TX + XA - XBB^TX.$
- Frechét derivative of $\mathcal{R}(X)$ at X:

$$\mathcal{R}'_X : Z \to (A - BB^TX)^TZ + Z(A - BB^TX).$$

- Newton-Kantorovich method:

$$X_{j+1} = X_j - \left(\mathcal{R}'_{X_j}\right)^{-1}\mathcal{R}(X_j), \quad j = 0, 1, 2, \ldots$$

Newton's method (with line search) for AREs

FOR $j = 0, 1, \ldots$

1. $A_j \leftarrow A - BB^TX_j =: A - BK_j.$
2. Solve the Lyapunov equation $A_j^TN_j + N_jA_j = -\mathcal{R}(X_j).$
3. $X_{j+1} \leftarrow X_j + t_jN_j.$

END FOR j

Newton-ADI for AREs

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Factored Galerkin-ADI Iteration

Properties and Implementation

- Convergence for K_0 stabilizing:
  - $A_j = A - BK_j = A - BB^T X_j$ is stable $\forall\ j \geq 0$.
  - $\lim_{j\to\infty} \|\mathcal{R}(X_j)\|_F = 0$ (monotonically).
  - $\lim_{j\to\infty} X_j = X_* \geq 0$ (locally quadratic).
- Need large-scale Lyapunov solver; here, ADI iteration: linear systems with dense, but "sparse+low rank" coefficient matrix A_j:

$$A_j = A - B \cdot K_j$$

= sparse − m · [ ]

- $m \ll n \implies$ efficient "inversion" using Sherman-Morrison-Woodbury formula:

$$(A - BK_j + p_k^{(j)} I)^{-1} = (I_n + (A + p_k^{(j)} I)^{-1} B (I_m - K_j (A + p_k^{(j)} I)^{-1} B)^{-1} K_j)(A + p_k^{(j)} I)^{-1}.$$

- BUT: $X = X^T \in \mathbb{R}^{n\times n} \implies n(n+1)/2$ unknowns!

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Low-Rank Newton-ADI for AREs

Re-write Newton's method for AREs

$$A_j^T N_j + N_j A_j = -\mathcal{R}(X_j)$$

$$\Longleftrightarrow$$

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + \underbrace{(X_j + N_j)}_{=X_{j+1}} A_j = \underbrace{-C^T C - X_j B B^T X_j}_{=:-W_j W_j^T}$$

Set $X_j = Z_j Z_j^T$ for $\operatorname{rank}(Z_j) \ll n \Longrightarrow$

$$A_j^T \left(Z_{j+1} Z_{j+1}^T\right) + \left(Z_{j+1} Z_{j+1}^T\right) A_j = -W_j W_j^T$$

Factored Newton Iteration [B./Li/Penzl 1999/2008]

Solve Lyapunov equations for Z_{j+1} directly by factored ADI iteration and use 'sparse + low-rank' structure of A_j.

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Optimal feedback

$$K_* = B^T X_* = B^T Z_* Z_*^T$$

can be computed by direct feedback iteration:

- jth Newton iteration:

$$K_j = B^T Z_j Z_j^T = \sum_{k=1}^{k_{\max}} (B^T V_{j,k}) V_{j,k}^T \xrightarrow{j\to\infty} K_* = B^T Z_* Z_*^T$$

- K_j can be updated in ADI iteration, no need to even form Z_j, need only fixed workspace for $K_j \in \mathbb{R}^{m\times n}$!

Related to earlier work by [Banks/Ito 1991].

Basic ideas

- Hybrid method of Galerkin projection methods for AREs [Jaimoukha/Kasenally '94, Jbilou '06, Heyouni/Jbilou '09] and Newton-ADI, i.e., use column space of current Newton iterate for projection, solve projected ARE, and prolongate.
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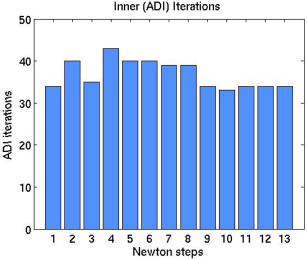
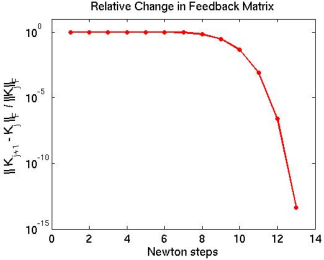
- Independence of good parameters observed for Galerkin-ADI applied to Lyapunov equations $\rightsquigarrow$ fix ADI parameters for all Newton iterations.

- Linear 2D heat equation with homogeneous Dirichlet boundary and point control/observation.
- FD discretization on uniform 150×150 grid.
- $n = 22.500$, $m = p = 1$, 10 shifts for ADI iterations.
- Convergence of large-scale matrix equation solvers:





Numerical Results

LQR Problem for 2D Geometry

- FDM for 2D heat/convection-diffusion equations on $[0,1]^2$ (LYAPACK benchmarks, $m = p = 1$) $\rightsquigarrow$ symmetric/nonsymmetric $A \in \mathbb{R}^{n \times n}$, $n = 10,000$.
- 15 shifts chosen by Penzl's heuristic from 50/25 Ritz/harmonic Ritz values of A.
- Computations using Intel Core 2 Quad CPU of type Q9400 at 2.66GHz with 4 GB RAM and 64Bit-MATLAB.

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Newton-ADI

step	rel. change	rel. residual	ADI
1	1	9.99e–01	200
2	9.99e–01	3.41e+01	23
3	5.25e–01	6.37e+00	20
4	5.37e–01	1.52e+00	20
5	7.03e–01	2.64e–01	23
6	5.57e–01	1.56e–02	23
7	6.59e–02	6.30e–05	23
8	4.02e–04	9.68e–10	23
9	8.45e–09	1.09e–11	23
10	1.52e–14	1.09e–11	23

CPU time: 76.9 sec.

- FDM for 2D heat/convection-diffusion equations on $[0,1]^2$ (LYAPACK benchmarks, $m = p = 1$) $\rightsquigarrow$ symmetric/nonsymmetric $A \in \mathbb{R}^{n \times n}$, $n = 10,000$.
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5	7.03e–01	2.64e–01	23
6	5.57e–01	1.56e–02	23
7	6.59e–02	6.30e–05	23
8	4.02e–04	9.68e–10	23
9	8.45e–09	1.09e–11	23
10	1.52e–14	1.09e–11	23

CPU time: 76.9 sec.

Newton-Galerkin-ADI

step	rel. change	rel. residual	ADI
1	1	3.56e–04	20
2	5.25e–01	6.37e+00	10
3	5.37e–01	1.52e+00	6
4	7.03e–01	2.64e–01	10
5	5.57e–01	1.57e–02	10
6	6.59e–02	6.30e–05	10
7	4.03e–04	9.79e–10	10
8	8.45e–09	1.43e–15	10

CPU time: 38.0 sec.

Numerical Results

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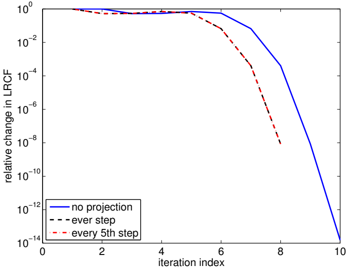

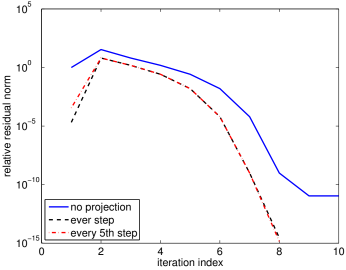

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1	1	9.99e–01	200
2	9.99e–01	3.56e+01	60
3	3.11e–01	3.72e+00	39
4	2.88e–01	9.62e–01	40
5	3.41e–01	1.68e–01	45
6	1.22e–01	5.25e–03	42
7	3.88e–03	2.96e–06	47
8	2.30e–06	6.09e–13	47

CPU time: 185.9 sec.

Numerical Results

LQR Problem for 2D Geometry

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8	2.30e–06	6.09e–13	47

CPU time: 185.9 sec.

Newton-Galerkin-ADI

step	rel. change	rel. residual	ADI it.
1	1	1.78e–02	35
2	3.11e–01	3.72e+00	15
3	2.88e–01	9.62e–01	20
4	3.41e–01	1.68e–01	15
5	1.22e–01	5.25e–03	20
6	3.89e–03	2.96e–06	15
7	2.30e–06	6.14e–13	20

CPU time: 75.7 sec.

Numerical Results
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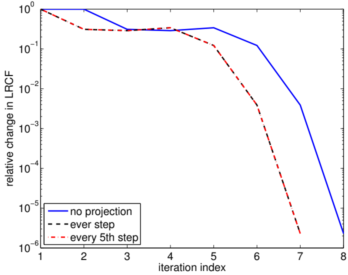
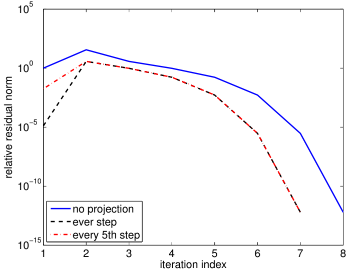
- Computations using Intel Core 2 Quad CPU of type Q9400 at 2.66GHz with 4 GB RAM and 64Bit-MATLAB.





Control problem for 3d Convection-Diffusion Equation

- FDM for 3D convection-diffusion equation on $[0,1]^3$
- proposed in [Simoncini '07], $q = p = 1$
- non-symmetric $A \in \mathbb{R}^{n \times n}$, $n = 10\,648$

Test system:

INTEL Xeon 5160 3.00GHz ; 16 GB RAM; 64Bit-MATLAB (R2010a) using threaded BLAS; stopping tolerance: 10^{-10}

Numerical Results

LQR Problem for 3D Geometry

Newton-ADI

NWT	rel. change	rel. residual	ADI
1	$1.0 \cdot 10^{0}$	$9.3 \cdot 10^{-01}$	100
2	$3.7 \cdot 10^{-02}$	$9.6 \cdot 10^{-02}$	94
3	$1.4 \cdot 10^{-02}$	$1.1 \cdot 10^{-03}$	98
4	$3.5 \cdot 10^{-04}$	$1.0 \cdot 10^{-07}$	97
5	$6.4 \cdot 10^{-08}$	$1.3 \cdot 10^{-10}$	97
6	$7.5 \cdot 10^{-16}$	$1.3 \cdot 10^{-10}$	97

CPU time: 4 805.8 sec.

NG-ADI inner= 5, outer= 1

NWT	rel. change	rel. residual	ADI
1	$1.0 \cdot 10^{0}$	$5.0 \cdot 10^{-11}$	80

CPU time: 497.6 sec.

NG-ADI inner= 1, outer= 1

NWT	rel. change	rel. residual	ADI
1	$1.0 \cdot 10^{0}$	$7.4 \cdot 10^{-11}$	71

CPU time: 856.6 sec.

NG-ADI inner= 0, outer= 1

NWT	rel. change	rel. residual	ADI
1	$1.0 \cdot 10^{0}$	$6.5 \cdot 10^{-13}$	100

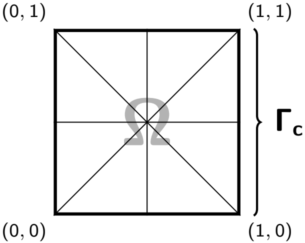
CPU time: 506.6 sec.

Test system:

INTEL Xeon 5160 3.00GHz ; 16 GB RAM; 64Bit-MATLAB (R2010a) using threaded BLAS; stopping tolerance: 10^{-10}

Numerical Results

Scaling of CPU times / Mesh Independence

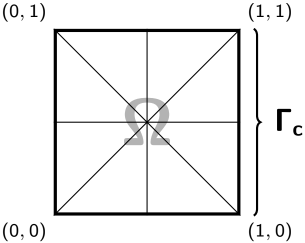


$$
\begin{aligned}
\partial_t x(\xi,t) &= \Delta x(\xi,t) && \text{in } \Omega \\
\partial_\nu x &= b(\xi)\cdot u(t) - x && \text{on } \Gamma_c \\
\partial_\nu x &= -x && \text{on } \partial\Omega \setminus \Gamma_c
\end{aligned}
$$

$$x(\xi,0) = 1$$

Note:

Here $b(\xi) = 4\,(1-\xi_2)\,\xi_2$ for $\xi \in \Gamma_c$ and 0 otherwise, thus $\forall t \in \mathbb{R}_{>0}$, we have $u(t) \in \mathbb{R}$.



$$\begin{aligned} \partial_t x(\xi,t) &= \Delta x(\xi,t) && \text{in } \Omega \\ \partial_\nu x &= b(\xi)\cdot u(t) - x && \text{on } \Gamma_c \\ \partial_\nu x &= -x && \text{on } \partial\Omega \setminus \Gamma_c \end{aligned}$$

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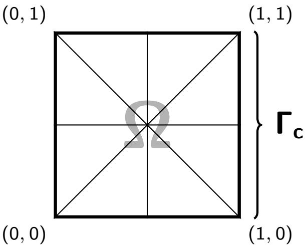
$$\Rightarrow B_h = M_{\Gamma,h} \cdot b.$$

Numerical Results

Scaling of CPU times / Mesh Independence



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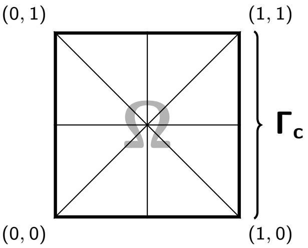
$$x(\xi, 0) = 1$$

Consider: output equation $y = Cx$, where

$$
\begin{aligned}
C : \mathcal{L}^2(\Omega) &\to \mathbb{R} \\
x(\xi, t) &\mapsto y(t) = \int_\Omega x(\xi, t)\, d\xi.
\end{aligned}
$$

Numerical Results

Scaling of CPU times / Mesh Independence



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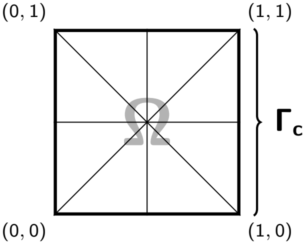
\end{aligned}
\quad \Rightarrow C_h = \underline{1} \cdot M_h.
$$

Numerical Results

Scaling of CPU times / Mesh Independence



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\end{aligned}
$$

$$x(\xi, 0) = 1$$

Cost Function:

$$\mathcal{J}(u) = \int_0^\infty y^2(t) + u^2(t)\, dt.$$

Simplified Low Rank Newton-Galerkin ADI

- generalized state space form implementation
- Penzl shifts (16/50/25) with respect to initial matrices
- projection acceleration in every outer iteration step
- projection acceleration in every 5-th inner iteration step

Test system:

INTEL Xeon 5160 @ 3.00 GHz; 16 GB RAM; 64Bit-MATLAB (R2010a) using threaded BLAS (`romulus`)
stopping criterion tolerances: 10^{-10}

Numerical Results

Scaling of CPU times / Mesh Independence

Computation Times

discretization level	problem size	time in seconds
3	81	$4.87 \cdot 10^{-2}$
4	289	$2.81 \cdot 10^{-1}$
5	1 089	$5.87 \cdot 10^{-1}$
6	4 225	2.63
7	16 641	$2.03 \cdot 10^{+1}$
8	66 049	$1.22 \cdot 10^{+2}$
9	263 169	$1.05 \cdot 10^{+3}$
10	1 050 625	$1.65 \cdot 10^{+4}$
11	4 198 401	$1.35 \cdot 10^{+5}$

Test system:

INTEL Xeon 5160 @ 3.00 GHz; 16 GB RAM; 64Bit-MATLAB (R2010a) using threaded BLAS (`romulus`)
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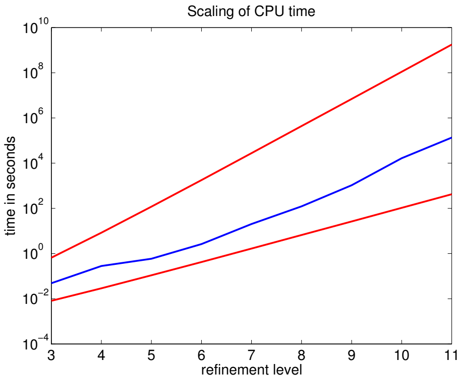
Numerical Results
Scaling of CPU times / Mesh Independence



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Basic QADI iteration [Wong/Balakrishnan et al. '05–'08]

$$\left((A - BB^T X_k)^T + p_k I\right) X_{k+\frac{1}{2}} = -W - X_k((A - p_k I)$$

$$\left((A - BB^T X_{k+\frac{1}{2}}^T)^T + p_k I\right) X_{k+1} = -W - X_{k+\frac{1}{2}}^T (A - p_k I)$$

Derivation of complicated Cholesky factor version, but requires square and invertible Cholesky factors.

Idea of low-rank Galerkin-QADI [B./Saak '09]

$V_1 \leftarrow \sqrt{-2\mathrm{Re}\,(p_1)}(A - B(B^T Y_0)Y_0^T + p_1 I)^{-T} B, \qquad Y_1 \leftarrow V_1$

FOR $k = 2, 3, \ldots$

$V_k \leftarrow V_{k-1} - (p_k + \overline{p_{k-1}})(A - B(B^T Y_{k-1})Y_{k-1}^T + p_k I)^{-T} V_{k-1}$

$Y_k \leftarrow \left[\; Y_{k-1} \quad \sqrt{\frac{\mathrm{Re}\,(p_k)}{\mathrm{Re}\,(p_{k-1})}} V_k \;\right]$

$Y_k \leftarrow \mathrm{rrlq}(Y_k, \tau)$ % column compression

If desired, project ARE onto $\mathrm{range}(Y_k)$, solve and prolongate.

Quadratic ADI for AREs

$0 = \mathcal{R}(X) = A^T X + XA - XBB^T X + W$

Basic QADI iteration [Wong/Balakrishnan et al. '05–'08]

$$
\begin{aligned}
\left((A - BB^T X_k)^T + p_k I\right) X_{k+\frac{1}{2}} &= -W - X_k((A - p_k I) \\
\left((A - BB^T X_{k+\frac{1}{2}}^T)^T + p_k I\right) X_{k+1} &= -W - X_{k+\frac{1}{2}}^T (A - p_k I)
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If desired, project ARE onto $\mathrm{range}(Y_k)$, solve and prolongate.

AREs with High-Rank Constant Term

Consider ARE

$$0 = \mathcal{R}(X) = W + A^T X + XA - XBB^T X$$

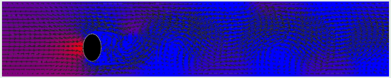
with $\operatorname{rank}(W) \not\ll n$, e.g., stabilization of flow problems described by Navier-Stokes eqns. requires solution of

$$0 = \mathcal{R}(X) = M_h - S_h^T X M_h - M_h X S_h - M_h X B_h B_h^T X M_h,$$

where M_h = mass matrix of FE velocity test functions.

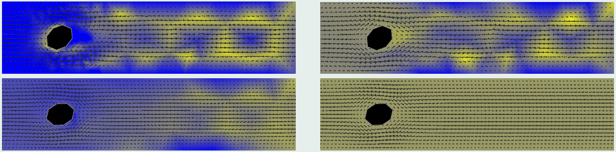
Example: von Kármán vortex street, Re= 500

uncontrolled:



controlled using ARE:

$t = 1$ $t = 5$



$t = 8$ $t = 10$

AREs with High-Rank Constant Term

Solution: remove W from r.h.s. of Lyapunov eqns. in Newton-ADI

One step of Newton-Kleinman iteration for ARE:

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + X_{j+1}A_j = -W - \underbrace{(X_j B)}_{=K_j^T} \underbrace{B^T X_j}_{=K_j} \qquad \text{for } j = 1, 2, \ldots$$

Subtract two consecutive equations $\Longrightarrow$

$$A_j^T N_j + N_j A_j = -N_{j-1}^T B B^T N_{j-1} \qquad \text{for } j = 1, 2, \ldots$$

See [Banks/Ito '91, B./Hernández/Pastor '03, Morris/Navasca '05] for details and applications of this variant.

But: need $B^T N_0 = K_1 - K_0$!

Assuming K_0 is known, need to compute K_1.

Solution idea:

$$\begin{aligned} K_1 &= B^T X_1 \\ &= B^T \int_0^\infty e^{(A-BK_0)^T t}(W + K_0^T K_0)e^{(A-BK_0)t}\, dt \\ &= \int_0^\infty g(t)\, dt \approx \sum_{\ell=0}^{N} \gamma_\ell g(t_\ell), \end{aligned}$$

where $g(t) = \left(\left(e^{(A-BK_0)t} B\right)^T (W + K_0^T K_0)\right) e^{(A-BK_0)t}$.

[Borgggaard/Stoyanov '08]:
evaluate $g(t_\ell)$ using ODE solver applied to $\dot{x} = (A-BK_0)x$ + adjoint eqn.

Better solution idea:

(related to frequency domain POD [Willcox/Peraire '02])

$$\begin{aligned} K_1 &= B^T X_1 \qquad\qquad \text{(Notation: } A_0 := A - BK_0) \\ &= B^T \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} (\jmath\omega I_n - A_0)^{-H} (W + K_0^T K_0)(\jmath\omega I_n - A_0)^{-1}\, d\omega \\ &= \int_{-\infty}^{\infty} f(\omega)\, d\omega \approx \sum_{\ell=0}^{N} \gamma_\ell f(\omega_\ell), \end{aligned}$$

where $f(\omega) = \left(-\left((\jmath\omega I_n + A_0)^{-1} B\right)^T (W + K_0^T K_0)\right) (\jmath\omega I_n - A_0)^{-1}$.

Evaluation of $f(\omega_\ell)$ requires

- 1 sparse LU decmposition (complex!),
- $2m$ forward/backward solves,
- m sparse and $2m$ low-rank matrix-vector products.

Use adaptive quadrature with high accuracy, e.g. Gauß-Kronrod (MATLAB's `quadgk`).

Software

Lyapack [Penzl 2000]

MATLAB toolbox for solving

- Lyapunov equations and algebraic Riccati equations,
- model reduction and LQR problems.

Main work horse: Low-rank ADI and Newton-ADI iterations.

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MESS – Matrix Equations Sparse Solvers [Saak/Mena/B. 2008]

- Extended and revised version of LYAPACK.
- Includes solvers for large-scale differential Riccati equations (based on Rosenbrock and BDF methods).
- Many algorithmic improvements:
  - new ADI parameter selection,
  - column compression based on RRQR,
  - more efficient use of direct solvers,
  - treatment of generalized systems without factorization of the mass matrix.
- C version CMESS under development (Martin Köhler).

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Conclusions and Open Problems

- Galerkin projection can significantly accelerate ADI iteration for Lyapunov equations.
- Low-rank Galerkin-QADI may become a viable alternative to Newton-ADI.
- High-rank constant terms in ARE can be handled using quadrature rules.
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- High-rank constant terms in ARE can be handled using quadrature rules.
- Software is available in MATLAB toolbox LYAPACK and its successor MESS.
- To-Do list:
  - computation of stabilizing initial guess.
    (If hierarchical grid structure is available, a multigrid approach is possible, other approaches based on "cheaper" matrix equations under development.)
  - Implementation of coupled Riccati solvers for LQG controller design and balancing-related model reduction.

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