MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

# Model Order Reduction for Nonlinear Systems Using Transfer Function Concepts

**Peter Benner**

11. Elgersburg Workshop
February 19–23, 2017

# Joint work with . . .

Mian Ilyas Ahmad
*National University of Science and Technology, Islamabad*

Tobias Breiten
*Karl-Franzens-Universität Graz*

**Pawan Goyal**
*MPI Magdeburg*

Jan Heiland
*MPI Magdeburg*

Imad Jaimoukha
*Imperial College London*

# Overview

# Outline

## Model Reduction for Control Systems

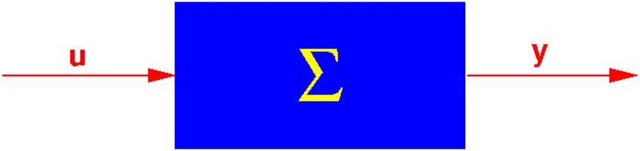
### Nonlinear Control Systems

$$\Sigma : \begin{cases} E\dot{x}(t) &= f(t, x(t), u(t)), \quad Ex(t_0) = Ex_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

with

- (generalized) states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^q$.

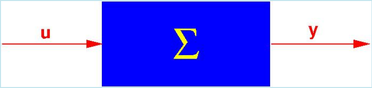
If $E$ singular $\rightsquigarrow$ descriptor system. Here, $E = I_n$ for simplicity.



## Original System ($E = I_n$)

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), \\ y(t) = g(t, x(t), u(t)). \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^q$.



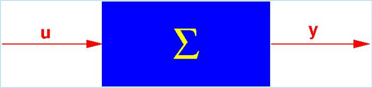

## Goals:

$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\|$ for all admissible input signals.

## Original System ($E = I_n$)

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), \\ y(t) = g(t, x(t), u(t)). \end{cases}$$

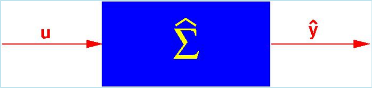
- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^q$.




## Reduced-Order Model (ROM)

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t) \in \mathbb{R}^q$.



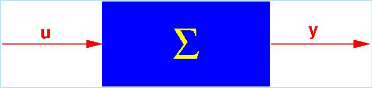

**Goals:**

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

## Original System ($E = I_n$)

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), \\ y(t) = g(t, x(t), u(t)). \end{cases}$$

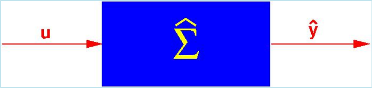
- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^q$.




## Reduced-Order Model (ROM)

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t) \in \mathbb{R}^q$.



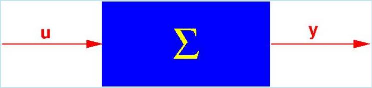

## Goals:

$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\|$ for all admissible input signals.

## Original System ($E = I_n$)

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), \\ y(t) = g(t, x(t), u(t)). \end{cases}$$

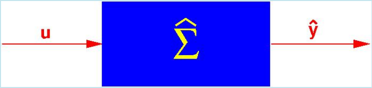
- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^q$.




## Reduced-Order Model (ROM)

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t) \in \mathbb{R}^q$.




## Goals:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Secondary goal: reconstruct approximation of $x$ from $\hat{x}$.

# System Classes

## Control-Affine (Autonomous) Systems

$$\begin{aligned}
\dot{x}(t) &= f(t,x,u) &= \mathcal{A}(x(t)) + \mathcal{B}(x(t))u(t), \quad \mathcal{A}:\mathbb{R}^n \to \mathbb{R}^{n\times n},\ \mathcal{B}:\mathbb{R}^n \to \mathbb{R}^{n\times m},\\
y(t) &= g(t,x,u) &= \mathcal{C}(x(t)) + \mathcal{D}(x(t))u(t), \quad \mathcal{C}:\mathbb{R}^n \to \mathbb{R}^{q\times n},\ \mathcal{D}:\mathbb{R}^n \to \mathbb{R}^{q\times m}.
\end{aligned}$$

# System Classes

## Control-Affine (Autonomous) Systems

$$
\begin{aligned}
\dot{x}(t) &= f(t,x,u) = \mathcal{A}(x(t)) + \mathcal{B}(x(t))u(t), \quad \mathcal{A}: \mathbb{R}^n \to \mathbb{R}^{n\times n},\ \mathcal{B}: \mathbb{R}^n \to \mathbb{R}^{n\times m},\\
y(t) &= g(t,x,u) = \mathcal{C}(x(t)) + \mathcal{D}(x(t))u(t), \quad \mathcal{C}: \mathbb{R}^n \to \mathbb{R}^{q\times n},\ \mathcal{D}: \mathbb{R}^n \to \mathbb{R}^{q\times m}.
\end{aligned}
$$

## Linear, Time-Invariant (LTI) Systems

$$
\begin{aligned}
\dot{x}(t) &= f(t,x,u) = Ax(t) + Bu(t), \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\\
y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}.
\end{aligned}
$$

## Control-Affine (Autonomous) Systems

$$\begin{aligned}\dot{x}(t) &= f(t,x,u) = \mathcal{A}(x(t)) + \mathcal{B}(x(t))u(t), \quad \mathcal{A}:\mathbb{R}^n \to \mathbb{R}^{n\times n},\ \mathcal{B}:\mathbb{R}^n \to \mathbb{R}^{n\times m},\\ y(t) &= g(t,x,u) = \mathcal{C}(x(t)) + \mathcal{D}(x(t))u(t), \quad \mathcal{C}:\mathbb{R}^n \to \mathbb{R}^{q\times n},\ \mathcal{D}:\mathbb{R}^n \to \mathbb{R}^{q\times m}.\end{aligned}$$

## Linear, Time-Invariant (LTI) Systems

$$\begin{aligned}\dot{x}(t) &= f(t,x,u) = Ax(t) + Bu(t), \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}.\end{aligned}$$

## Bilinear Systems

$$\begin{aligned}\dot{x}(t) &= f(t,x,u) = Ax(t) + \sum_{i=1}^{m} u_i(t)A_i x(t) + Bu(t), \quad A, A_i \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}.\end{aligned}$$

# System Classes

## Linear, Time-Invariant (LTI) Systems

$$\begin{aligned}\dot{x}(t) &= f(t,x,u) = Ax(t) + Bu(t), \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}.\end{aligned}$$

## Bilinear Systems

$$\begin{aligned}\dot{x}(t) &= f(t,x,u) = Ax(t) + \sum_{i=1}^{m} u_i(t)A_i x(t) + Bu(t), \qquad A, A_i \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}.\end{aligned}$$

## Quadratic-Bilinear (QB) Systems

$$\begin{aligned}\dot{x}(t) &= f(t,x,u) = Ax(t) + H\left(x(t)\otimes x(t)\right) + \sum_{i=1}^{m} u_i(t)A_i x(t) + Bu(t),\\ &\qquad A, A_i \in \mathbb{R}^{n\times n},\ H \in \mathbb{R}^{n\times n^2},\ B \in \mathbb{R}^{n\times m},\\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}.\end{aligned}$$

# System Classes

## Control-Affine (Autonomous) Systems

$$\begin{aligned}
\dot{x}(t) &= f(t,x,u) = \mathcal{A}(x(t)) + \mathcal{B}(x(t))u(t), \quad \mathcal{A}:\mathbb{R}^n \to \mathbb{R}^{n\times n},\ \mathcal{B}:\mathbb{R}^n \to \mathbb{R}^{n\times m}, \\
y(t) &= g(t,x,u) = \mathcal{C}(x(t)) + \mathcal{D}(x(t))u(t), \quad \mathcal{C}:\mathbb{R}^n \to \mathbb{R}^{q\times n},\ \mathcal{D}:\mathbb{R}^n \to \mathbb{R}^{q\times m}.
\end{aligned}$$

## Quadratic-Bilinear (QB) Systems

$$\begin{aligned}
\dot{x}(t) &= f(t,x,u) = Ax(t) + H\left(x(t)\otimes x(t)\right) + \textstyle\sum_{i=1}^m u_i(t)A_i x(t) + Bu(t), \\
& \qquad A, A_i \in \mathbb{R}^{n\times n},\ H \in \mathbb{R}^{n\times n^2},\ B \in \mathbb{R}^{n\times m}, \\
y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}.
\end{aligned}$$

Written in control-affine form:

$$\begin{aligned}
\mathcal{A}(x) &:= Ax + H\left(x\otimes x\right), & \mathcal{B}(x) &:= [A_1,\ldots,A_m]\left(I_m\otimes x\right) + B \\
\mathcal{C}(x) &:= Cx, & \mathcal{D}(x) &:= Dx.
\end{aligned}$$

# Carleman Bilinearization

Consider smooth nonlinear, control-affine system with $m = 1$:

$$\begin{aligned} \dot{x} &= \mathcal{A}(x) + Bu \qquad \text{with } \mathcal{A}(0) = 0, \\ y &= Cx + Du. \end{aligned}$$

# Carleman Bilinearization

Consider smooth nonlinear, control-affine system with $m = 1$:

$$\dot{x} = \mathcal{A}(x) + Bu \qquad \text{with } \mathcal{A}(0) = 0.$$

Taylor expansion of state equation about $x = 0$ yields

$$\dot{x} = Ax + H\,(x \otimes x) + \ldots + Bu.$$

# Carleman Bilinearization

Consider smooth nonlinear, control-affine system with $m = 1$:

$$\dot{x} = \mathcal{A}(x) + Bu \qquad \text{with } \mathcal{A}(0) = 0.$$

Taylor expansion of state equation about $x = 0$ yields

$$\dot{x} = Ax + H\,(x \otimes x) + \ldots + Bu.$$

Instead of truncating Taylor expansion, Carleman (bi)linearization takes into account $K$ higher order terms (h.o.t.) by introducing new variables:

$$x^{(k)} := x \underbrace{\otimes \cdots \otimes}_{(k-1)\ \text{times}} x, \qquad k = 1, \ldots, K.$$

Here: $K = 2$, i.e., $z := x^{(2)} = x \otimes x$.

# Carleman Bilinearization

Consider smooth nonlinear, control-affine system with $m = 1$:

$$\dot{x} = \mathcal{A}(x) + Bu \qquad \text{with } \mathcal{A}(0) = 0.$$

Taylor expansion of state equation about $x = 0$ yields

$$\dot{x} = Ax + H(x \otimes x) + \ldots + Bu.$$

Instead of truncating Taylor expansion, Carleman (bi)linearization takes into account $K = 2$ higher order terms (h.o.t.) by introducing new variables: $z := x^{(2)} = x \otimes x$.
Then $z$ satisfies

$$\dot{z} = \dot{x} \otimes x + x \otimes \dot{x} = (Ax + Hz + \ldots + Bu) \otimes x + x \otimes (Ax + Hz + \ldots + Bu)$$

Consider smooth nonlinear, control-affine system with $m = 1$:

$$\dot{x} = \mathcal{A}(x) + Bu \qquad \text{with } \mathcal{A}(0) = 0,$$
$$y = Cx + Du.$$

Instead of truncating Taylor expansion, Carleman (bi)linearization takes into account $K = 2$ higher order terms (h.o.t.) by introducing new variables: $z := x^{(2)} = x \otimes x$. Then $z$ satisfies

$$\dot{z} = \dot{x} \otimes x + x \otimes \dot{x} = (Ax + Hz + \ldots + Bu) \otimes x + x \otimes (Ax + Hz + \ldots + Bu)$$

Ignoring h.o.t. $\Longrightarrow$ bilinear system with state $x^{\otimes} := \left[x^T, z^T\right]^T \in \mathbb{R}^{n+n^2}$:

$$\frac{d}{dt}x^{\otimes} = \begin{bmatrix} A & H \\ 0 & A \otimes I_n + I_n \otimes A \end{bmatrix} x^{\otimes} + \begin{bmatrix} 0 & 0 \\ B \otimes I_n + I_n \otimes B & 0 \end{bmatrix} (x^{\otimes}) u + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$
$$y^{\otimes} = \begin{bmatrix} C & 0 \end{bmatrix} x^{\otimes} + Du.$$

# Carleman Bilinearization

Consider smooth nonlinear, control-affine system with $m = 1$:

$$\dot{x} = \mathcal{A}(x) + Bu \qquad \text{with } \mathcal{A}(0) = 0,$$
$$y = Cx + Du.$$

Instead of truncating Taylor expansion, Carleman (bi)linearization takes into account $K = 2$ higher order terms (h.o.t.) by introducing new variables: $z := x^{(2)} = x \otimes x$.
Then $z$ satisfies

$$\dot{z} = \dot{x} \otimes x + x \otimes \dot{x} = (Ax + Hz + \ldots + Bu) \otimes x + x \otimes (Ax + Hz + \ldots + Bu)$$

Ignoring h.o.t. $\Longrightarrow$ bilinear system with state $x^{\otimes} := \left[x^T, z^T\right]^T \in \mathbb{R}^{n+n^2}$:

$$\frac{d}{dt}x^{\otimes} = \begin{bmatrix} A & H \\ 0 & A \otimes I_n + I_n \otimes A \end{bmatrix} x^{\otimes} + \begin{bmatrix} 0 & 0 \\ B \otimes I_n + I_n \otimes B & 0 \end{bmatrix} (x^{\otimes})u + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$
$$y^{\otimes} = \begin{bmatrix} C & 0 \end{bmatrix} x^{\otimes} + Du.$$

**Remark**

Bilinear systems directly occur, e.g., in biological systems, PDE control problems with mixed boundary conditions, "control via coefficients", networked control systems, . . .

# Quadratic-Bilinearization

QB systems can be obtained as approximation (by truncating Taylor expansion) to weakly nonlinear systems [Phillips '03].


C. Gu. QLMOR: A Projection-Based Nonlinear Model Order Reduction Approach Using Quadratic-Linear Representation of Nonlinear Systems. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 30(9):1307–1320, 2011.

L. Feng, X. Zeng, C. Chiang, D. Zhou, and Q. Fang. Direct nonlinear order reduction with variational analysis. In: Proceedings of DATE 2004, pp. 1316-1321.

J. R. Phillips. Projection-based approaches for model reduction of weakly nonlinear time-varying systems. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 22(2):171-187, 2003.

# Quadratic-Bilinearization

QB systems can be obtained as approximation (by truncating Taylor expansion) to weakly nonlinear systems [PHILLIPS '03].

But exact representation of smooth nonlinear systems possible:

**Theorem [GU '09/'11]**

Assume that the state equation of a nonlinear system is given by

$$\dot{x} = a_0 x + a_1 g_1(x) + \ldots + a_k g_k(x) + Bu,$$

where $g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$ are compositions of uni-variable rational, exponential, logarithmic, trigonometric or root functions, respectively. Then, by iteratively taking derivatives and adding algebraic equations, respectively, the nonlinear system can be transformed into a QB(DAE) system.


C. Gu. QLMOR: A Projection-Based Nonlinear Model Order Reduction Approach Using Quadratic-Linear Representation of Nonlinear Systems. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 30(9):1307–1320, 2011.

L. Feng, X. Zeng, C. Chiang, D. Zhou, and Q. Fang. Direct nonlinear order reduction with variational analysis. In: Proceedings of DATE 2004, pp. 1316-1321.

J. R. Phillips. Projection-based approaches for model reduction of weakly nonlinear time-varying systems. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 22(2):171-187, 2003.

# Quadratic-Bilinearization

## McCormick Relaxation

Idea borrowed from non-convex optimization:

- **Lift to higher dimensions using** $const. \cdot n$ **additional variables,**
- convex relaxation.

G. P. McCormick. Computability of global solutions to factorable nonconvex programs: Part I, convex underestimating problems. Mathematical Programming, 10(1):147-175, 1976.

# Quadratic-Bilinearization

## McCormick Relaxation

Idea borrowed from non-convex optimization:

- **Lift to higher dimensions using** $const. \cdot n$ **additional variables,**
- convex relaxation.

**Example**

$$\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1}, \qquad \dot{x}_2 = -x_2 + u.$$

G. P. McCormick. Computability of global solutions to factorable nonconvex programs: Part I, convex underestimating problems. Mathematical Programming, 10(1):147-175, 1976.

## McCormick Relaxation

Idea borrowed from non-convex optimization:

- **Lift to higher dimensions using** *const.* $\cdot\, n$ **additional variables,**
- convex relaxation.

**Example**

$$\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1}, \qquad \dot{x}_2 = -x_2 + u.$$

$$z_1 := \exp(-x_2), \qquad z_2 := \sqrt{x_1^2 + 1}.$$

G. P. McCormick. Computability of global solutions to factorable nonconvex programs: Part I, convex underestimating problems. Mathematical Programming, 10(1):147-175, 1976.

## McCormick Relaxation

Idea borrowed from non-convex optimization:

- **Lift to higher dimensions using $const. \cdot n$ additional variables,**
- convex relaxation.

**Example**

$$\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1}, \qquad \dot{x}_2 = -x_2 + u.$$

$$z_1 := \exp(-x_2), \qquad z_2 := \sqrt{x_1^2 + 1}.$$

$$\dot{x}_1 = z_1 \cdot z_2, \qquad \dot{x}_2 = -x_2 + u,$$


G. P. McCormick. Computability of global solutions to factorable nonconvex programs: Part I, convex underestimating problems. Mathematical Programming, 10(1):147-175, 1976.

## McCormick Relaxation

Idea borrowed from non-convex optimization:

- **Lift to higher dimensions using $const. \cdot n$ additional variables,**
- convex relaxation.

**Example**

$$\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1}, \qquad \dot{x}_2 = -x_2 + u.$$

$$z_1 := \exp(-x_2), \qquad z_2 := \sqrt{x_1^2 + 1}.$$

$$\dot{x}_1 = z_1 \cdot z_2, \qquad \dot{x}_2 = -x_2 + u,$$

$$\dot{z}_1 = -z_1 \cdot (-x_2 + u), \qquad \dot{z}_2 = \frac{2 \cdot x_1 \cdot z_1 \cdot z_2}{2 \cdot z_2} = x_1 \cdot z_1.$$


G. P. McCormick. Computability of global solutions to factorable nonconvex programs: Part I, convex underestimating problems. Mathematical Programming, 10(1):147-175, 1976.

# Quadratic-Bilinearization

## McCormick Relaxation

Idea borrowed from non-convex optimization:

- **Lift to higher dimensions using $const. \cdot n$ additional variables,**
- convex relaxation.

**Example**

$$\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1}, \qquad \dot{x}_2 = -x_2 + u.$$

$$z_1 := \exp(-x_2), \qquad z_2 := \sqrt{x_1^2 + 1}.$$

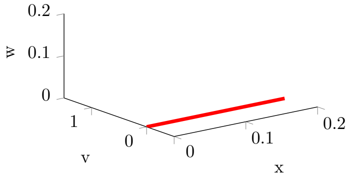
$$\dot{x}_1 = z_1 \cdot z_2, \qquad \dot{x}_2 = -x_2 + u,$$

$$\dot{z}_1 = -z_1 \cdot (-x_2 + u), \qquad \dot{z}_2 = \frac{2 \cdot x_1 \cdot z_1 \cdot z_2}{2 \cdot z_2} = x_1 \cdot z_1.$$

Alternatively, polynomial-bilinear system can be obtained using iterated Lie brackets [Gu '11].

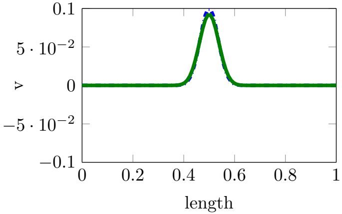
G. P. McCormick. Computability of global solutions to factorable nonconvex programs: Part I, convex underestimating problems. Mathematical Programming, 10(1):147-175, 1976.

## FitzHugh-Nagumo model




- Model describes activation and de-activation of neurons.
- It contains a cubic nonlinearity, which can be transformed to QB form.

## Sine-Gordon equation




- Applications in biomedical studies, mechanical transmission lines, etc.
- It contains sin function, which can also be rewritten into QB form.

## The Laplace transform

**Definition**

The Laplace transform of a time domain function $f \in L_{1,\mathrm{loc}}$ with $\mathrm{dom}(f) = \mathbb{R}_0^+$ is

$$\mathcal{L} : f \mapsto F, \quad F(s) := \mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\, dt, \quad s \in \mathbb{C}.$$

$F$ is a function in the (Laplace or) frequency domain.

**Note:** With $\mathfrak{R}s = 0$ and $\mathfrak{I}s \geq 0$, $\omega := \mathfrak{I}s$ takes the role of a frequency (in [rad/s], i.e., $\omega = 2\pi\nu$ with $\nu$ measured in [Hz]).

## The Laplace transform

**Definition**

The Laplace transform of a time domain function $f \in L_{1,\mathrm{loc}}$ with $\mathrm{dom}\,(f) = \mathbb{R}_0^+$ is

$$\mathcal{L} : f \mapsto F, \quad F(s) := \mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\,dt, \quad s \in \mathbb{C}.$$

$F$ is a function in the (Laplace or) frequency domain.

**Note:** With $\Re s = 0$ and $\Im s \geq 0$, $\omega := \Im s$ takes the role of a frequency (in [rad/s], i.e., $\omega = 2\pi\nu$ with $\nu$ measured in [Hz]).

**Lemma**

$$\mathcal{L}\{\dot{f}(t)\}(s) = sF(s) - f(0).$$

## The Laplace transform

**Definition**

The Laplace transform of a time domain function $f \in L_{1,\text{loc}}$ with $\operatorname{dom}(f) = \mathbb{R}_0^+$ is

$$\mathcal{L} : f \mapsto F, \quad F(s) := \mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\, dt, \quad s \in \mathbb{C}.$$

$F$ is a function in the (Laplace or) frequency domain.

**Lemma**

$$\mathcal{L}\{\dot{f}(t)\}(s) = sF(s) - f(0).$$

Note: for ease of notation, in the following we will use lower-case letters for both, a function and its Laplace transform!

## Transfer functions of linear systems

**Linear Systems in Frequency Domain**

Application of Laplace transform $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sx(s) - x(0))$ to linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

with $x(0) = 0$ yields:

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s),$$

## Transfer functions of linear systems

### Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sx(s) - x(0))$ to linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

with $x(0) = 0$ yields:

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s),$$

$\Longrightarrow$ I/O-relation in frequency domain:

$$y(s) = \Big( \underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)} \Big) u(s).$$

$G(s)$ is the **transfer function** of $\Sigma$.

## Transfer functions of linear systems

### Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sx(s) - x(0))$ to linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

with $x(0) = 0$ yields:

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s),$$

$\Longrightarrow$ I/O-relation in frequency domain:

$$y(s) = \Big( \underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)} \Big) u(s).$$

$G(s)$ is the **transfer function** of $\Sigma$.

**Model reduction in frequency domain:** Fast evaluation of mapping $u \to y$.

# Linear Systems and their Transfer Functions

## Formulating model reduction in frequency domain

Approximate the dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y &= Cx + Du, \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}, \end{aligned}$$

by reduced-order system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \quad \hat{A} \in \mathbb{R}^{r\times r},\ \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, \quad \hat{C} \in \mathbb{R}^{q\times r},\ \hat{D} \in \mathbb{R}^{q\times m} \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.$$

## Formulating model reduction in frequency domain

Approximate the dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y &= Cx + Du, \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}, \end{aligned}$$

by reduced-order system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \quad \hat{A} \in \mathbb{R}^{r\times r},\ \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, \quad \hat{C} \in \mathbb{R}^{q\times r},\ \hat{D} \in \mathbb{R}^{q\times m} \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \le \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.$$

$\Longrightarrow$ Approximation problem: $\displaystyle\min_{\text{order}(\hat{G}) \le r} \|G - \hat{G}\|.$

# Outline

# Balanced Truncation for Linear Systems

## Basic concept

- System $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with $A$ stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions $P, Q$ of the Lyapunov equations

  $$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

## Basic concept

- System $\Sigma$ : $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with $A$ stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions $P, Q$ of the Lyapunov equations

  $$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.
- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of $\Sigma$.

# Balanced Truncation for Linear Systems

## Basic concept

- System $\Sigma: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with $A$ stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,
  is balanced, if system Gramians, i.e., solutions $P, Q$ of the Lyapunov equations
  $$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$
  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$.
- $\{\sigma_1, \dots, \sigma_n\}$ are the Hankel singular values (HSVs) of $\Sigma$.
- Compute balanced realization (needs $P, Q$!) of the system via state-space transformation
  $$\begin{aligned} \mathcal{T} : (A, B, C) \quad &\mapsto \quad (TAT^{-1}, TB, CT^{-1}) \\ &= \quad \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right). \end{aligned}$$

# Balanced Truncation for Linear Systems

## Basic concept

- System $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with $A$ stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions $P, Q$ of the Lyapunov equations

  $$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.
- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of $\Sigma$.
- Compute balanced realization (needs $P, Q$!) of the system via state-space transformation

  $$\begin{aligned} \mathcal{T} : (A, B, C) &\mapsto (TAT^{-1}, TB, CT^{-1}) \\ &= \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right). \end{aligned}$$

- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}) = (A_{11}, B_1, C_1)$.

# Balanced Truncation for Linear Systems

**Motivation:**

HSV are system invariants: they are preserved under $\mathcal{T}$ and determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

**"functional analyst's point of view"**

# Balanced Truncation for Linear Systems

## Motivation:

HSV are system invariants: they are preserved under $\mathcal{T}$ and determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

**"functional analyst's point of view"**

Minimum energy to reach $x_0$ in balanced coordinates:

$$\inf_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \int_{-\infty}^{0} u(t)^T u(t)\, dt = x_0^T P^{-1} x_0 = \sum_{j=1}^{n} \frac{1}{\sigma_j} x_{0,j}^2$$

**Motivation:**

HSV are system invariants: they are preserved under $\mathcal{T}$ and determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

**"functional analyst's point of view"**

Minimum energy to reach $x_0$ in balanced coordinates:

$$\inf_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \int_{-\infty}^{0} u(t)^T u(t)\, dt = x_0^T P^{-1} x_0 = \sum_{j=1}^{n} \frac{1}{\sigma_j} x_{0,j}^2$$

Energy contained in the system if $x(0) = x_0$ and $u(t) \equiv 0$ in balanced coordinates:

$$\|y\|_2^2 = \int_0^{\infty} y(t)^T y(t)\, dt = x_0^T Q x_0 = \sum_{j=1}^{n} \sigma_j x_{0,j}^2$$

# Balanced Truncation for Linear Systems

**Motivation:**

HSV are system invariants: they are preserved under $\mathcal{T}$ and determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

**"functional analyst's point of view"**

In balanced coordinates, energy transfer from $u_-$ to $y_+$ is

$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int_0^\infty y(t)^T y(t)\, dt}{\int_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2.$$

**"engineer's point of view"**

# Balanced Truncation for Linear Systems

**Motivation:**

HSV are system invariants: they are preserved under $\mathcal{T}$ and determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

**"functional analyst's point of view"**

In balanced coordinates, energy transfer from $u_-$ to $y_+$ is

$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int\limits_0^\infty y(t)^T y(t)\, dt}{\int\limits_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2.$$

**"engineer's point of view"**

$\Longrightarrow$ Truncate states corresponding to "small" HSVs

# Balanced Truncation for Linear Systems

## Properties

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.

# Balanced Truncation for Linear Systems

**Properties**

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of $r$ via computable error bound:

$$\|y - \hat{y}\|_2 \leq \|G - \hat{G}\|_{\mathcal{H}_\infty} \|u\|_2 \leq \left(2 \sum_{k=r+1}^{n} \sigma_k\right) \|u\|_2.$$

# Balanced Truncation for Linear Systems

## Properties

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of $r$ via computable error bound:

$$\|y - \hat{y}\|_2 \leq \|G - \hat{G}\|_{\mathcal{H}_\infty} \|u\|_2 \leq \left(2 \sum_{k=r+1}^{n} \sigma_k\right) \|u\|_2.$$

## Practical implementation

- Rather than solving Lyapunov equations for $P, Q$ ($n^2$ unknowns!), find $S, R \in \mathbb{R}^{n \times s}$ with $s \ll n$ such that $P \approx SS^T$, $Q \approx RR^T$.
- Reduced-order model directly obtained via small-scale ($s \times s$) SVD of $R^T S$!
- No $\mathcal{O}(n^3)$ or $\mathcal{O}(n^2)$ computations necessary!

## Computation of reduced-order model by projection

Given linear (descriptor) system $E\dot{x} = Ax + Bu,\ y = Cx$ with transfer function

$$G(s) = C(sE - A)^{-1}B,$$

a ROM is obtained using truncation matrices $V, W \in \mathbb{R}^{n\times r}$ with $W^T V = I_r$
($\rightsquigarrow (VW^T)^2 = VW^T$ is projector) by computing

$$\hat{E} = W^T EV,\ \hat{A} = W^T AV,\ \hat{B} = W^T B,\ \hat{C} = CV.$$

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

# Interpolatory Model Reduction

## Computation of reduced-order model by projection

Given linear (descriptor) system $E\dot{x} = Ax + Bu,\ y = Cx$ with transfer function

$$G(s) = C(sE - A)^{-1}B,$$

a ROM is obtained using truncation matrices $V, W \in \mathbb{R}^{n \times r}$ with $W^T V = I_r$ ($\rightsquigarrow$ $(VW^T)^2 = VW^T$ is projector) by computing

$$\hat{E} = W^T EV,\ \hat{A} = W^T AV,\ \hat{B} = W^T B,\ \hat{C} = CV.$$

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

## Rational Interpolation/Moment-Matching

Choose $V, W$ such that

$$G(s_j) = \hat{G}(s_j), \quad j = 1, \ldots, k,$$

and

$$\frac{d^i}{ds^i} G(s_j) = \frac{d^i}{ds^i} \hat{G}(s_j), \quad i = 1, \ldots, K_j, \quad j = 1, \ldots, k.$$

# Interpolatory Model Reduction

**Theorem (simplified)** [Grimme '97, Villemagne/Skelton '87]

If

$$\operatorname{span}\left\{(s_1 E - A)^{-1} B, \dots, (s_k E - A)^{-1} B\right\} \subset \operatorname{Ran}(V),$$
$$\operatorname{span}\left\{(s_1 E - A)^{-T} C^T, \dots, (s_k E - A)^{-T} C^T\right\} \subset \operatorname{Ran}(W),$$

then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds} G(s_j) = \frac{d}{ds} \hat{G}(s_j), \quad \text{for } j = 1, \dots, k.$$

# Interpolatory Model Reduction

**Theorem (simplified)** [Grimme '97, Villemagne/Skelton '87]

If

$$\operatorname{span}\left\{(s_1E-A)^{-1}B,\dots,(s_kE-A)^{-1}B\right\} \subset \operatorname{Ran}(V),$$
$$\operatorname{span}\left\{(s_1E-A)^{-T}C^T,\dots,(s_kE-A)^{-T}C^T\right\} \subset \operatorname{Ran}(W),$$

then

$$G(s_j)=\hat{G}(s_j),\quad \frac{d}{ds}G(s_j)=\frac{d}{ds}\hat{G}(s_j),\quad \text{for } j=1,\dots,k.$$

Remarks:

computation of $V, W$ from rational Krylov subspaces, e.g.,

- dual rational Arnoldi/Lanczos [Grimme '97],
- **Iterative Rational Krylov Algorithm (IRKA)** computes $\mathcal{H}_2$-optimal model of given order $r$, i.e., solves transfer function approximation problem in $\mathcal{H}_2$-norm, using tangential rational interpolation [Antoulas/Beattie/Gugercin '06/'08].

**Theorem (simplified)** [Grimme '97, Villemagne/Skelton '87]

If

$$\begin{aligned} \operatorname{span}\left\{(s_1 E - A)^{-1}B, \ldots, (s_k E - A)^{-1}B\right\} &\subset \operatorname{Ran}(V), \\ \operatorname{span}\left\{(s_1 E - A)^{-T}C^T, \ldots, (s_k E - A)^{-T}C^T\right\} &\subset \operatorname{Ran}(W), \end{aligned}$$

then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds}G(s_j) = \frac{d}{ds}\hat{G}(s_j), \quad \text{for } j = 1, \ldots, k.$$

Remarks:

using Galerkin/one-sided projection $(W \equiv V)$ yields $G(s_j) = \hat{G}(s_j)$, but in general

$$\frac{d}{ds}G(s_j) \neq \frac{d}{ds}\hat{G}(s_j).$$

# Interpolatory Model Reduction

**Theorem (simplified)** [Grimme '97, Villemagne/Skelton '87]

If

$$\begin{aligned} \mathrm{span}\left\{(s_1E-A)^{-1}B,\ldots,(s_kE-A)^{-1}B\right\} &\subset \mathrm{Ran}(V),\\ \mathrm{span}\left\{(s_1E-A)^{-T}C^T,\ldots,(s_kE-A)^{-T}C^T\right\} &\subset \mathrm{Ran}(W),\end{aligned}$$

then

$$G(s_j)=\hat{G}(s_j),\quad \frac{d}{ds}G(s_j)=\frac{d}{ds}\hat{G}(s_j),\quad \text{for } j=1,\ldots,k.$$

Remarks:

$k=1$, standard Krylov subspace(s) of dimension $K$:

$$\mathrm{range}\,(V)=\mathcal{K}_K((s_1E-A)^{-1},(s_1E-A)^{-1}B).$$

$\rightsquigarrow$ moment-matching methods/Padé approximation [Freund/Feldmann '95],

$$\frac{d^i}{ds^i}G(s_1)=\frac{d^i}{ds^i}\hat{G}(s_1),\quad i=0,\ldots,K-1(+K).$$

# Interpolatory Model Reduction

Remarks:

$k = 1$, standard Krylov subspace(s) of dimension $K$:

$$\operatorname{range}(V) = \mathcal{K}_K((s_1E - A)^{-1}, (s_1E - A)^{-1}B).$$

$\rightsquigarrow$ moment-matching methods/Padé approximation [FREUND/FELDMANN '95],

$$\frac{d^i}{ds^i}G(s_1) = \frac{d^i}{ds^i}\hat{G}(s_1), \quad i = 0, \ldots, K-1(+K).$$

Recent developments:

Adaptive choice of interpolation points and number of moments to be matched based on dual-weighted residual based error estimate!

---


L. Feng, J. G. Korvink, P. Benner.
A Fully Adaptive Scheme for Model Order Reduction Based on Moment-Matching. *IEEE Transactions on Components, Packaging, and Manufacturing Technology,* 5(12):1872–1884, 2015.

L. Feng, A. C. Antoulas, P. Benner.
Some a posteriori error bounds for reduced order modelling of (non-)parametrized linear systems. *MPI Magdeburg Preprints* MPIMD/15-17, October 2015.

# Outline

## Approaches

- Nonlinear balancing based on energy functionals [Scherpen '93, Gray/Mesko '96].

**Definition** [Scherpen '93, Gray/Mesko '96]

The reachability energy functional, $L_c(x_0)$, and observability energy functional, $L_o(x_0)$ of a system are given as:

$$L_c(x_0) = \inf_{\substack{u \in L_2(-\infty,0] \\ x(-\infty)=0,\ x(0)=x_0}} \frac{1}{2} \int_{-\infty}^{0} \|u(t)\|^2 dt, \qquad L_o(x_0) = \frac{1}{2} \int_{0}^{\infty} \|y(t)\|^2 dt.$$

**Disadvantage:** energy functionals are the solutions of nonlinear Hamilton-Jacobi equations which are hardly solvable for large-scale systems.

## Approaches

- Nonlinear balancing based on energy functionals [Scherpen '93, Gray/Mesko '96].
  **Disadvantage:** energy functionals are the solutions of nonlinear Hamilton-Jacobi equations which are hardly solvable for large-scale systems.
- Empirical Gramians/frequency-domain POD [Lall et al '99, Willcox/Peraire '02].

**Example: controllability Gramian from time domain data (snapshots)**

1. Define reachability Gramian of the system
$$P = \int_0^\infty x(t)x(t)^T dt, \quad \text{where } x(t) \text{ solves } \dot{x} = f(x, \delta),\ x(0) = x_0.$$
2. Use time-domain integrator to produce snapshots $x_k \approx x(t_k)$, $k = 1, \ldots, K$.
3. Approximate $P \approx \sum_{k=0}^{K} w_k x_k x_k^T$ with positive weights $w_k$.
4. Analogously for observability Gramian.
5. Compute balancing transformation and apply it to nonlinear system.

**Disadvantage:** Depends on chosen training input (e.g., $\delta(t_0)$) like other POD approaches.

## Approaches

- Nonlinear balancing based on energy functionals [Scherpen '93, Gray/Mesko '96].
  **Disadvantage:** energy functionals are the solutions of nonlinear Hamilton-Jacobi equations which are hardly solvable for large-scale systems.
- Empirical Gramians/frequency-domain POD [Lall et al '99, Willcox/Peraire '02].
  **Disadvantage:** Depends on chosen training input (e.g., $\delta(t_0)$) like other POD approaches.
- $\rightsquigarrow$ **Goal:** computationally efficient and input-independent method!


W. S. Gray and J. P. Mesko. Controllability and observability functions for model reduction of nonlinear systems. In Proc. of the Conf. on Information Sci. and Sys., pp. 1244–1249, 1996.

S. Lall, J. Marsden, and S. Glavaški. A subspace approach to balanced truncation for model reduction of nonlinear control systems. International Journal of Robust and Nonlinear Control, 12:519-535, 2002.

J. M. A. Scherpen. Balancing for nonlinear systems. Systems & Control Letters, 21:143–153, 1993.

K. Willcox and J. Peraire, Balanced model reduction via the proper orthogonal decomposition. AIAA Journal, 40:2323-2330, 2002.

# New "Gramians"

- A possible solution is to obtain bounds for the energy functionals, instead of computing them exactly.

# New "Gramians"

- A possible solution is to obtain bounds for the energy functionals, instead of computing them exactly.
- For example, (locally) $L_c(x_0) \geq \frac{1}{2} x_0^T \tilde{P}^{-1} x_0$, where $\tilde{P} = \tilde{P}^T > 0$ [Gray/Mesko '96].
- For bilinear systems, such local bounds were derived in [B./Damm '11] using the solutions to the Lyapunov-plus-positive equations:

$$AP + PA^T + \sum_{i=1}^m A_i P A_i^T + BB^T = 0,$$
$$A^T Q + QA^T + \sum_{i=1}^m A_i^T Q A_i + C^T C = 0.$$

(If their solutions exist, they define reachability and observability Gramians of BIBO stable bilinear system.)

- Efficient solution methods for Lyapunov-plus-positive equations are derived in [B./Breiten '13, Shank/Simoncini/Szyld '16, Kürschner '17].

# New "Gramians"

- A possible solution is to obtain bounds for the energy functionals, instead of computing them exactly.
- For example, (locally) $L_c(x_0) \geq \frac{1}{2} x_0^T \tilde{P}^{-1} x_0$, where $\tilde{P} = \tilde{P}^T > 0$ [Gray/Mesko '96].
- For bilinear systems, such local bounds were derived in [B./Damm '11] using the solutions to the Lyapunov-plus-positive equations:

$$AP + PA^T + \sum_{i=1}^{m} A_i P A_i^T + BB^T = 0,$$
$$A^T Q + QA^T + \sum_{i=1}^{m} A_i^T Q A_i + C^T C = 0.$$

  (If their solutions exist, they define reachability and observability Gramians of BIBO stable bilinear system.)
- Efficient solution methods for Lyapunov-plus-positive equations are derived in [B./Breiten '13, Shank/Simoncini/Szyld '16, Kürschner '17].
- **Here we aim at determining algebraic Gramians for QB systems, which**
  - **provide bounds for the energy functionals of QB systems,**
  - **generalize the Gramians of linear and bilinear systems, and**
  - **allow us to find the states that are hard to control as well as hard to observe in an efficient and reliable way.**

# Gramians for QB Systems

## Controllability Gramians

- Consider input → state map of QB system ($m = 1,\ N \equiv A_1$):

$$\dot{x}(t) = Ax(t) + Hx(t) \otimes x(t) + Nx(t)u(t) + Bu(t), \qquad x(0) = 0.$$

- Integration yields

$$x(t) = \int_0^t e^{A\sigma_1} Bu(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Nx(t-\sigma_1)u(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Hx(t-\sigma_1) \otimes x(t-\sigma_1)d\sigma_1$$

[Rugh '81]

## Controllability Gramians

- Consider input → state map of QB system ($m = 1,\ N \equiv A_1$):

$$\dot{x}(t) = Ax(t) + Hx(t) \otimes x(t) + Nx(t)u(t) + Bu(t), \qquad x(0) = 0.$$

- Integration yields

$$x(t) = \int_0^t e^{A\sigma_1} Bu(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Nx(t-\sigma_1)u(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Hx(t-\sigma_1) \otimes x(t-\sigma_1)d\sigma_1$$

$$= \int_0^t e^{A\sigma_1} Bu(t-\sigma_1)d\sigma_1 + \int_0^t \int_0^{t-\sigma_1} e^{A\sigma_1} N e^{A\sigma_2} Bu(t-\sigma_1)u(t-\sigma_1-\sigma_2)d\sigma_1 d\sigma_2$$

$$+ \int_0^t \int_0^{t-\sigma_1} \int_0^{t-\sigma_1} e^{A\sigma_1} H(e^{A\sigma_2} B \otimes e^{A\sigma_3} B)u(t-\sigma_1-\sigma_2)u(t-\sigma_1-\sigma_3)d\sigma_1 d\sigma_2 d\sigma_3 + \ldots$$

[Rugh '81]

# Gramians for QB Systems

## Controllability Gramians

- Consider input → state map of QB system ($m = 1,\ N \equiv A_1$):

$$\dot{x}(t) = Ax(t) + Hx(t) \otimes x(t) + Nx(t)u(t) + Bu(t), \qquad x(0) = 0.$$

- Integration yields

$$x(t) = \int_0^t e^{A\sigma_1} Bu(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Nx(t-\sigma_1)u(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Hx(t-\sigma_1) \otimes x(t-\sigma_1)d\sigma_1$$

$$= \int_0^t e^{A\sigma_1} Bu(t-\sigma_1)d\sigma_1 + \int_0^t \int_0^{t-\sigma_1} e^{A\sigma_1} Ne^{A\sigma_2} Bu(t-\sigma_1)u(t-\sigma_1-\sigma_2)d\sigma_1 d\sigma_2$$

$$+ \int_0^t \int_0^{t-\sigma_1} \int_0^{t-\sigma_1} e^{A\sigma_1} H(e^{A\sigma_2} B \otimes e^{A\sigma_3} B)u(t-\sigma_1-\sigma_2)u(t-\sigma_1-\sigma_3)d\sigma_1 d\sigma_2 d\sigma_3 + \ldots$$

- By iteratively inserting expressions for $x(t-\bullet)$, we obtain the **Volterra series expansion** for the QB system. [Rugh '81]

## Controllability Gramians

Using the *Volterra kernels*, we can define the controllability mappings

$$\Pi_1(t_1) := e^{At_1}B, \qquad \Pi_2(t_1,t_2) := e^{At_1}N\Pi_1(t_2),$$

$$\Pi_3(t_1,t_2,t_3) := e^{At_1}[H(\Pi_1(t_2)\otimes\Pi_1(t_3)), N\Pi_2(t_1,t_2)], \ldots$$

and a candidate for a new Gramian:

$$P := \sum_{k=1}^{\infty} P_k, \qquad \text{where} \quad P_k = \int_0^{\infty}\cdots\int_0^{\infty}\Pi_k(t_1,\ldots,t_k)\Pi_k(t_1,\ldots,t_k)^T\,dt_1\ldots dt_k.$$

## Controllability Gramians

Using the *Volterra kernels*, we can define the controllability mappings

$$\Pi_1(t_1) := e^{At_1}B, \qquad \Pi_2(t_1,t_2) := e^{At_1}N\Pi_1(t_2),$$
$$\Pi_3(t_1,t_2,t_3) := e^{At_1}[H(\Pi_1(t_2)\otimes\Pi_1(t_3)), N\Pi_2(t_1,t_2)], \ldots$$

and a candidate for a new Gramian:

$$P := \sum_{k=1}^{\infty} P_k, \qquad \text{where} \quad P_k = \int_0^{\infty}\cdots\int_0^{\infty}\Pi_k(t_1,\ldots,t_k)\Pi_k(t_1,\ldots,t_k)^T\,dt_1\ldots dt_k.$$

**Theorem** [B./Goyal '16]

If it exists, the new controllability Gramian $P$ for QB (MIMO) systems with stable $A$ solves the **quadratic Lyapunov equation**

$$AP + PA^T + \sum_{k=1}^{m} A_k P A_k^T + H(P\otimes P)H^T + BB^T = 0.$$

**Note:** $H = 0 \rightsquigarrow$ "bilinear reachability Gramian"; if additionally, all $A_k = 0 \rightsquigarrow$ linear one.

# Gramians for QB Systems

## Dual systems and observability Gramians [Fujimoto et al. '02]

- Controllability energy functional (Gramian) of the dual system $\Leftrightarrow$ observability energy functional (Gramian) of the original system.

# Gramians for QB Systems

## Dual systems and observability Gramians [Fujimoto et al. '02]

- Controllability energy functional (Gramian) of the dual system $\Leftrightarrow$ observability energy functional (Gramian) of the original system.
- Employ close relation between port-Hamiltonian systems and dual systems of nonlinear systems.

# Gramians for QB Systems

## Dual systems and observability Gramians [Fujimoto et al. '02]

- Controllability energy functional (Gramian) of the dual system ⇔ observability energy functional (Gramian) of the original system.
- Employ close relation between port-Hamiltonian systems and dual systems of nonlinear systems.
- Allows to define dual systems for QB systems:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Hx(t) \otimes x(t) + \textstyle\sum_{k=1}^{m} A_k x(t)u(t) + Bu(t), && x(0) = 0,\\
\dot{x}_d(t) &= -A^T x_d(t) - H^{(2)} x(t) \otimes x_d(t) - \textstyle\sum_{k=1}^{m} A_k^T x_d(t)u(t) - C^T u_d(t), && x_d(\infty) = 0,\\
y_d(t) &= B^T x_d(t),
\end{aligned}$$

where $H^{(2)}$ is the mode-2 matricization of the QB Hessian.

# Gramians for QB Systems

**Dual systems and observability Gramians for QB systems** [B./Goyal '16]

- Writing down the Volterra series for the dual system $\rightsquigarrow$ observability mapping.

- This provides the observability Gramian $Q$ for the QB system. It solves

$$A^T Q + QA + \sum_{k=1}^{m} A_k^T Q A_k + H^{(2)} (P \otimes Q) \left(H^{(2)}\right)^T + C^T C = 0.$$

# Gramians for QB Systems

## Dual systems and observability Gramians for QB systems [B./Goyal '16]

- Writing down the Volterra series for the dual system $\rightsquigarrow$ observability mapping.
- This provides the observability Gramian $Q$ for the QB system. It solves

$$A^T Q + QA + \sum_{k=1}^{m} A_k^T Q A_k + H^{(2)}(P \otimes Q)\left(H^{(2)}\right)^T + C^T C = 0.$$

Remarks:

- Observability Gramian depends on controllability Gramian!
- For $H = 0$, obtain "bilinear observability Gramian", and if also all $A_k = 0$, the linear one.

# Gramians and Energy Functionals

Bounding the energy functionals:

**Lemma** [B./Goyal '16]

In a neighborhood of the stable equilibrium, $B_\varepsilon(0)$,

$$L_c(x_0) \geq \frac{1}{2} x_0^T P^{-1} x_0, \qquad L_o(x_0) \leq \frac{1}{2} x_0^T Q x_0, \qquad x_0 \in B_\varepsilon(0),$$

for "small signals" and $x_0$ pointing in unit directions.

# Gramians and Energy Functionals

Bounding the energy functionals:

**Lemma** [B./Goyal '16]

In a neighborhood of the stable equilibrium, $B_\varepsilon(0)$,

$$L_c(x_0) \geq \frac{1}{2} x_0^T P^{-1} x_0, \qquad L_o(x_0) \leq \frac{1}{2} x_0^T Q x_0, \qquad x_0 \in B_\varepsilon(0),$$

for "small signals" and $x_0$ pointing in unit directions.

**Another interpretation of Gramians in terms of energy functionals**

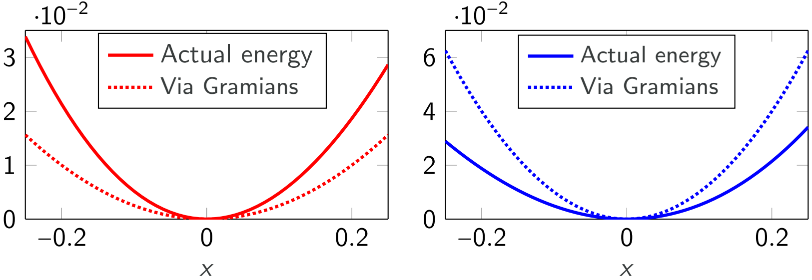
1. If the system is to be steered from 0 to $x_0$, where $x_0 \notin \mathrm{range}(P)$, then $L_c(x_0) = \infty$ for all input functions $u$.
2. If the system is (locally) controllable and $x_0 \in \ker(Q)$, then $L_o(x_0) = 0$.

## Illustration using a scalar system

$$\dot{x}(t) = ax(t) + hx^2(t) + nx(t)u(t) + bu(t), \quad y(t) = cx(t).$$

## Illustration using a scalar system

$$\dot{x}(t) = ax(t) + hx^2(t) + nx(t)u(t) + bu(t), \quad y(t) = cx(t).$$




(a) Input energy lower bound.

(b) Output energy upper bound.

Figure: Comparison of energy functionals for $-a = b = c = 2, h = 1, n = 0$.

# Truncated Gramians

- Now, the main obstacle for using the new Gramians is the solution of the (quadratic) Lyapunov-type equations.

# Truncated Gramians

- Now, the main obstacle for using the new Gramians is the solution of the (quadratic) Lyapunov-type equations.
- Fix point iteration scheme can be employed but it still very expensive.

  [Damm '08]

# Truncated Gramians

- Now, the main obstacle for using the new Gramians is the solution of the (quadratic) Lyapunov-type equations.
- Fix point iteration scheme can be employed but it still very expensive.

  [Damm '08]
- To overcome this issue, we propose truncated Gramians for QB systems.

# Truncated Gramians

- Now, the main obstacle for using the new Gramians is the solution of the (quadratic) Lyapunov-type equations.
- Fix point iteration scheme can be employed but it still very expensive.

  [Damm '08]

- To overcome this issue, we propose truncated Gramians for QB systems.

**Definition (Truncated Gramians)** [B./Goyal '16]

The truncated Gramians $P_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ for QB systems satisfy

$$AP_{\mathcal{T}} + P_{\mathcal{T}}A^T = -BB^T - \sum_{k=1}^m N_k P_l N_k^T - H(P_l \otimes P_l)H^T,$$

$$A^T Q_{\mathcal{T}} + Q_{\mathcal{T}}A = -C^T C - \sum_{k=1}^m N_k^T Q_l N_k - H^{(2)}(P_l \otimes Q_l)(H^{(2)})^T,$$

where

$$AP_l + P_l A^T = -BB^T \qquad \text{and} \qquad A^T Q_l + Q_l A = -C^T C.$$

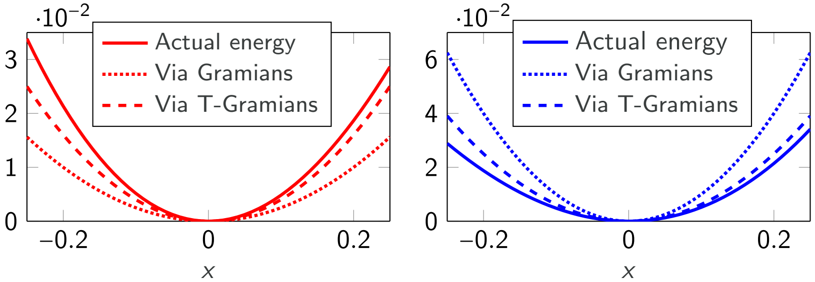
# Truncated Gramians

## Advantages of truncated Gramians (T-Gramians)

- T-Gramians approximate energy functionals better than the actual Gramians.

## Advantages of truncated Gramians (T-Gramians)

- T-Gramians approximate energy functionals better than the actual Gramians.




(a) Input energy lower bounds.  (b) Output energy upper bounds.

Figure: Comparison of energy functionals for $-a = b = c = 2, h = 1, n = 0$.

# Truncated Gramians

## Advantages of truncated Gramians (T-Gramians)

- T-Gramians approximate energy functionals better than the actual Gramians.

- $\sigma_i(P \cdot Q) > \sigma_i(P_{\mathcal{T}} \cdot Q_{\mathcal{T}}) \Rightarrow$ obtain smaller order of reduced system if truncation is done at the same cutoff threshold.

# Truncated Gramians

## Advantages of truncated Gramians (T-Gramians)

- T-Gramians approximate energy functionals better than the actual Gramians.
- $\sigma_i(P \cdot Q) > \sigma_i(P_{\mathcal{T}} \cdot Q_{\mathcal{T}}) \Rightarrow$ obtain smaller order of reduced system if truncation is done at the same cutoff threshold.
- Most importantly, we need solutions of only four standard Lyapunov equations.

# Truncated Gramians

## Advantages of truncated Gramians (T-Gramians)

- T-Gramians approximate energy functionals better than the actual Gramians.
- $\sigma_i(P \cdot Q) > \sigma_i(P_\mathcal{T} \cdot Q_\mathcal{T}) \Rightarrow$ obtain smaller order of reduced system if truncation is done at the same cutoff threshold.
- Most importantly, we need solutions of only four standard Lyapunov equations.
- Interpretation of controllability/observability of the system via T-Gramians:
  - If the system is to be steered from 0 to $x_0$, where $x_0 \notin \mathrm{range}\,(P_\mathcal{T})$, then $L_c(x_0) = \infty$.
  - If the system is controllable and $x_0 \in \ker\,(Q_\mathcal{T})$, then $L_o(x_0) = 0$.

# Balanced Truncation Algorithm

**Algorithm 1** Balanced Truncation MOR for QB Systems (Truncated Gramians).

1: **Input:** $A, H, A_k, B, C$.

**Algorithm 1** Balanced Truncation MOR for QB Systems (Truncated Gramians).

1: **Input:** $A, H, A_k, B, C$.

2: Compute low-rank factors of T-Gramians: $P_{\mathcal{T}} \approx SS^T$ and $Q_{\mathcal{T}} \approx RR^T$.

# Balanced Truncation Algorithm

**Algorithm 1** Balanced Truncation MOR for QB Systems (Truncated Gramians).

1: **Input:** $A, H, A_k, B, C$.

2: Compute low-rank factors of T-Gramians: $P_{\mathcal{T}} \approx SS^T$ and $Q_{\mathcal{T}} \approx RR^T$.

3: Compute SVD of $S^T R$:

$$S^T R = U\Sigma V^T = [U_1 \;\; U_2]\mathrm{diag}(\Sigma_1, \Sigma_2)[V_1 \;\; V_2]^T.$$

# Balanced Truncation Algorithm

**Algorithm 1** Balanced Truncation MOR for QB Systems (Truncated Gramians).

1: **Input:** $A, H, A_k, B, C$.

2: Compute low-rank factors of T-Gramians: $P_{\mathcal{T}} \approx SS^T$ and $Q_{\mathcal{T}} \approx RR^T$.

3: Compute SVD of $S^T R$:

$$S^T R = U\Sigma V^T = [U_1 \ \ U_2]\mathrm{diag}(\Sigma_1, \Sigma_2)[V_1 \ \ V_2]^T.$$

4: Construct the projection matrices $\mathcal{V}$ and $\mathcal{W}$:

$$\mathcal{V} = SU_1\Sigma_1^{-1/2} \text{ and } \mathcal{W} = RV_1\Sigma_1^{-1/2}.$$

# Balanced Truncation Algorithm

**Algorithm 1** Balanced Truncation MOR for QB Systems (Truncated Gramians).

1: **Input:** $A, H, A_k, B, C$.

2: Compute low-rank factors of T-Gramians: $P_{\mathcal{T}} \approx SS^T$ and $Q_{\mathcal{T}} \approx RR^T$.

3: Compute SVD of $S^T R$:

$$S^T R = U \Sigma V^T = [U_1 \;\; U_2] \mathrm{diag}(\Sigma_1, \Sigma_2) [V_1 \;\; V_2]^T.$$

4: Construct the projection matrices $\mathcal{V}$ and $\mathcal{W}$:

$$\mathcal{V} = S U_1 \Sigma_1^{-1/2} \text{ and } \mathcal{W} = R V_1 \Sigma_1^{-1/2}.$$

5: **Output:** reduced-order matrices:

$$\hat{A} = \mathcal{W}^T A \mathcal{V}, \quad \hat{H} = \mathcal{W}^T H (\mathcal{V} \otimes \mathcal{V}), \quad \hat{A}_k = \mathcal{W}^T A_k \mathcal{V},$$
$$\hat{B} = \mathcal{W}^T B, \quad \hat{C} = C \mathcal{V}.$$

Remark: There are efficient ways to compute $\hat{H}$, avoiding the explicit computation of $\mathcal{V} \otimes \mathcal{V}$. [B./Breiten '15, B./Goyal/Gugercin. '16]

## Chafee-Infante equation

$$
\begin{aligned}
v_t + v^3 &= v_{xx} + v, && (0, L) \times (0, T),\\
v(0, .) &= u(t), && (0, T),\\
v_x(L, .) &= 0, && (0, T),\\
v(x, 0) &= v_0(x), && (0, L).
\end{aligned}
$$

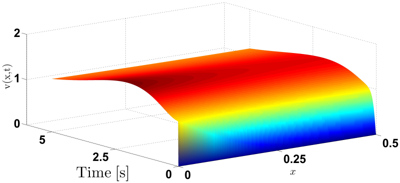



Figure: Chafee-Infante equation.

- Cubic nonlinearity that can be rewritten into QB form. [B./Breiten '15']

## Chafee-Infante equation

$$\begin{aligned} v_t + v^3 &= v_{xx} + v, && (0, L) \times (0, T), \\ v(0,.) &= u(t), && (0, T), \\ v_x(L,.) &= 0, && (0, T), \\ v(x,0) &= v_0(x), && (0, L). \end{aligned}$$

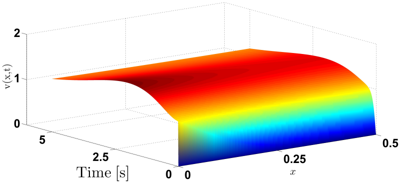



Figure: Chafee-Infante equation.

- Cubic nonlinearity that can be rewritten into QB form. [B./Breiten '15']
- The transformed QB system is of order $n = 1,000$.
- The output of interest is the response at right boundary at $x = L$.

## Chafee-Infante equation

$$\begin{aligned}
v_t + v^3 &= v_{xx} + v, && (0, L) \times (0, T),\\
v(0, .) &= u(t), && (0, T),\\
v_x(L, .) &= 0, && (0, T),\\
v(x, 0) &= v_0(x), && (0, L).
\end{aligned}$$

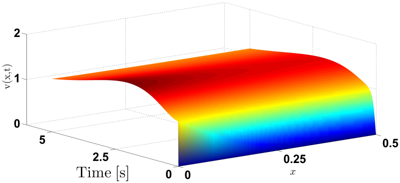



Figure: Chafee-Infante equation.

- Cubic nonlinearity that can be rewritten into QB form. [B./Breiten '15']
- The transformed QB system is of order $n = 1,000$.
- The output of interest is the response at right boundary at $x = L$.
- We determine the reduced-order system of order $r = 10$.

## Chafee-Infante equation

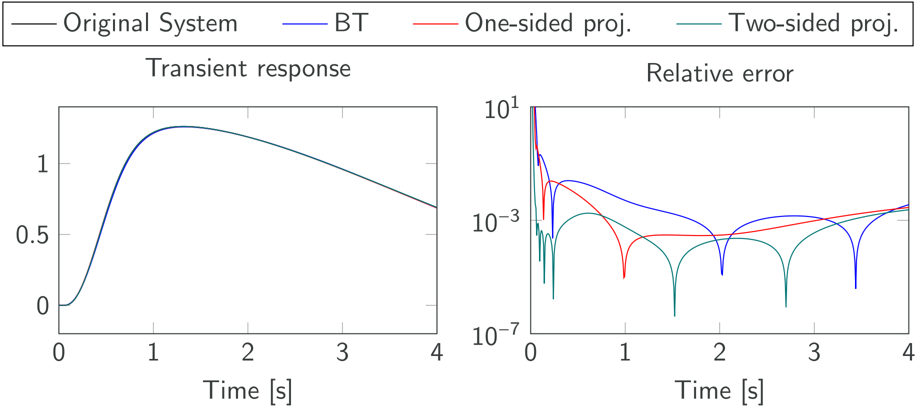



Figure: Boundary control for a control input $u(t) = 5t\exp(-t)$.

## Chafee-Infante equation

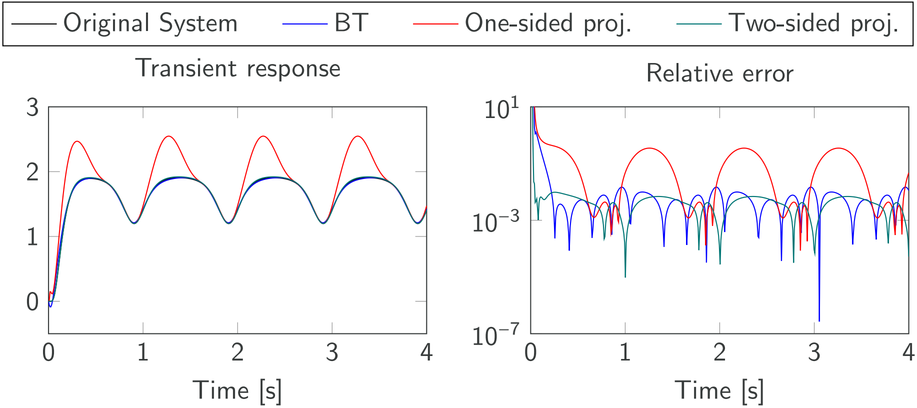



Figure: Boundary control for a control input $u(t) = 25(1 + \sin(2\pi t))/2$.

## FitzHugh-Nagumo (F-N) model

$$\epsilon v_t(x,t) = \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + q,$$
$$w_t(x,t) = hv(x,t) - \gamma w(x,t) + q,$$

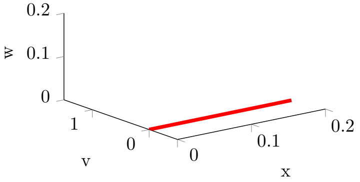
with a nonlinear function

$$f(v(x,t)) = v(v-0.1)(1-v).$$

The boundary conditions are as follows:

$$v_x(0,t) = i_0(t), \quad v_x(L,t) = 0, \quad t \geq 0,$$

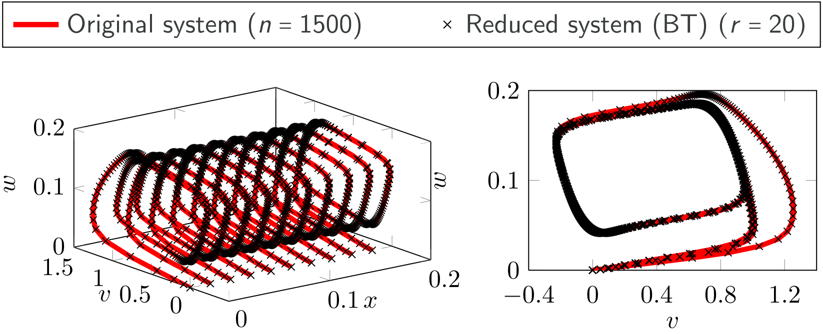
where $\epsilon = 0.015$, $h = 0.5$, $\gamma = 2$, $q = 0.05$, $L = 0.2$.




- Input $i_0(t) = 5 \cdot 10^4 t^3 \exp(-15t)$ serves as actuator.

# Numerical Results

## FitzHugh-Nagumo (F-N) model




(a) Limit-cycles at various $x$.

(b) Projection onto the $v$−$w$ plane.

Figure: Comparison of the limit-cycles obtained via the original and reduced-order (BT) systems. The reduced-order systems constructed by moment-matching methods were unstable.

# Conclusions — Balanced Truncation

- BT extended to bilinear and QB systems.
- Local Lyapunov stability is preserved.
- As of yet, only weak motivation by bounding energy functionals.
- No error bounds in terms of "Hankel" singular values.
- Computationally efficient (as compared to nonlinear balancing), and input independent.
- **To do:**
  - error bound,
  - conditions for existence of new QB Gramians,
  - extension to descriptor systems,
  - time-limited versions.

# Outline

- Applying multivariate Laplace transform to Volterra kernels yields generalized transfer functions.

# Rational Interpolation for Nonlinear Systems

- Applying multivariate Laplace transform to Volterra kernels yields generalized transfer functions.
- Rational interpolation of transfer functions using (rational) Krylov subspaces yields moment-matching **for bilinear systems:**
  - 2005–10: [Condon/Ivanov, Phillips, Bai/Skoogh, B./Feng, Breiten/Damm],
  - $H_2$-optimal model reduction via bilinear IRKA [B./Breiten '12],
  - extension to bilinear descriptor systems [B./Goyal '16, Ahmad/B./Goyal '17].

# Rational Interpolation for Nonlinear Systems

- Applying multivariate Laplace transform to Volterra kernels yields generalized transfer functions.
- Rational interpolation of transfer functions using (rational) Krylov subspaces yields moment-matching **for bilinear systems:**
  - 2005–10: [Condon/Ivanov, Phillips, Bai/Skoogh, B./Feng, Breiten/Damm],
  - $H_2$-optimal model reduction via bilinear IRKA [B./Breiten '12],
  - extension to bilinear descriptor systems [B./Goyal '16, Ahmad/B./Goyal '17].
- Analogously, for QB systems,
  - moment-matching via one-sided [Phillips '03, Feng et al '05, Gu '11] and two-sided (SISO case) [B./Breiten '12,'15] projection
    → extension to MIMO systems: talk by M. Cruz Varona, today, 16h,
  - extension to special descriptor systems ("Stokes-type") [Ahmad/B./Goyal/Heiland '15],
  - using Volterra series interpolation instead of transfer function interpolation [Ahmad/B./Jaimoukha '16, Ahmad/Baur/B. '17],
  - $H_2$-quasi-optimal model reduction via TQB-IRKA [B./Goyal/Gugercin '16]
    → talk by P. Goyal, today, 16:30h,

# Rational Interpolation for Nonlinear Systems

- Applying multivariate Laplace transform to Volterra kernels yields generalized transfer functions.
- Rational interpolation of transfer functions using (rational) Krylov subspaces yields moment-matching **for bilinear systems:**
  - 2005–10: [Condon/Ivanov, Phillips, Bai/Skoogh, B./Feng, Breiten/Damm],
  - $H_2$-optimal model reduction via bilinear IRKA [B./Breiten '12],
  - extension to bilinear descriptor systems [B./Goyal '16, Ahmad/B./Goyal '17].
- Analogously, for QB systems,
  - moment-matching via one-sided [Phillips '03, Feng et al '05, Gu '11] and two-sided (SISO case) [B./Breiten '12,'15] projection
    → extension to MIMO systems: talk by M. Cruz Varona, today, 16h,
  - extension to special descriptor systems ("Stokes-type") [Ahmad/B./Goyal/Heiland '15],
  - using Volterra series interpolation instead of transfer function interpolation [Ahmad/B./Jaimoukha '16, Ahmad/Baur/B. '17],
  - $H_2$-quasi-optimal model reduction via TQB-IRKA [B./Goyal/Gugercin '16]
    → talk by P. Goyal, today, 16:30h,
- Rational interpolation of bilinear and QB systems using **Loewner pencil framework** [Antoulas/Gosea/Ionita '16, Gosea '17].

# Own References — Balanced Truncation

P. Benner and T. Damm.
Lyapunov Equations, Energy Functionals, and Model Order Reduction of Bilinear and Stochastic Systems.
SIAM Journal on Control and Optimiztion, 49(2):686–711, 2011.

P. Benner and T. Breiten.
Low Rank Methods for a Class of Generalized Lyapunov Equations and Related Issues.
Numerische Mathematik, 124(3):441–470, 2013.

P. Benner, P. Goyal, and M. Redmann.
Truncated Gramians for Bilinear Systems and their Advantages in Model Order Reduction.
In P. Benner, M. Ohlberger, T. Patera, G. Rozza, K. Urban (Eds.), Model Reduction of Parametrized Systems, MS & A — Modeling, Simulation and Applications, Springer International Publishing, Cham, 2017.

P. Benner and P. Goyal.
Balanced Truncation Model Order Reduction for Quadratic-Bilinear Control Systems.
In preparation.

# Own References — Rational Interpolation


- P. Benner and T. Breiten.
  Krylov-Subspace Based Model Reduction of Nonlinear Circuit Models Using Bilinear and Quadratic-Linear Approximations.
  In M. Günther, A. Bartel, M. Brunk, S. Schöps, M. Striebel (Eds.), *Progress in Industrial Mathematics at ECMI 2010*, Mathematics in Industry, 17:153–159, Springer-Verlag, Berlin, 2012.
- P. Benner and T. Breiten.
  Interpolation-Based $H_2$-Model Reduction of Bilinear Control Systems.
  SIAM Journal on Matrix Analysis and Applications, 33(3):859–885, 2012.
- P. Benner and T. Breiten.
  Two-Sided Moment Matching Methods for Nonlinear Model Reduction.
  SIAM Journal on Scientific Computing, 37(2):B239-B260, 2015.
- M. I. Ahmad, P. Benner, P. Goyal, and J. Heiland.
  Moment-Matching Based Model Reduction for Stokes-Type Quadratic-Bilinear Descriptor Systems.
  MPI Magdeburg Preprint MPIMD/15-18, October 2015.
- P. Benner and P. Goyal.
  Multipoint Interpolation of Volterra Series and $H_2$-Model Reduction for a Family of Bilinear Descriptor Systems.
  Systems & Control Letters, 97:1–11, 2016.
- M. I. Ahmad, P. Benner, and I. M. Jaimoukha.
  Krylov Subspace Projection Methods for Model Reduction of Quadratic-Bilinear Systems.
  IET Control Theory & Applications, 10(16);2010-2018, 2016.
- P. Goyal, M. I. Ahmad, and P. Benner.
  Krylov Subspace-based Model Reduction for a Class of Bilinear Descriptor Systems.
  Journal of Computational and Applied Mathematics, 315:303-318, 2017.
- M. I. Ahmad, P. Benner, and U. Baur.
  Implicit Volterra Series Interpolation for Model Reduction of Bilinear Systems.
  Journal of Computational and Applied Mathematics, 316:15–28, 2017.
- P. Benner, P. Goyal, and S. Gugercin.
  $H_2$-Quasi-Optimal Model Order Reduction for Quadratic-Bilinear Control Systems.
  arXiv:1610.03279v1, October 2016.