Householder Symposium XVIII, Tahoe City, California
June 13, 2011

Rational Krylov Subspaces for Nonlinear Model Reduction

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NETWORK THEORY

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Overview

Introduction

Model Order Reduction

Here, we consider large-scale nonlinear control systems of the form

$$\Sigma : \quad \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

with $f, g : \mathbb{R}^n \to \mathbb{R}^n$, $C \in \mathbb{R}^{p \times n}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$.

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$$\hat{\Sigma} : \quad \begin{cases} \dot{\hat{x}}(t) = \hat{f}(\hat{x}(t)) + \hat{g}(\hat{x}(t))u(t), \\ \hat{y}(t) = \hat{C}\hat{x}(t), \quad \hat{x}(0) = \hat{x}_0, \end{cases}$$

with $\hat{f}, \hat{g} : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{n}}$, $C \in \mathbb{R}^{p \times \hat{n}}$, $x \in \mathbb{R}^{\hat{n}}$, $u \in \mathbb{R}^m$, $\hat{y} \in \mathbb{R}^p$, $\hat{n} \ll n$.

Goal

$\hat{y} \approx y$ for all admissible u.

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Linear System Norms

Let us start with linear systems, i.e. $f(x) = Ax$ and $g(x) = B$.

Two common system norms for measuring approximation quality:

- $\mathcal{H}_2$-norm, $||\Sigma||_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_0^{2\pi} \operatorname{tr}\left(H^*(-i\omega) H(i\omega) \right) d\omega \right)^{\frac{1}{2}}$,

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- $\mathcal{H}_\infty$-norm, $||\Sigma||_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{max}\left(H(i\omega) \right)$,

where

$$H(s) = C\left(sI - A\right)^{-1} B$$

denotes the corresponding transfer function of the linear system.

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We focus on the first one $\rightsquigarrow$ interpolation-based model reduction approaches.

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Error system and $\mathcal{H}_2$-Optimality

In order to find an $\mathcal{H}_2$-optimal reduced system, define the error system:

$$A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.$$

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$\rightsquigarrow$ first-order necessary $\mathcal{H}_2$-optimality conditions (SISO)

$$H(-\lambda_i) = \hat{H}(-\lambda_i),$$
$$H'(-\lambda_i) = \hat{H}'(-\lambda_i),$$

where λ_i are the poles of the reduced system $\hat{\Sigma}$.

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$$\begin{aligned} H(-\lambda_i)\tilde{B}_i &= \hat{H}(-\lambda_i)\tilde{B}_i, && \text{for } i = 1, \dots, \hat{n}, \\ \tilde{C}_i^T H(-\lambda_i) &= \tilde{C}_i^T \hat{H}(-\lambda_i), && \text{for } i = 1, \dots, \hat{n}, \\ \tilde{C}_i^T H'(-\lambda_i)\tilde{B}_i &= \tilde{C}_i^T \hat{H}'(-\lambda_i)\tilde{B}_i && \text{for } i = 1, \dots, \hat{n}, \end{aligned}$$

where $\hat{A} = R\Lambda R^{-T}$ is the spectral decomposition of the reduced system and $\tilde{B} = \hat{B}^T R^{-T}$, $\tilde{C} = \hat{C}R$.

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$$\begin{aligned} &\operatorname{vec}(I_p)^T \left(e_j e_i^T \otimes C\right) (-\Lambda \otimes I_n - I_{\hat{n}} \otimes A)^{-1} \left(\tilde{B}^T \otimes B\right) \operatorname{vec}(I_m) \\ &= \operatorname{vec}(I_p)^T \left(e_j e_i^T \otimes \hat{C}\right) \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A}\right)^{-1} \left(\tilde{B}^T \otimes \hat{B}\right) \operatorname{vec}(I_m), \end{aligned}$$

for $i = 1, \ldots, \hat{n}$ and $j = 1, \ldots, p$.

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Interpolation of the Transfer Function [Grimme '97]

Construct reduced transfer function by Petrov-Galerkin projection $\mathcal{P} = VW^T$, i.e.

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where V and W are given as

$$V = \left[(\sigma_1 I - A)^{-1}B, \ldots, (\sigma_r I - A)^{-1}B\right],$$
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$$H(\sigma_i) = \hat{H}(\sigma_i) \quad \text{and} \quad H'(\sigma_i) = \hat{H}'(\sigma_i),$$

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$\rightsquigarrow$ iterative algorithms (IRKA/MIRIAm) that yield $\mathcal{H}_2$-optimal models.

[Gugercin et al. '08], [Bunse-Gerstner et al. '07], [Van Dooren et al. '08]

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Bilinear Control Systems

Let us now focus on the special case $f(x) = A$ and

$$g(x) = B + \left[N_1, \ldots, N_m\right](I_m \otimes x),$$

i.e. bilinear control systems:

$$\Sigma: \quad \begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} N_i x(t) u_i(t) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

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- Approximation of weakly nonlinear systems $\rightsquigarrow$ Carleman linearization.
- A lot of linear concepts can be extended, e.g. transfer functions, Gramians, Lyapunov equations, ...
- An equivalent structure arise for some stochastic control systems.

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Some Basic Facts

Output Characterization (SISO): Volterra series

$$y(t) = \sum_{k=1}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} K(t_1, \dots, t_k) u(t - t_1 - \dots - t_k) \cdots u(t - t_k) dt_k \cdots dt_1,$$

with kernels $K(t_1, \dots, t_k) = Ce^{At_k} N_1 \cdots e^{At_2} N_1 e^{At_1} B$.

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Multivariate Laplace-transform (SISO):

$$H_k(s_1,\dots,s_k) = C(s_k I - A)^{-1} N_1 \cdots (s_2 I - A)^{-1} N_1 (s_1 I - A)^{-1} B.$$

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Bilinear $\mathcal{H}_2$-norm (MIMO):

$$||\Sigma||_{\mathcal{H}_2} := \left(\operatorname{tr} \left(\sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^k} \overline{H_k(i\omega_1, \dots, i\omega_k)} H_k^T(i\omega_1, \dots, i\omega_k) \right) \right)^{\frac{1}{2}}.$$

[Zhang/Lam. '02]

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

$\mathcal{H}_2$-Norm Computation

Lemma [B./Breiten '11]

Let Σ denote a bilinear system. Then, the $\mathcal{H}_2$-norm is given as:

$$||\Sigma||^2_{\mathcal{H}_2} = (\text{vec}(I_p))^T \, (C \otimes C) \left(-A \otimes I - I \otimes A - \sum_{i=1}^{m} N_i \otimes N_i \right)^{-1} (B \otimes B) \, \text{vec}(I_m).$$

Error System

In order to find an $\mathcal{H}_2$-optimal reduced system, define the error system $\Sigma^{err} := \Sigma - \hat{\Sigma}$ as follows:

$$A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad N_i^{err} = \begin{bmatrix} N_i & 0 \\ 0 & \hat{N}_i \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.$$

$\mathcal{H}_2$-Model Reduction

$\mathcal{H}_2$-Optimality Conditions

Let us assume $\hat{\Sigma}$ is given by its eigenvalue decomposition:

$$\hat{A} = R\Lambda R^{-1}, \quad \tilde{N}_i = R^{-1}\hat{N}_i R, \quad \tilde{B} = R^{-1}\hat{B}, \quad \tilde{C} = \hat{C}R.$$

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Using Λ, $\tilde{N}_i$, $\tilde{B}$, $\tilde{C}$ as optimization parameters, we can derive necessary conditions for $\mathcal{H}_2$-optimality, e.g.:

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$$(\operatorname{vec}(I_q))^T \left(e_j e_\ell^T \otimes C\right)\left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{i=1}^{m} \tilde{N}_i \otimes N_i\right)^{-1}\left(\tilde{B} \otimes B\right)\operatorname{vec}(I_m)$$

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Where is the connection to the interpolation of transfer functions?

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$$(\mathrm{vec}(I_q))^T \left(e_j e_\ell^T \otimes C\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A\right)^{-1} \mathrm{vec}(B\tilde{B}^T)$$
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$$(\mathrm{vec}(I_q))^T \left(e_j e_\ell^T \otimes C\right) \begin{pmatrix} -\lambda_1 I - A & & \\ & \ddots & \\ & & -\lambda_{\hat{n}} I - A \end{pmatrix}^{-1} \begin{pmatrix} B\tilde{B}_1^T \\ \vdots \\ B\tilde{B}_{\hat{n}}^T \end{pmatrix}$$

$$= (\mathrm{vec}(I_q))^T \left(e_j e_\ell^T \otimes \hat{C}\right) \begin{pmatrix} -\lambda_1 I - \hat{A} & & \\ & \ddots & \\ & & -\lambda_{\hat{n}} I - \hat{A} \end{pmatrix}^{-1} \begin{pmatrix} \hat{B}\tilde{B}_1^T \\ \vdots \\ \hat{B}\tilde{B}_{\hat{n}}^T \end{pmatrix}.$$

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$$H(-\lambda_\ell)\tilde{B}_\ell^T = \hat{H}(-\lambda_\ell)\tilde{B}_\ell^T$$

$\rightsquigarrow$ tangential interpolation at mirror images of reduced system poles

A First Iterative Approach

Algorithm 1 Bilinear IRKA

Input: $A, N_i, B, C, \hat{A}, \hat{N}_i, \hat{B}, \hat{C}$

Output: $A^{opt}, N_i^{opt}, B^{opt}, C^{opt}$

1: **while** (change in $\Lambda > \epsilon$) **do**

2: $R\Lambda R^{-1} = \hat{A}$, $\tilde{B} = R^{-1}\hat{B}$, $\tilde{C} = \hat{C}R$, $\tilde{N}_i = R^{-1}\hat{N}_i R$

3: $\mathrm{vec}(V) = \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{i=1}^{m} \tilde{N}_i \otimes N_i\right)^{-1} \left(\tilde{B} \otimes B\right) \mathrm{vec}(I_m)$

4: $\mathrm{vec}(W) = \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A^T - \sum_{i=1}^{m} \tilde{A}_i^T \otimes N_i^T\right)^{-1} \left(\tilde{C}^T \otimes C^T\right) \mathrm{vec}(I_q)$

5: $V = \mathrm{orth}(V)$, $W = \mathrm{orth}(W)$

6: $\hat{A} = \left(W^T V\right)^{-1} W^T A V$, $\hat{N}_i = \left(W^T V\right)^{-1} W^T N_i V$,
$\hat{B} = \left(W^T V\right)^{-1} W^T B$, $\hat{C} = CV$

7: **end while**

8: $A^{opt} = \hat{A}$, $N_i^{opt} = \hat{N}_i$, $B^{opt} = \hat{B}$, $C^{opt} = \hat{C}$

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

A Heat Transfer Model

- 2-dimensional heat distribution
  [Benner, Saak '05]
- Boundary control by spraying intensities of a cooling fluid

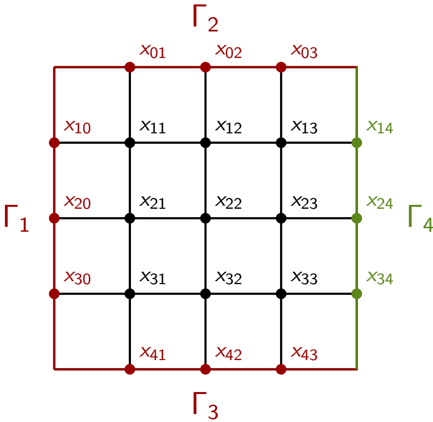
$$
\begin{aligned}
\Omega &= (0,1)\times(0,1), & \\
x_t &= \Delta x & \text{in } \Omega,\\
n\cdot\nabla x &= c\cdot u_{1,2,3}(x-1) & \text{on } \Gamma_1,\Gamma_2,\Gamma_3,\\
x &= u_4 & \text{on } \Gamma_4.
\end{aligned}
$$

- Spatial discretization $k\times k$-grid

$$\Rightarrow \dot{x} \approx A_1 x + \sum_{i=1}^{3} N_i x u_i + Bu$$

$$\Rightarrow A_2 = 0.$$

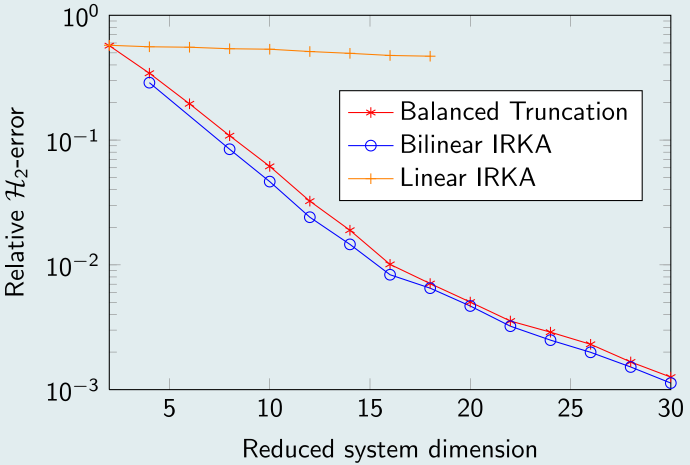
- Output: $y = \frac{1}{k^2}\begin{bmatrix}1 & \dots & 1\end{bmatrix}.$



$\mathcal{H}_2$-Model Reduction for Bilinear Systems

A Heat Transfer Model

Comparison of relative $\mathcal{H}_2$-error for $n = 10.000$



$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Fokker-Planck Equation

As a second example, we consider a dragged Brownian particle whose one-dimensional motion is given by

$$dX_t = -\nabla V(X_t, t)dt + \sqrt{2\sigma}dW_t,$$

with $\sigma = \frac{2}{3}$ and $V(x, u) = W(x, t) + \Phi(x, u_t) = (x^2 - 1)^2 - xu - x$.
Alternatively, one can consider ([HARTMANN ET AL. '10]) ,

$$\rho(x, t)dx = \mathbf{P}\left[X_t \in [x, x + dx)\right]$$

which is described by the Fokker-Planck equation

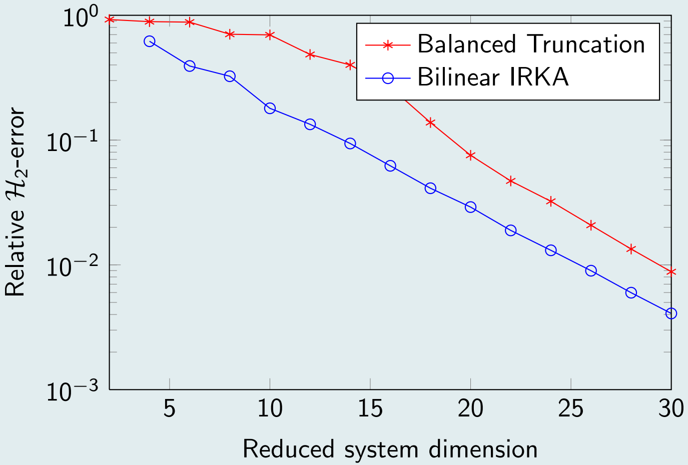
$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \sigma\Delta\rho + \nabla \cdot (\rho\nabla V), && (x, t) \in (-2, 2) \times (0, T],\\
0 &= \sigma\nabla\rho + \rho\nabla B, && (x, t) \in \{-2, 2\} \times [0, T],\\
\rho_0 &= \rho, && (x, t) \in (-2, 2) \times 0.
\end{aligned}$$

Output C discrete characteristic function of the interval $[0.95, 1.05]$.

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Fokker-Planck Equation

Comparison of relative $\mathcal{H}_2$-error for $n = 500$



Nonlinear Model Reduction

Quadratic-Bilinear Differential Algebraic Equations (QBDAEs)

Finally, we come back to the more general case with $f(x)$ nonlinear and $g(x) = B$. Here, the class of quadratic-bilinear differential algebraic equations

$$\Sigma : \quad \begin{cases} E\dot{x}(t) = A_1 x(t) + A_2 x(t) \otimes x(t) + N x(t) u(t) + B u(t), \\ \quad y(t) = C x(t), \quad x(0) = x_0, \end{cases}$$

where $E, A_1, N \in \mathbb{R}^{n \times n}, A_2 \in \mathbb{R}^{n \times n^2}$ (Hessian tensor), $B, C^T \in \mathbb{R}^n$ are quite helpful.

- A large class of smooth nonlinear control-affine systems can be transformed into the above type of control system.
- The transformation is exact, but a slight increase of the state dimension has to be accepted.
- Input-output behavior can be characterized by generalized transfer functions $\rightsquigarrow$ enables us to use Krylov-based reduction techniques.

Nonlinear Model Reduction

Transformation via McCormick Relaxation

Theorem [Gu'09]

Assume that the state equation of a nonlinear system Σ is given by

$$\dot{x} = a_0 x + a_1 g_1(x) + \ldots + a_k g_k(x) + Bu,$$

where $g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$ are compositions of uni-variable rational, exponential, logarithmic, trigonometric or root functions, respectively. Then, by iteratively taking derivatives and adding algebraic equations, respectively, Σ can be transformed into a system of QBDAEs.

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Example

- $\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1}, \quad \dot{x}_2 = -x_2 + u.$

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- $\dot{x}_1 = z_1 \cdot z_2, \quad \dot{x}_2 = -x_2 + u, \quad \dot{z}_1 = -z_1 \cdot (-x_2 + u),$
  $\dot{z}_2 = \frac{2 \cdot x_1 \cdot z_1 \cdot z_2}{2 \cdot z_2} = x_1 \cdot z_1.$

Nonlinear Model Reduction

Variational Analysis and Linear Subsystems

Analysis of nonlinear systems by variational equation approach:

Nonlinear Model Reduction

Variational Analysis and Linear Subsystems

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- consider input of the form $\alpha u(t)$,

Nonlinear Model Reduction

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- nonlinear system is assumed to be a series of homogeneous nonlinear subsystems, i.e. response should be of the form

$$x(t) = \alpha x_1(t) + \alpha^2 x_2(t) + \alpha^3 x_3(t) + \dots .$$

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- comparison of terms $\alpha^i, i = 1, 2, \dots$ leads to series of systems

$$\begin{aligned} E\dot{x}_1 &= A_1 x_1 + Bu, \\ E\dot{x}_2 &= A_1 x_2 + A_2 x_1 \otimes x_1 + N x_1 u, \\ E\dot{x}_3 &= A_1 x_3 + A_2 (x_1 \otimes x_2 + x_2 \otimes x_1) + N x_2 u \\ &\ \ \vdots \end{aligned}$$

Nonlinear Model Reduction

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- although i-th subsystem is coupled nonlinearly to preceding systems, linear systems are obtained if terms x_j, $j < i$, are interpreted as pseudo-inputs.

Nonlinear Model Reduction

Generalized Transfer Functions

In a similar way, a series of generalized symmetric transfer functions can be obtained via the growing exponential approach:

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$$H_2(s_1, s_2) = \frac{1}{2!} C\left((s_1 + s_2)E - A_1\right)^{-1} \left[N\left(G_1(s_1) + G_1(s_2)\right) + A_2\left(G_1(s_1) \otimes G_1(s_2) + G_1(s_2) \otimes G_1(s_1)\right)\right],$$

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$$\left. +A_2\left(G_1(s_1)\otimes G_1(s_2) + G_1(s_2)\otimes G_1(s_1)\right)\right],$$

$$H_3(s_1, s_2, s_3) = \frac{1}{3!}C\left((s_1+s_2+s_3)E - A_1\right)^{-1}$$
$$\Big[N(G_2(s_1,s_2) + G_2(s_2,s_3) + G_2(s_1,s_3))$$
$$+ A_2\big(G_1(s_1)\otimes G_2(s_2,s_3) + G_1(s_2)\otimes G_2(s_1,s_3)$$
$$+ G_1(s_3)\otimes G_2(s_1,s_3) + G_2(s_2,s_3)\otimes G_1(s_1)$$
$$+ G_2(s_1,s_3)\otimes G_1(s_2) + G_2(s_1,s_2)\otimes G_1(s_3)\big)\Big].$$

Nonlinear Model Reduction

Characterization via Multimoments

For simplicity, focus on the first two transfer functions. For $H_1(s_1)$, choosing σ and making use of the Neumann lemma leads to

$$H_1(s_1) = \sum_{i=0}^{\infty} C \underbrace{\left((A_1 - \sigma E)^{-1} E\right)^i (A_1 - \sigma E)^{-1} B \, (s_1 - \sigma)^i}_{m^i_{s_1,\sigma}}.$$

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Similarly, specifying an expansion point (τ, ξ) yields

$$H_2(s_1, s_2) = \frac{1}{2} \sum_{i=0}^{\infty} C \left((A_1 - (\tau + \xi)E)^{-1} E\right)^i (A_1 - (\tau + \xi)E)^{-1} \, (s_1 + s_2 - \tau - \xi)^i \cdot$$
$$\left[A_2 \left(\sum_{j=0}^{\infty} m^j_{s_1,\tau} \otimes \sum_{k=0}^{\infty} m^k_{s_2,\xi} + \sum_{k=0}^{\infty} m^k_{s_2,\xi} \otimes \sum_{j=0}^{\infty} m^j_{s_1,\tau}\right) + N \left(\sum_{p=0}^{\infty} m^p_{s_1,\tau} + \sum_{p=0}^{\infty} m^q_{s_2,\xi}\right)\right]$$

Nonlinear Model Reduction

Constructing the Projection Matrix

For derivatives around $\sigma = \tau = \xi$ up to order $q - 1$, construct the Krylov spaces:

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$$U = \mathcal{K}_q\left((A_1 - \sigma E)^{-1}E, (A_1 - \sigma E)^{-1}B\right)$$

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\begin{aligned}
U =& \mathcal{K}_q\left((A_1 - \sigma E)^{-1} E, (A_1 - \sigma E)^{-1} B\right) \\
&\textbf{for } i = 1 : q \\
&\quad W_i = \mathcal{K}_{q-i+1}\left((A_1 - 2\sigma E)^{-1} E, (A_1 - 2\sigma E)^{-1} N U_i\right),
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&\qquad \textbf{for } j = 1 : \min(q-i+1, i) \\
&\qquad\quad Z_i = \mathcal{K}_{q-i-j+2} \left((A_1 - 2\sigma E)^{-1} E, (A_1 - 2\sigma E)^{-1} A_2 U_i \otimes U_j\right),
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U_i denoting the i-*th* column of U.

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&\qquad \textbf{for } j = 1:\min(q-i+1, i) \\
&\qquad\quad Z_i = \mathcal{K}_{q-i-j+2}\left((A_1-2\sigma E)^{-1}E, (A_1-2\sigma E)^{-1}A_2U_i\otimes U_j\right),
\end{aligned}
$$

U_i denoting the i-*th* column of U. Set $V = \mathrm{orth}([U, W, Z])$ and construct $\hat{\Sigma}$ by the Galerkin-Projection $\mathcal{P} = VV^T$:

$$
\hat{A}_1 = V^TA_1V \in \mathbb{R}^{\hat{n}\times\hat{n}}, \quad \hat{A}_2 = V^TA_2V\otimes V \in \mathbb{R}^{\hat{n}\times\hat{n}^2},
$$

$$
\hat{N} = V^TNV \in \mathbb{R}^{\hat{n}\times\hat{n}}, \quad \hat{B} = V^TB \in \mathbb{R}^{\hat{n}}, \quad \hat{C}^T = V^TC \in \mathbb{R}^{\hat{n}}.
$$

Nonlinear Model Reduction

Two-Sided Projection Methods

Similarly to the linear case, one can exploit duality concepts, in order to construct two-sided projection methods.

Nonlinear Model Reduction

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Similarly to the linear case, one can exploit duality concepts, in order to construct two-sided projection methods.

Interpreting A_2 as the matricization of a 3-tensor from $\mathbb{R}^{n\times n\times n}$. one can show that the dual Krylov spaces have to be constructed as follows

$$\tilde{U} = \mathcal{K}_q\left((A_1-\sigma E)^{-T}E^T, (A_1-\sigma E)^{-T}C^T\right)$$

for $i = 1 : q$

$$\tilde{W}_i = \mathcal{K}_{q-i+1}\left((A_1-2\sigma E)^{-T}E^T, (A_1-2\sigma E)^{-T}N^T\tilde{U}_i\right),$$

for $j = 1 : \min(q-i+1, i)$

$$\tilde{Z}_i = \mathcal{K}_{q-i-j+2}\left((A_1-2\sigma E)^{-T}E^T, (A_1-2\sigma E)^{-T}\tilde{A}_2 U_i \otimes \tilde{U}_j\right),$$

where $\tilde{A}_2$ now is another matricization of the Hessian tensor.

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\end{aligned}
$$

where $\tilde{A}_2$ now is another matricization of the Hessian tensor.
Note: If one uses the third matricization, then $U_i \otimes \tilde{U}_j$ has to be replaced by $\tilde{U}_i \otimes U_j$. For matricizations, see e.g. [Kressner/Tobler '10].

Nonlinear Model Reduction

The FitzHugh-Nagumo System

- FitzHugh-Nagumo system modeling a neuron

  [Chaturantabut, Sorensen '09]

$$
\begin{aligned}
\epsilon v_t(x,t) &= \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g, \\
w_t(x,t) &= hv(x,t) - \gamma w(x,t) + g,
\end{aligned}
$$

  with $f(v) = v(v-0.1)(1-v)$ and initial and boundary conditions

$$
\begin{aligned}
v(x,0) &= 0, & w(x,0) &= 0, & x &\in [0,1], \\
v_x(0,t) &= -i_0(t), & v_x(1,t) &= 0, & t &\geq 0,
\end{aligned}
$$

  where $\epsilon = 0.015,\ h = 0.5,\ \gamma = 2,\ g = 0.05,\ i_0(t) = 5 \cdot 10^4 t^3 \exp(-15t)$
- original state dimension $n = 2 \cdot 400$, reduced state dimension $\hat{n} = 26$, chosen interpolation point $\sigma = 1$
- 3D phase space

Nonlinear Model Reduction

Two-Dimensional Burgers Equation

- 2D-Burgers Equation on $\underbrace{(0,1)\times(0,1)}_{:=\Omega}\times[0,T]$

$$u_t = -\left(u\cdot\nabla\right)u + \nu\Delta u$$

with $u(x,y,t)\in\mathbb{R}^2$ describing the motion of a compressible fluid.

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$$u_t = -\left(u\cdot\nabla\right)u + \nu\Delta u$$

with $u(x,y,t)\in\mathbb{R}^2$ describing the motion of a compressible fluid.

- Consider initial and boundary conditions

$$\begin{aligned}
&u_x(x,y,0)=\frac{\sqrt{2}}{2}, && u_y(x,y,0)=\frac{\sqrt{2}}{2}, && \text{for } x,y\in\Omega_1:=(0,0.5],\\
&u_x(x,y,0)=0, && u_y(x,y,0)=0, && \text{for } x,y,\in\Omega\backslash\Omega_1,\\
&u_x=0, && u_y=0, && \text{for } x,y,\in\partial\Omega.
\end{aligned}$$

Nonlinear Model Reduction

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  with $u(x,y,t)\in\mathbb{R}^2$ describing the motion of a compressible fluid.
- Consider initial and boundary conditions

$$\begin{aligned}
u_x(x,y,0) &= \frac{\sqrt{2}}{2}, & u_y(x,y,0) &= \frac{\sqrt{2}}{2}, && \text{for } x,y\in\Omega_1 := (0,0.5],\\
u_x(x,y,0) &= 0, & u_y(x,y,0) &= 0, && \text{for } x,y,\in\Omega\backslash\Omega_1,\\
u_x &= 0, & u_y &= 0, && \text{for } x,y,\in\partial\Omega.
\end{aligned}$$

- Spatial discretization $\rightsquigarrow$ QBDAE system with nonzero i.c. and $N=0$. $\rightsquigarrow$ reformulate as system with zero i.c. and constant input.

Nonlinear Model Reduction

Two-Dimensional Burgers Equation

- 2D-Burgers Equation on $\underbrace{(0,1)\times(0,1)}_{:=\Omega}\times[0,T]$

$$u_t = -\left(u\cdot\nabla\right)u + \nu\Delta u$$

  with $u(x,y,t)\in\mathbb{R}^2$ describing the motion of a compressible fluid.
- Consider initial and boundary conditions

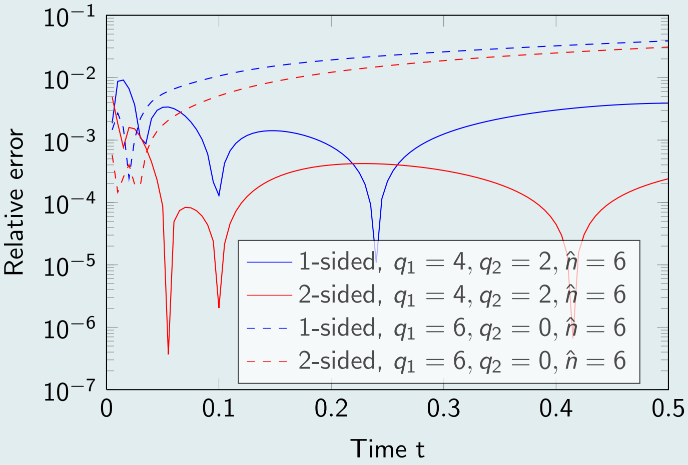
$$\begin{aligned} u_x(x,y,0) &= \frac{\sqrt{2}}{2}, & u_y(x,y,0) &= \frac{\sqrt{2}}{2}, & &\text{for } x,y\in\Omega_1 := (0,0.5], \\ u_x(x,y,0) &= 0, & u_y(x,y,0) &= 0, & &\text{for } x,y,\in\Omega\backslash\Omega_1, \\ u_x &= 0, & u_y &= 0, & &\text{for } x,y,\in\partial\Omega. \end{aligned}$$

- Spatial discretization $\rightsquigarrow$ QBDAE system with nonzero i.c. and $N=0$. $\rightsquigarrow$ reformulate as system with zero i.c. and constant input.
- Output C chosen to be average x-velocity.

Nonlinear Model Reduction

Two-Dimensional Burgers Equation

Comparison of relative time-domain error for $n = 1600$



Nonlinear Model Reduction

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- Now consider initial and boundary conditions

$$\begin{aligned}
&u_x(x,y,0)=0, && u_y(x,y,0)=0, && \text{for } x,y\in\Omega,\\
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- Spatial discretization $\rightsquigarrow$ QBDAE system with zero i.c. and 4 inputs $B\in\mathbb{R}^{n\times 4}$, N_1,N_2,N_3,N_4, ROM with $q_1=5, q_2=2, \sigma=0, \hat{n}=52$.

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- State reconstruction by reduced model $x\approx V\hat{x}$, max. rel. err $< 3\%$.

Conclusions

- A lot of linear reduction techniques can be transferred to the nonlinear case.
- We have shown a generalized interpolation theory for bilinear control systems $\rightsquigarrow$ $\mathcal{H}_2$-optimal model reduction.
- Many nonlinear dynamics can be expressed by a system of quadratic-bilinear differential algebraic equations.
- There exist Krylov subspace methods that extend the concept of moment-matching $\rightsquigarrow$ using basic tools from tensor theory allow for better approximations.

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Thank you for your attention. . .