MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Feedback Stabilization of Unsteady Flow Problems

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Felix-Klein-Kolloquium
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Optimal Control

is used for the optimization of **dynamical processes**,

described by ordinary or partial differential equations.

This is achieved by minimizing a **cost functional**

(penalizing, e.g. energy consumption, deviation from reference trajectory),

such that a prescribed target
is reached in given or **minimal time**

whilst complying with given control and state constraints.

Let (x_*, u_*) solve $\min_{u \in \mathcal{U}_{ad}} J(x,u)$ **s.t.** $\dot{x}(t) = f(x(t), u(t))$.

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Fundamental observation

Optimized trajectory $x_*(t; u_*)$ and precomputed optimal control $u_*(t)$ will not be attainable in practice due to

- modeling errors and/or unmodeled dynamics,
- model uncertainties,
- external perturbations,
- measurement errors.

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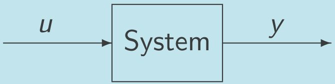
Consequence: need **feedback control**

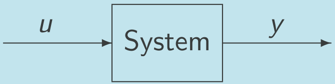
$$u(t) = u_*(t) + U(t, x(t) - x_*(t))$$

in order to attenuate perturbations/errors!

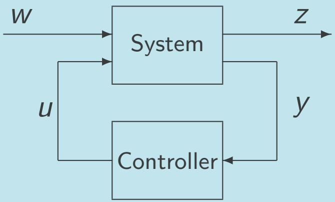
Open-loop control/optimization



Open-loop control/optimization



Closed-loop/feedback control/optimization



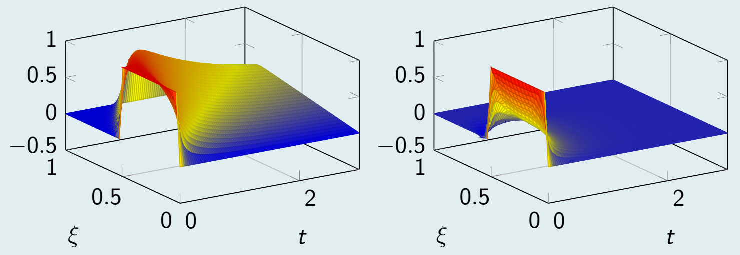
Open-loop control/optimization

Closed-loop/feedback control/optimization

Example: Optimal control of a simple transport model

Burgers' equation:

$$\begin{aligned} \partial_t x(t,\xi) &= \nu\, \partial_{\xi\xi} x(t,\xi) - x(t,\xi)\, \partial_\xi x(t,\xi) + B(\xi)u(t), \\ x(t,0) &= x(t,1) = 0, \quad x(0,\xi) = x_0(\xi), \quad \xi \in (0,1), \\ y(t,\xi) &= C\, x(t,\xi). \end{aligned}$$



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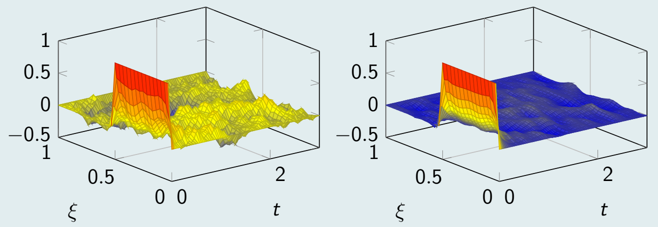
Nonlinear control (here: MPC-LQG):



Reduction of tracking error $\int_0^T \|x(t) - x_*(t)\|_2^2\,dt$ by factor > 10.

[Benner/Grner, PAMM 2006]; [Benner/Grner/Saak, Springer LNCSE 2006].

The Linear-Quadratic Regulator (LQR) Problem

Minimize $\mathcal{J}(u) = \frac{1}{2}\int_0^\infty \left(y^T Q y + u^T R u\right) dt$ for $u \in \mathcal{L}_2(0,\infty;\mathbb{R}^m)$,

subject to

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t), \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$.

The LQR/LQG Controller

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Solution of finite-dimensional LQR problem: **feedback control**

$$u_*(t) = -B^T X_* x(t) =: -K_* x(t),$$

where $X_* = X_*^T \geq 0$ is the unique **stabilizing**[1] solution of the **algebraic Riccati equation (ARE)**

$$0 = \mathcal{R}(X) := C^T QC + A^T X + XA - XBR^{-1}B^T X.$$

[1] X is stabilizing $\Leftrightarrow \Lambda\left(A - BB^T X\right) \subset \mathbb{C}^-$.

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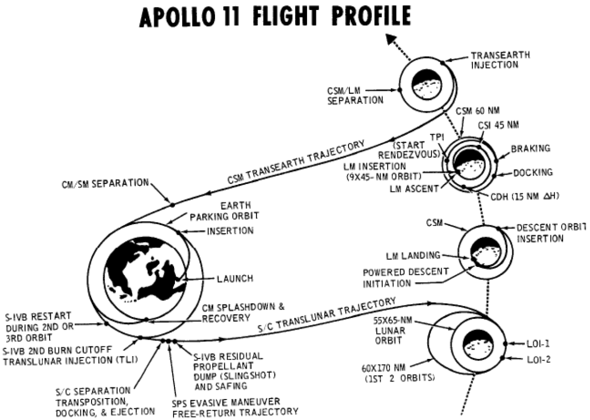
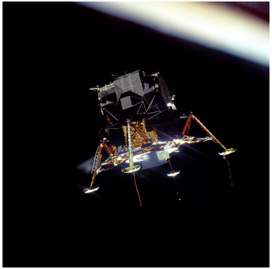
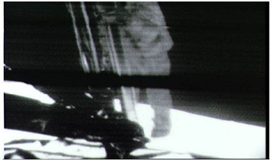
Linear-quadratic Gaussian (LQG) controller: in LQR feedback law, replace state x by **state estimation** $\hat{x}$ obtained by **Kalman-Bucy filter.**







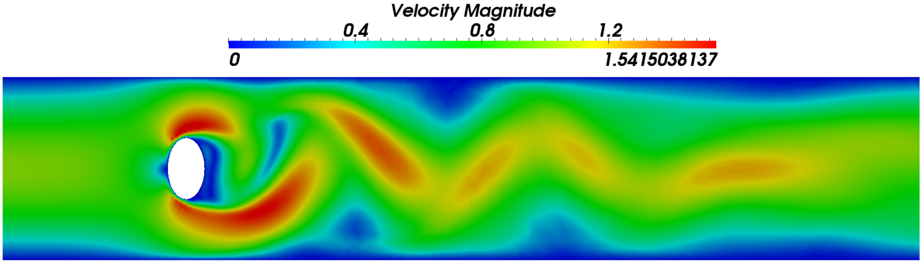
By courtesy of:

middle	Apollo 11 Mission Operations report, http://history.nasa.gov/alsj/a11/A11_MissionOpReport.pdf
left/right, top	http://de.wikipedia.org/wiki/Apollo_11,
left/right, bottom	NASA photo IDs AS11-44-6552, S69-42583.

- **Physical transport** is one of the most fundamental dynamical processes in nature.
- **Prediction** and **manipulation** of transport processes are important research topics, e.g., to
  - avoid stall — for stable and safe flight;
  - save energy (or increase attainable speed) by minimizing drag coefficient;
  - use fluid flow for optimal transport (e.g., in blood veins).
- **Open-loop** controllers are widely used in various engineering fields.
  $\rightarrow$ **Not robust** regarding perturbation
- Dynamical systems are often influenced via so called **distributed control**.
  $\rightarrow$ **Unfeasible** in many real-world areas

$\Rightarrow$ **Boundary feedback stabilization (closed-loop)**
should be used to increase robustness and feasibility.

- Consider 2D flow problems described by **incompressible Navier–Stokes equations**.
- Riccati feedback approach requires the solution of an **algebraic Riccati equation**.
- Conservation of mass introduces a **divergence-freeness** condition $\rightsquigarrow$ problems with mathematical basis of control design schemes.



Kármán vortex street

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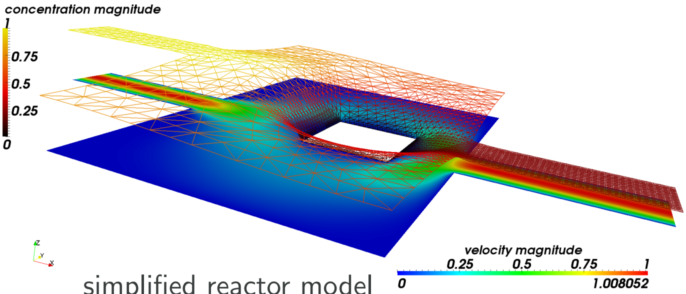
- **Coupling** flow problems with a **scalar reaction-advection-diffusion equation**.



simplified reactor model

— Available Tools and Necessary Tasks at Project Start[1] —

1. Functional analytic control approach by Raymond ([Raymond '05–'07]) works in subspace of divergence-free functions.

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   Develop techniques to deal with complex-shifted multi-field flow systems.

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⇒ develop suitable preconditioner to be used with GMRES

⇒ efficient preconditioner uses various approximation methods

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- include feedback into forward simulation within NAVIER

⇒ closed-loop forward flow simulation

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Overview

Navier–Stokes Equations

$$\frac{\partial \vec{v}}{\partial t} - \frac{1}{\mathrm{Re}}\Delta \vec{v} + (\vec{v} \cdot \nabla)\vec{v} + \nabla p = \vec{f}$$

$$\operatorname{div} \vec{v} = 0$$

- defined for time $t \in (0, \infty)$ and space $\vec{x} \in \Omega \subset \mathbb{R}^2$ bounded with $\Gamma = \partial\Omega$
- $+$ boundary and initial conditions
- initial boundary value problem with additional algebraic constraints

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Linearize + Discretize → Index-2 DAE

$$M\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = A\mathbf{v}(t) + \widehat{G}\mathbf{p}(t) + B\mathbf{u}(t)$$

$$0 = \widehat{G}^T\mathbf{v}(t)$$

$$\mathbf{y}(t) = C\mathbf{v}(t)$$

$$A,\ M \in \mathbb{R}^{n\times n},\ \widehat{G} \in \mathbb{R}^{n\times n_\mathbf{p}}$$

$$B \in \mathbb{R}^{n\times n_r},\ C \in \mathbb{R}^{n_a\times n}$$

$$\mathbf{u}(t) \in \mathbb{R}^{n_r},\ \mathbf{y}(t) \in \mathbb{R}^{n_a}$$

$$\operatorname{rank}\left(\widehat{G}\right) = n_\mathbf{p}$$

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Show that projection in [Hei/Sor/Sun '08] is discretized version of Leray projector in [Ray '06].

$$M\Pi^T = \Pi M \quad \wedge \quad \Pi^T\mathbf{v} = \mathbf{v}_{\mathrm{div},0}$$

[**Bänsch**/B./Saak/**Stoll**/Weichelt **'13,'15**]

Feedback Stabilization for Index-2 DAE Systems — Physics of Multi-Field Flow —

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Concentration Equation

$$\frac{\partial c^{(\vec{v})}}{\partial t} - \frac{1}{\mathrm{Re}\,\mathrm{Sc}}\Delta c^{(\vec{v})} + (\vec{v}\cdot\nabla)c^{(\vec{v})} = 0$$

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$A,\ M \in \mathbb{R}^{n\times n},\ \widehat{G} \in \mathbb{R}^{n\times n_\mathbf{p}}$

$B \in \mathbb{R}^{n\times n_r},\ C \in \mathbb{R}^{n_a\times n}$

$\mathbf{u}(t) \in \mathbb{R}^{n_r},\ \mathbf{y}(t) \in \mathbb{R}^{n_a}$

$\operatorname{rank}\left(\widehat{G}\right) = n_\mathbf{p}$

Linearize + Discretize → Index-2 DAE

$$M\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = A\mathbf{x}(t) + \widehat{G}\mathbf{p}(t) + B\mathbf{u}(t)$$

$$0 = \widehat{G}^T\mathbf{x}(t)$$

$$\mathbf{y}(t) = C\mathbf{x}(t)$$

$M = M^T \succ 0$

$$\mathbf{x}(t) = \begin{bmatrix}\mathbf{v}(t)\\ \mathbf{c}(t)\end{bmatrix} \in \mathbb{R}^n$$

$n = n_\mathbf{v} + n_\mathbf{c},\ N = n + n_\mathbf{p}$

Show that projection in [Hei/Sor/Sun '08] is discretized version of Leray projector in [Ray '06].

$$M\Pi^T = \Pi M \quad \wedge \quad \Pi^T\mathbf{v} = \mathbf{v}_{\mathrm{div},0}$$

[**Bänsch**/B./Saak/**Stoll**/Weichelt **'13**,**'15**]

Navier–Stokes Equations

$$\frac{\partial \vec{v}}{\partial t} - \frac{1}{\mathrm{Re}}\Delta\vec{v} + (\vec{v}\cdot\nabla)\vec{v} + \nabla p = \vec{f}$$

$$\operatorname{div}\vec{v} = 0$$

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Extension to coupled flow case, i.e.,

$$\widehat{\Pi} := \begin{bmatrix}\Pi & 0\\ 0 & I\end{bmatrix} \quad \wedge \quad \begin{bmatrix}\Pi^T & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\mathbf{v}\\ \mathbf{c}\end{bmatrix} = \begin{bmatrix}\mathbf{v}_{\mathrm{div},0}\\ \mathbf{c}\end{bmatrix}.$$

[Bänsch/B./Saak/Weichelt '14]

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[Bänsch/B./Saak/Weichelt '14]

Helmholtz Decomposition [Girault/Raviart '86]

- Splitting:

$$(L^2(\Omega))^2 = \mathbf{H}(\mathrm{div}, 0) \perp \mathbf{H}(\mathrm{div}, 0)^{\perp}$$

Divergence-free: $\mathbf{H}(\mathrm{div}, 0) := \{\vec{v} \in (L^2(\Omega))^2 \mid \mathrm{div}\,\vec{v} = 0, \vec{v} \cdot \vec{n}_{|\Gamma} = 0\}$

Curl-free: $\mathbf{H}(\mathrm{div}, 0)^{\perp} = \{\nabla p \mid p \in H^1(\Omega)\}$

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Leray Projector P

This splitting is equivalent to $\vec{v} = \vec{v}_{\mathrm{div},0} + \nabla p$, where $\vec{v}_{\mathrm{div},0}$ and p fulfill

$$\begin{aligned} \vec{v}_{\mathrm{div},0} + \nabla p &= \vec{v} \quad \text{in } \Omega, \\ \mathrm{div}\,\vec{v}_{\mathrm{div},0} &= 0 \quad \text{in } \Omega, \\ \vec{v}_{\mathrm{div},0}\cdot\vec{n} &= 0 \quad \text{on } \Gamma. \end{aligned}$$

$P : (L^2(\Omega))^2 \to \mathbf{H}(\mathrm{div}, 0)$ with $P : \vec{v} \mapsto \vec{v}_{\mathrm{div},0}$.

Projection Method [Heinkenschloss/Sorensen/Sun '08]

- Index reduction for Lyapunov-solver.
- Projector:

$$\Pi^T := I_{n_v} - M_v^{-1} G (G^T M_v^{-1} G)^{-1} G^T.$$

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Recall $P : \vec{v} \mapsto \vec{w}$:

$$\begin{aligned} \vec{w} + \nabla p &= \vec{v}, \\ \mathrm{div}\, \vec{w} &= 0 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} M_v & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} M_v \mathbf{v} \\ 0 \end{bmatrix}$$

$$\mathbf{p} = (G^T M_v^{-1} G)^{-1} G^T \mathbf{v}$$

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Leray vs. Π^T

$$\begin{aligned} \vec{w} &= P(\vec{v}), \\ 0 &= \operatorname{div} \vec{w} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{w} &= \Pi^T \mathbf{v}, \\ 0 &= G^T \mathbf{w} \end{aligned}$$

Minimize

$$\mathcal{J}(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty \lambda^2 ||\mathbf{y}||^2 + ||\mathbf{u}||^2 \,\mathrm{d}t$$

subject to

$$\widehat{\Theta}_r^T M \widehat{\Theta}_r \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\boldsymbol{x}}(t) = \widehat{\Theta}_r^T A \widehat{\Theta}_r \widetilde{\boldsymbol{x}}(t) + \widehat{\Theta}_r^T B\mathbf{u}(t)$$

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with $\widehat{\Pi} = \widehat{\Theta}_l \widehat{\Theta}_r^T$ such that $\widehat{\Theta}_r^T \widehat{\Theta}_l = I \in \mathbb{R}^{(n-n_\mathbf{p})\times(n-n_\mathbf{p})}$ and $\widetilde{\boldsymbol{x}} = \widehat{\Theta}_l^T \mathbf{x}$.

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Riccati Based Feedback Approach

- Optimal control: $\mathbf{u}(t) = -\mathcal{K}\widetilde{\mathbf{x}}(t)$, with feedback: $\mathcal{K} = \mathcal{B}^T\mathcal{X}\mathcal{M}$,

  where $\mathcal{X}$ is the solution of the generalized continuous-time algebraic Riccati equation (GCARE)

$$\mathcal{R}(\mathcal{X}) = \lambda^2\mathcal{C}^T\mathcal{C} + \mathcal{A}^T\mathcal{X}\mathcal{M} + \mathcal{M}\mathcal{X}\mathcal{A} - \mathcal{M}\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}\mathcal{M} = 0.$$

Determine $\mathcal{X} = \mathcal{X}^T \succeq 0$ such that $\mathcal{R}(\mathcal{X}) = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T\mathcal{X}\mathcal{M} + \mathcal{M}\mathcal{X}\mathcal{A} - \mathcal{M}\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}\mathcal{M} = 0.$

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Step $\mathbf{m+1}$: Solve the Lyapunov equation

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)})^T\mathcal{X}^{(m+1)}\mathcal{M} + \mathcal{M}\mathcal{X}^{(m+1)}(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)}) = -(\mathcal{W}^{(m)})^T\mathcal{W}^{(m)} \qquad (1)$$

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low-rank ADI method

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Step ℓ: Solve the projected and shifted linear system

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)} + q_\ell\mathcal{M})^T\mathcal{V}_\ell = \mathcal{Y} \quad (2)$$

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linear solver

Feedback Stabilization for Index-2 DAE Systems
— Nested Iteration without Projection —

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Replace (2) and **solve instead** the saddle point system (SPS)

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for different ADI shifts $q_\ell \in \mathbb{C}^-$ for a couple of rhs Y.

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$$\begin{bmatrix} A^T + q_\ell M & \widehat{G} \\ \widehat{G}^T & 0 \end{bmatrix} \begin{bmatrix} V_\ell \\ * \end{bmatrix} = \begin{bmatrix} \tilde{Y} \\ 0 \end{bmatrix}$$

for different ADI shifts $q_\ell \in \mathbb{C}^-$ for a couple of rhs $\tilde{Y}$.

Theorem [B./Heinkenschloss/Saak/Weichelt '16]

- assume $(\mathbf{A},\mathbf{B};\mathbf{M})$ stabilizable, $(\mathbf{C},\mathbf{A};\mathbf{M})$ detectable
- $\Rightarrow \exists$ unique, symmetric solution $X^{(*)} = \widehat{\Theta}_r \mathcal{X}^{(*)} \widehat{\Theta}_r^T$ with $\mathcal{R}(\mathcal{X}^{(*)}) = 0$ that stabilizes
$$\left(\begin{bmatrix} A - BB^T X^{(*)} M & \widehat{G} \\ \widehat{G}^T & 0 \end{bmatrix}, \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \right)$$
- for $\{X^{(k)}\}_{k=0}^{\infty}$ defined by $X^{(k)} := \widehat{\Theta}_r \mathcal{X}^{(k)} \widehat{\Theta}_r^T$, (1), and $X^{(0)}$ symmetric with $\left(\mathbf{A} - \mathbf{B}\left(\mathbf{K}^{(0)}\right)^T, \mathbf{M}\right)$ stable, it holds that, for $k \geq 1$,
$$X^{(1)} \succeq X^{(2)} \succeq \cdots \succeq X^{(k)} \succeq 0 \quad \text{and} \quad \lim_{k\to\infty} X^{(k)} = X^{(*)}$$
- $\exists 0 < \widetilde{\kappa} < \infty$ such that, for $k \geq 1$,
$$||X^{(k+1)} - X^{(*)}||_F \leq \widetilde{\kappa} ||X^{(k)} - X^{(*)}||_F^2$$

Additional Contributions [Bänsch/B./Saak/Weichelt '15,'16]

- Suitable approximation framework for Raymond's projected boundary control input.
- Proposed method directly iterates on the feedback matrix $K \in \mathbb{R}^{n \times n_r}$.
- Initial feedback for index-2 DAE systems using a special eigenvalue shifting technique.
- Improved ADI shift computation for index-2 DAE systems (Penzl- and projection shifts).

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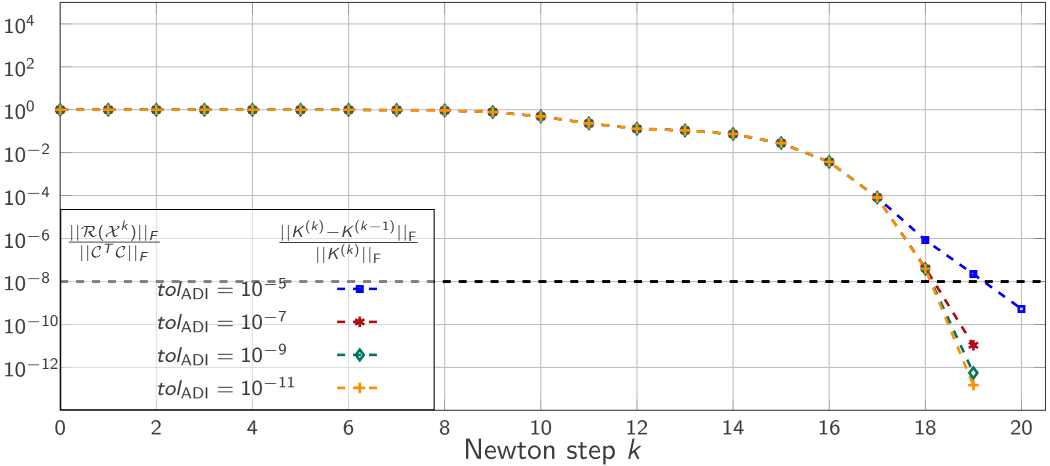
Current Problems

- Determination of suitable stopping criteria/tolerances.
- Computation of projected residuals is very costly ($\approx$ 10x ADI step).
  $\Rightarrow$ use relative change of feedback matrix [B./Li/Penzl '08]

NSE scenario: $\mathrm{Re} = 500$, $n = 5\,468$, $\lambda = 10^2$, $tol_{\mathrm{Newton}} = 10^{-8}$



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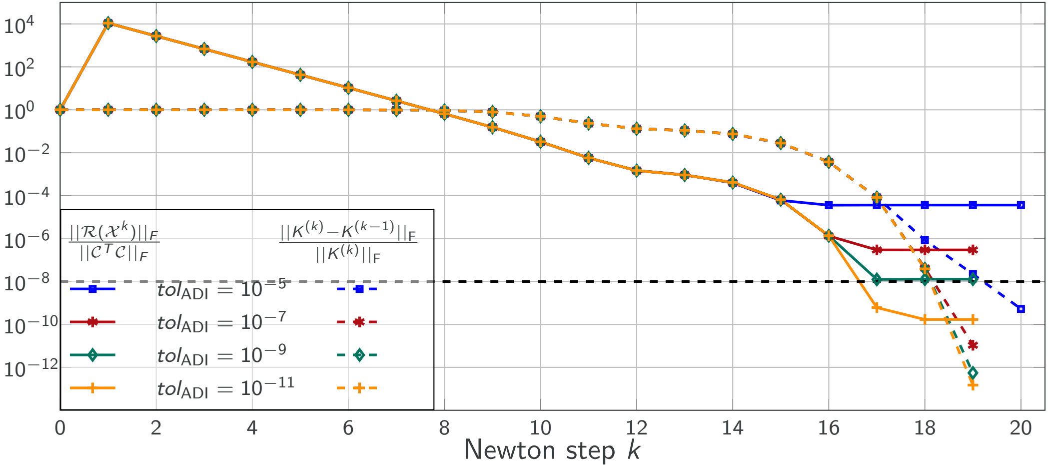

Overview

- Coefficients of GCARE are large-scale matrices (resulting from FE discretization).
- Quadratic system matrices A, $M = M^T \in \mathbb{R}^{n \times n}$ are sparse.

$$\mathcal{R}(X) = C^T C + A^T XM + MXA - MXBB^T XM$$

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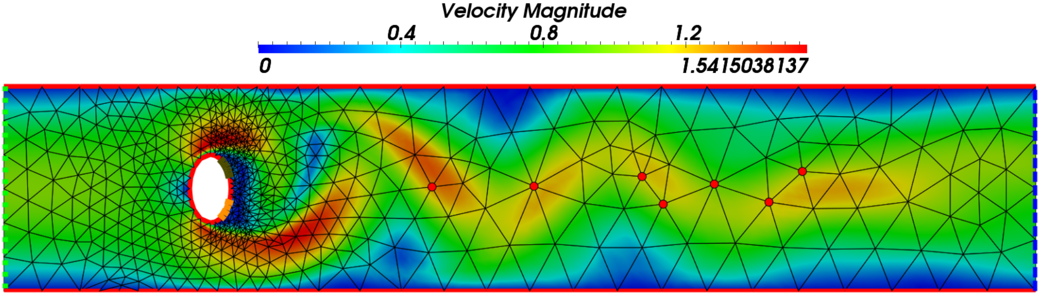

Kármán vortex street

- Coefficients of GCARE are large-scale matrices (resulting from FE discretization).
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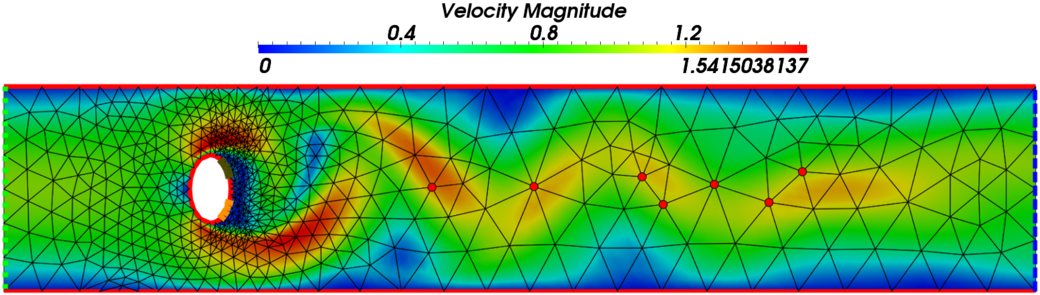
- In-/output matrices are rectangular and dense: $B \in \mathbb{R}^{n \times n_r}$, $C \in \mathbb{R}^{n_a \times n}$ with $n_r + n_a \ll n$.



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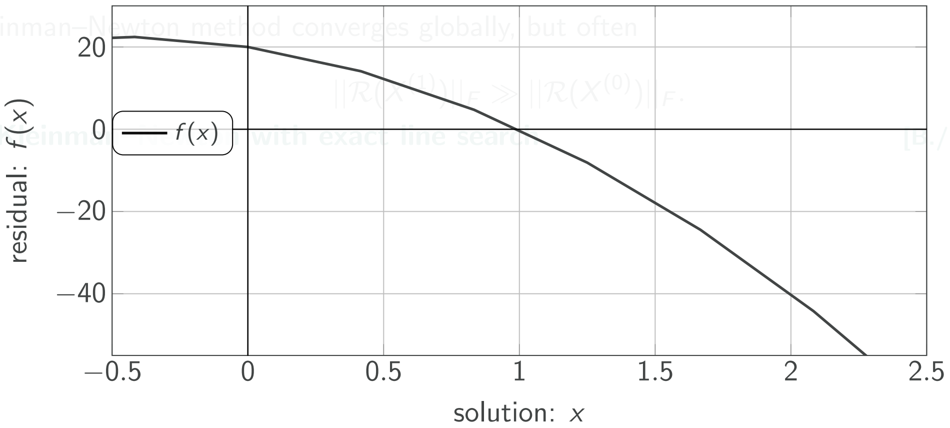

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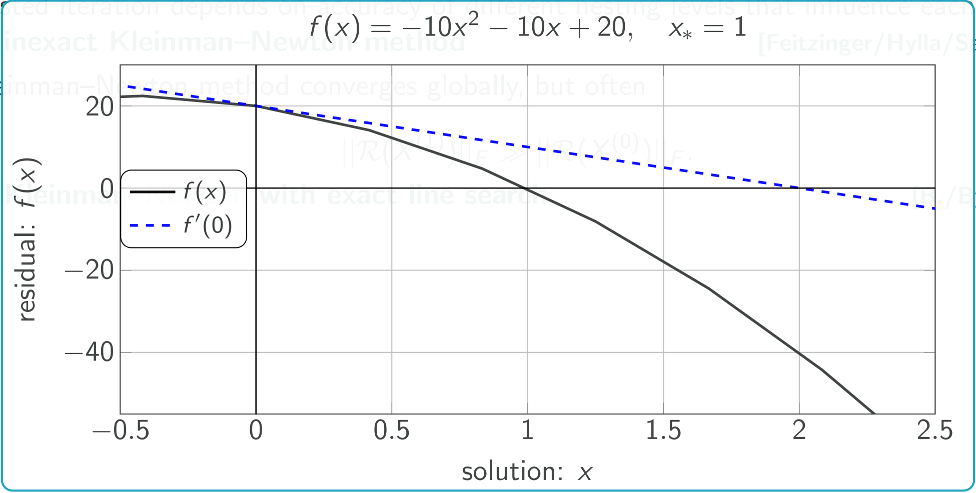

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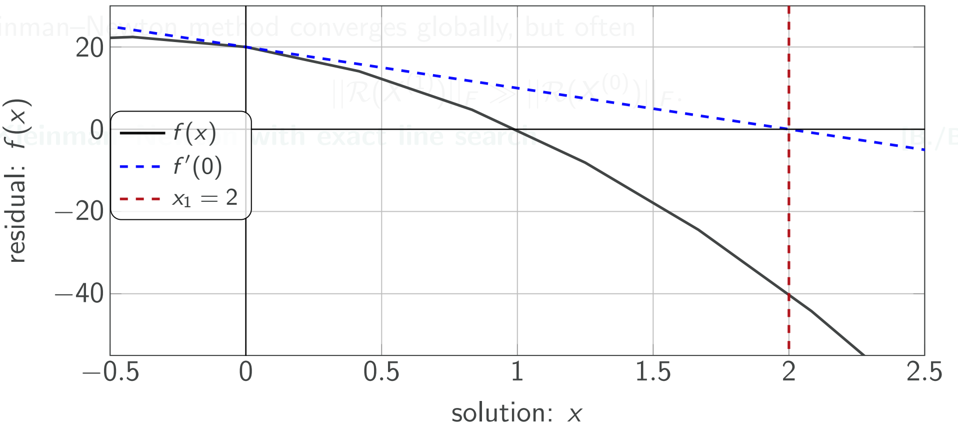

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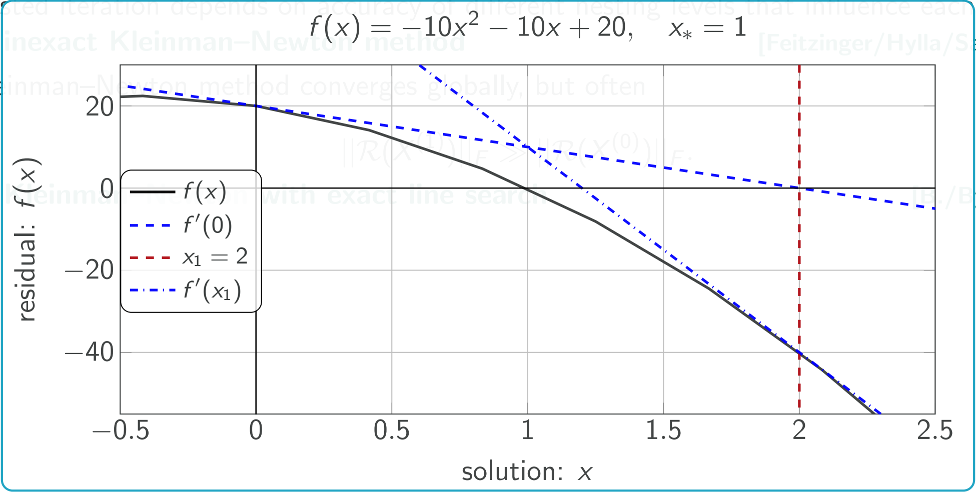

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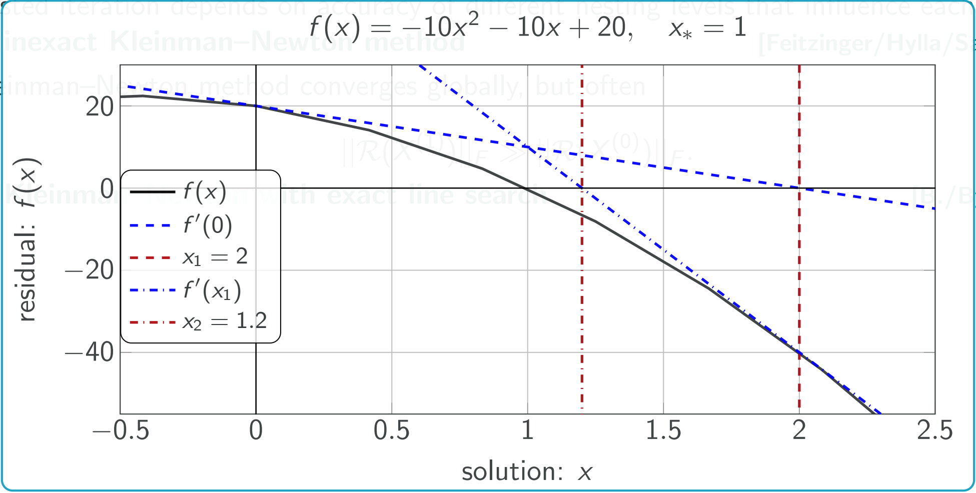

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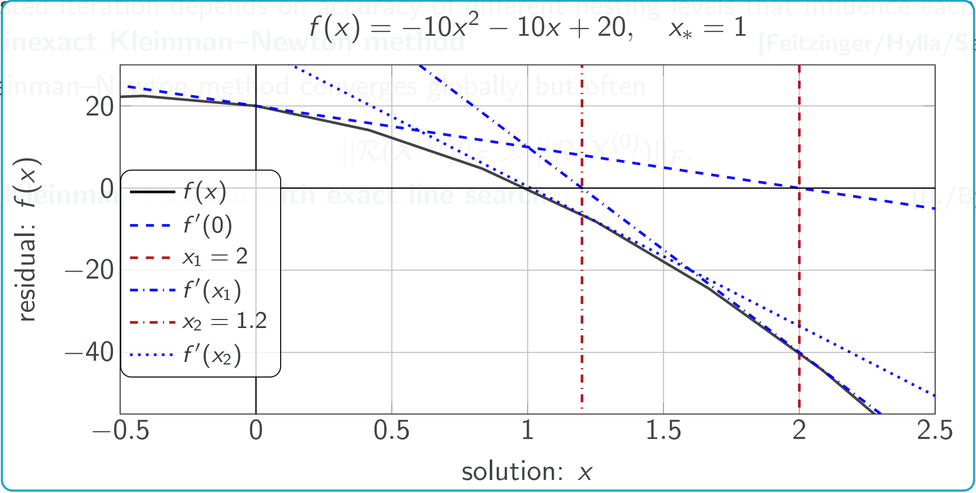

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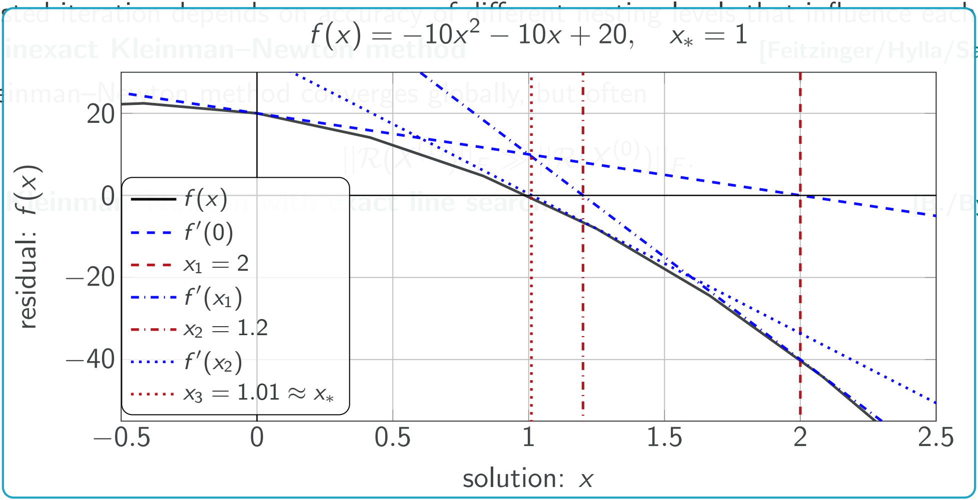

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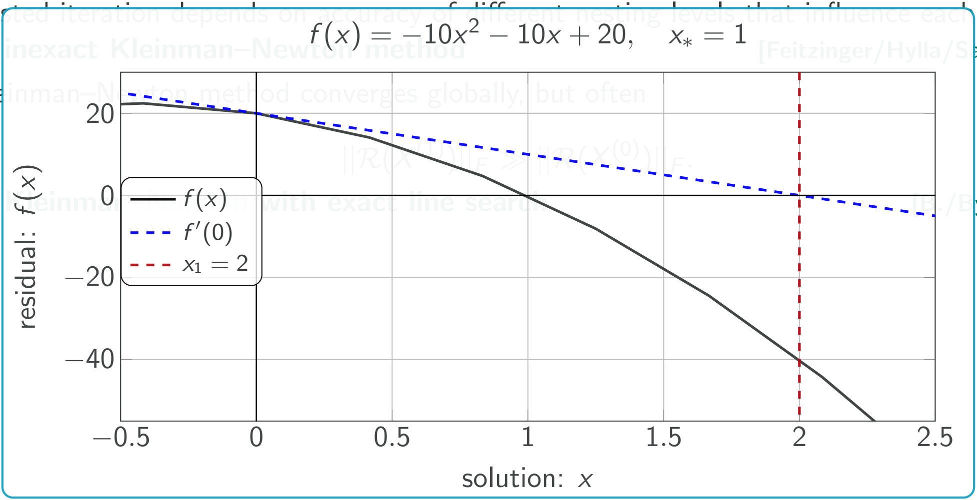

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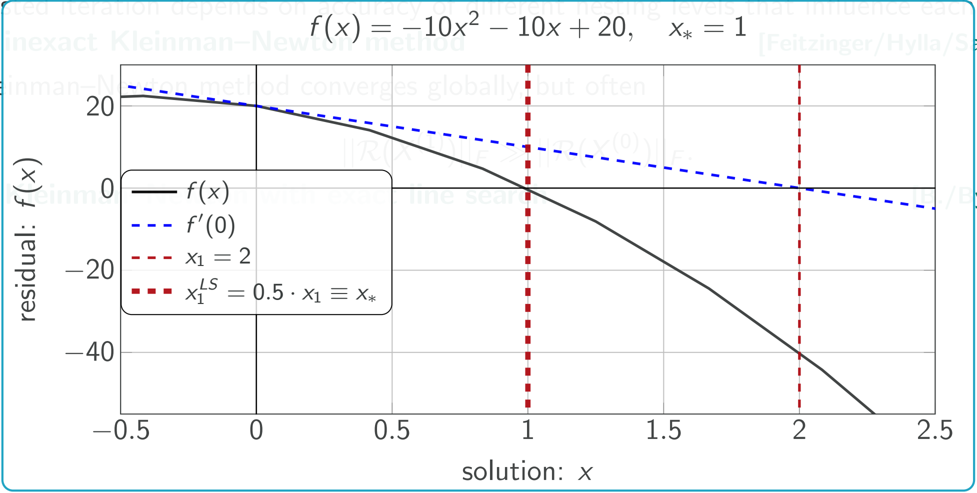

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- **extension to index-2 DAE case "straight forward"**

Theorem **[B./Heinkenschloss/Saak/Weichelt '16]**

Set $\tau_k \in (0,1)$ and assume: $(\mathcal{A}, \mathcal{B}; \mathcal{M})$ stabilizable, $(\mathcal{C}, \mathcal{A}; \mathcal{M})$ detectable, and $\exists\, \widetilde{\mathcal{X}}^{(k+1)} \succeq 0\ \forall k$ that solves

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(k)})^T \widetilde{\mathcal{X}}^{(k+1)} \mathcal{M} + \mathcal{M}\widetilde{\mathcal{X}}^{(k+1)}(\mathcal{A} - \mathcal{B}\mathcal{K}^{(k)}) = -\mathcal{C}^T\mathcal{C} - (\mathcal{K}^{(k)})^T\mathcal{K}^{(k)} + \mathcal{L}^{(k+1)}$$

such that

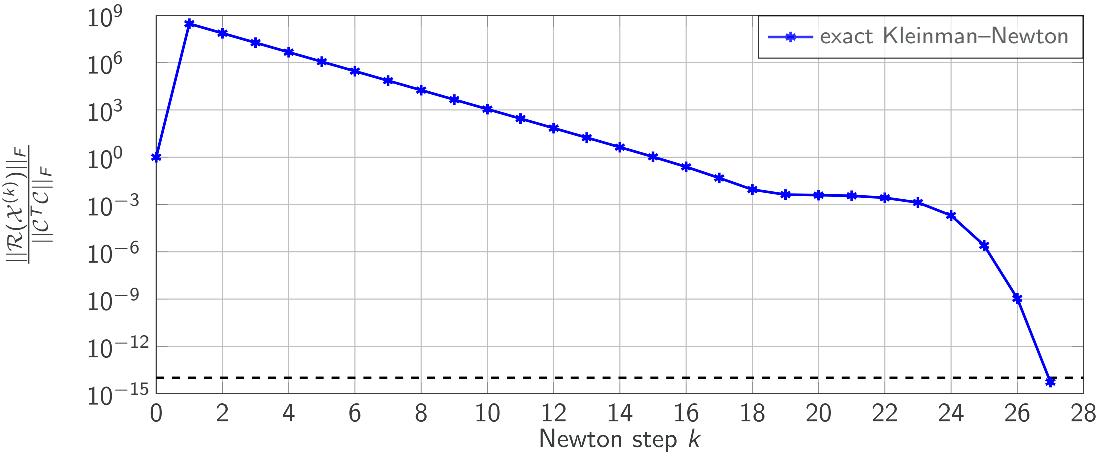
$$||\mathcal{L}^{(k+1)}||_F \leq \tau_k ||\mathcal{R}(\mathcal{X}^{(k)})||_F.$$

Find $\xi_k \in (0,1]$ such that $||\mathcal{R}(\mathcal{X}^{(k)} + \xi_k \mathcal{S}^{(k)})||_F \leq (1 - \xi_k \alpha)||\mathcal{R}(\mathcal{X}^{(k)})||_F$ and set

$$\mathcal{X}^{(k+1)} = (1 - \xi_k)\mathcal{X}^{(k)} + \xi_k \widetilde{\mathcal{X}}^{(k+1)}.$$

1. **IF** $\xi_k \geq \xi_{\min} > 0\ \forall k \ \Rightarrow\ \|\mathcal{R}(\mathcal{X}^{(k)})\|_F \to 0.$
2. **IF** $\mathcal{X}^{(k)} \succeq 0$, and $(\mathcal{A} - \mathcal{B}\mathcal{B}^T\mathcal{X}^{(k)}, \mathcal{M})$ stable for $k \geq K > 0 \ \Rightarrow\ \mathcal{X}^{(k)} \to \mathcal{X}^{(*)}$ ($\mathcal{X}^{(*)} \succeq 0$ the unique stabilizing solution).

NSE scenario: Re $= 500$, Level 1, $\lambda = 10^4$, $tol_{\text{Newton}} = 10^{-14}$



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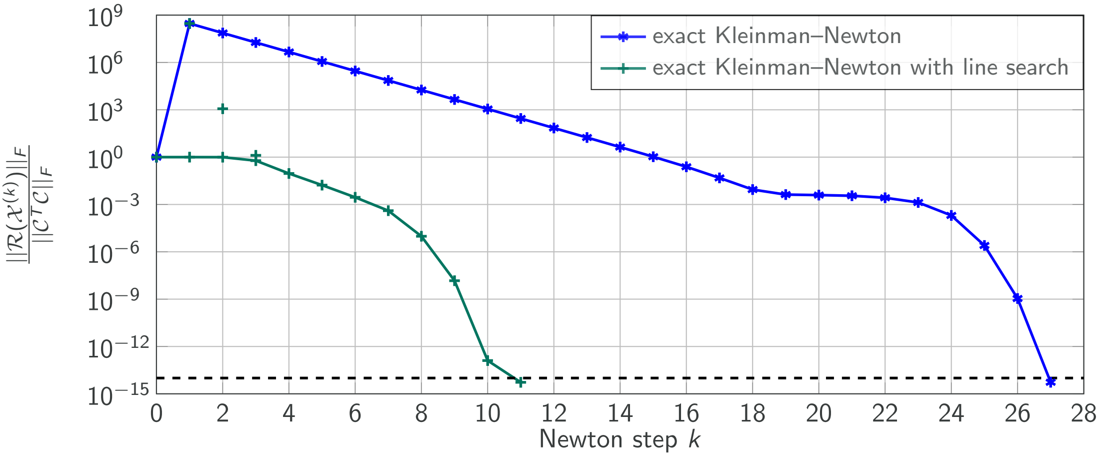

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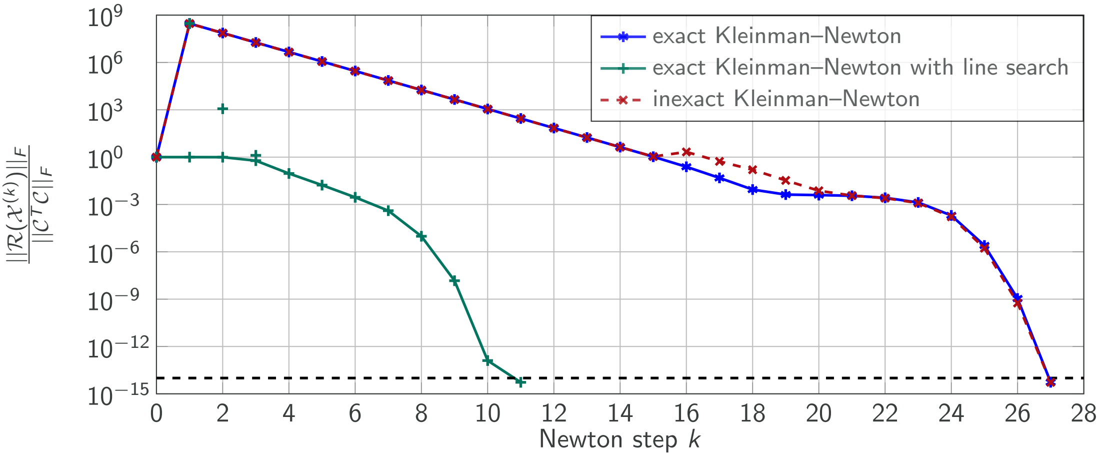

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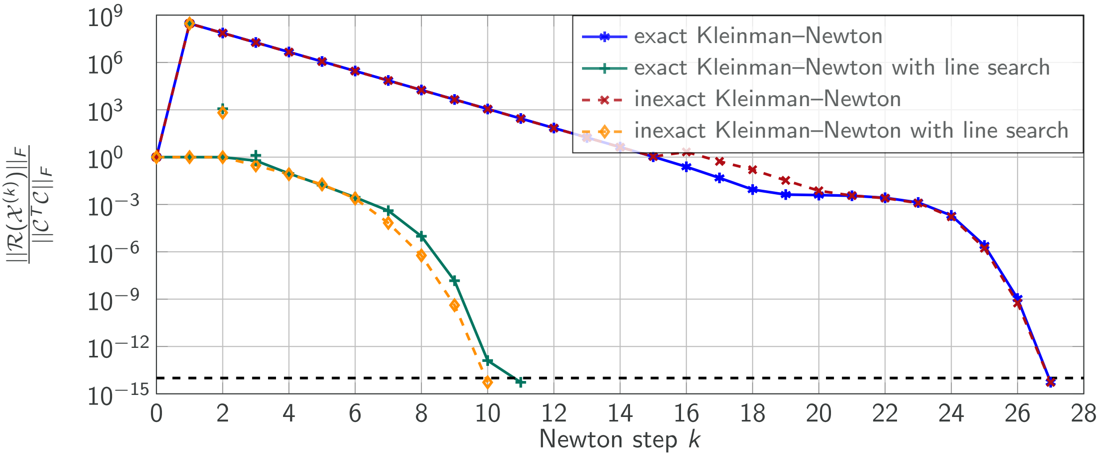

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$\#ADI$	3185	1351	852	549
$t_{\text{Newt-ADI}}$	1304.769	540.984	331.871	222.295
t_{shift}	29.998	12.568	7.370	5.507
t_{LS}	–		–	
t_{total}	**1334.767**	**553.581**	**339.241**	**227.824**

Table: Numbers of steps and timings in seconds.

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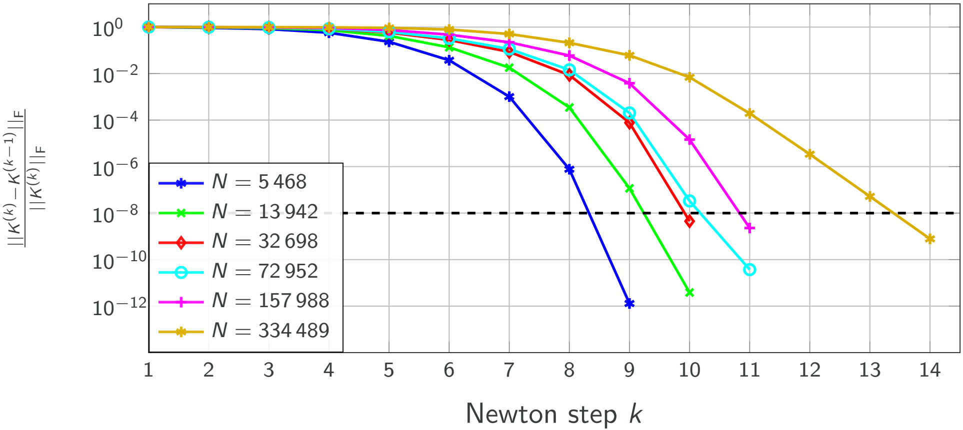

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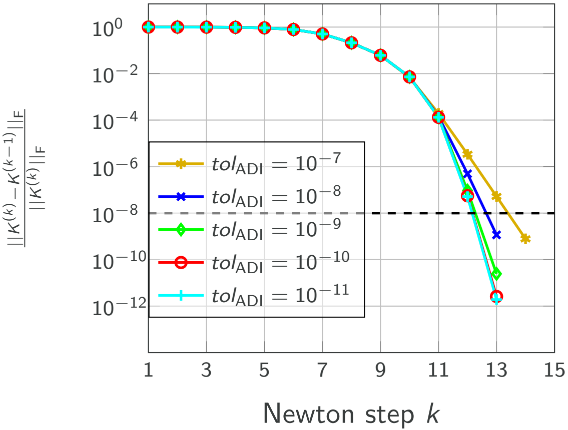

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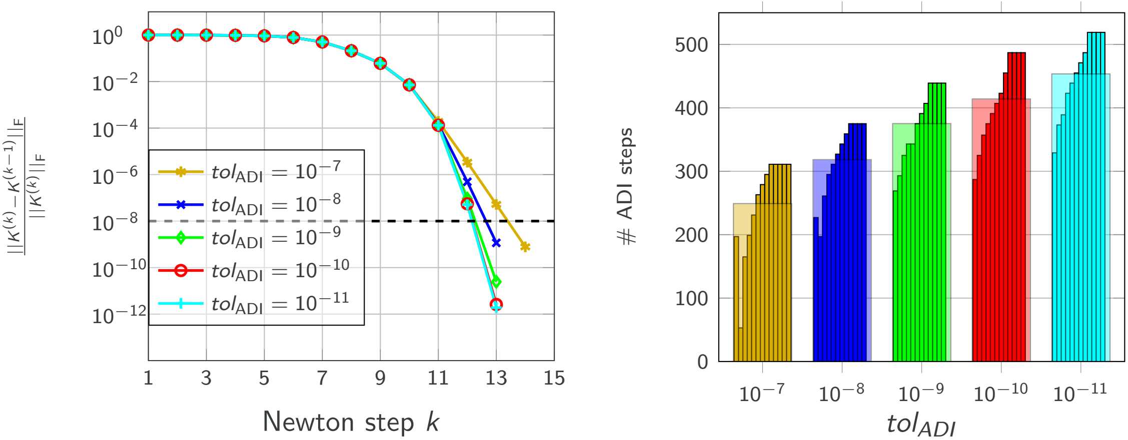

Conclusions

Main Contributions

- Analyzed **Riccati-based feedback** for **scalar** and **vector-valued transport** problems.
- Wide-spread usability tailored for standard **inf-sup stable finite element** discretizations.
- Established **specially tailored Kleinman–Newton-ADI** that **avoids explicit projections**.
- **Suitable preconditioners** for multi-field flow problems have been developed.
- **Ongoing research** in similar areas has been **incorporated**.
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$\Rightarrow$ Showed **overall usability** of new approach by a **closed-loop forward simulation**.

References

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