MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Fast Frequency Response Analysis using Model Order Reduction

Peter Benner

Seminar *Numerische Mathematik und Mechanik*

Universität zu Köln, 18 November 2016

Outline

Frequency Response Analysis

- *Frequency response is the quantitative measure of the output spectrum of a system or device in response to a stimulus, and is used to characterize the dynamics of the system.*[1]

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- It is based on the Laplace/Fourier transforms, mapping a time-domain signal to frequency domain.
- For linear time-invariant (LTI) system $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases}$

  this requires the solution of a sequence of linear systems

  $$H(:,:,k) = C(\jmath\omega_k I_n - A)^{-1}B + D, \quad k = 1, \ldots K,$$

  where $\{\omega_1, \ldots, \omega_K\}$ defines a frequency grid (in [rad/s]).

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Frequency Response Analysis

Linear time-invariant (LTI) system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m} \\ y(t) = Cx(t) + Du(t), & C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m} \end{cases}$$

Laplace (Fourier) transform (assuming $x(0) = 0$) $\rightsquigarrow$

$$Y(s) = \left(C(sI_n - A)^{-1}B + D\right) U(s) =: G(s)U(s), \quad s \in \mathbb{C},$$

where $G \in \mathbb{R}(s)^{q\times m}$ is the (rational) transfer function of Σ.

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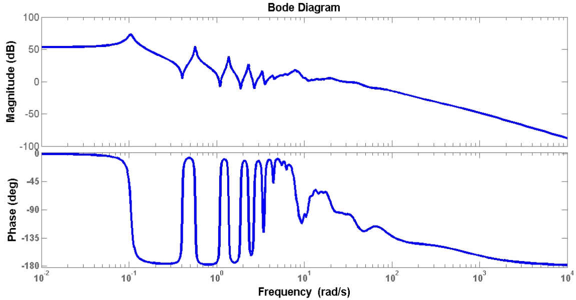
- Visualization often by Bode (amplitude and phase), Bode magnitude (amplitude only), or sigma (only $\sigma(G(\jmath\omega))$={nonzero singular values of G}) plots.
- Requires K evaluations of the transfer function $G(\jmath\omega_k)$, $k = 1, \ldots, K$, i.e., solution of K linear systems of equations with $\min\{q, m\}$ right-hand sides.

Example: Beam

- Clamped beam (discretized elasticity equation).
- $n = 348$, $m = q = 1$, MATLAB® automatically chooses $K = 389$.

Bode plot `bode(sys)`



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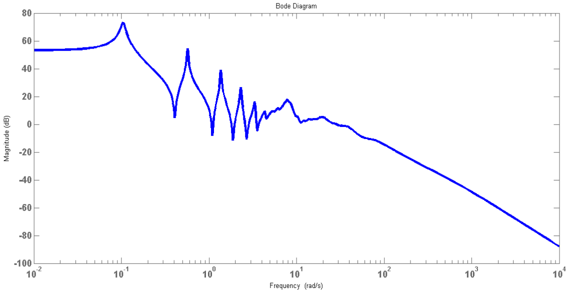

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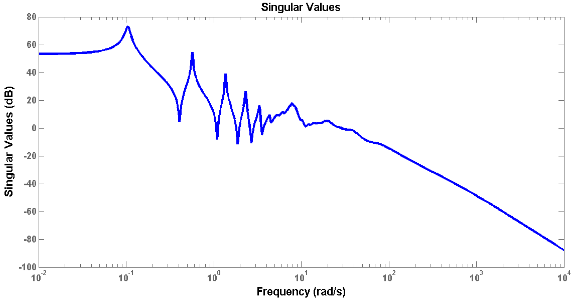
Sigma plot

`sigma(sys)`



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Example: CD Player

- Modal model of a rotating swing arm in a CD player.
- $n = 120$, $m = q = 2$, MATLAB automatically chooses $K = 445$.

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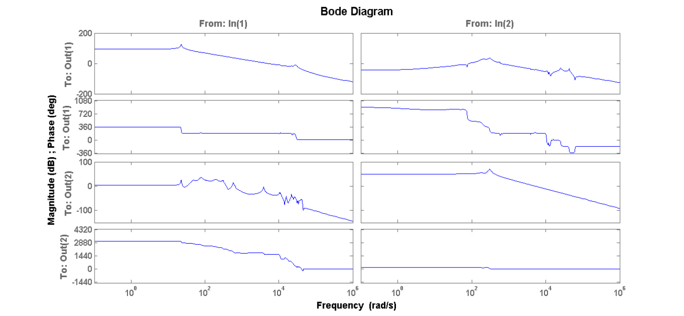

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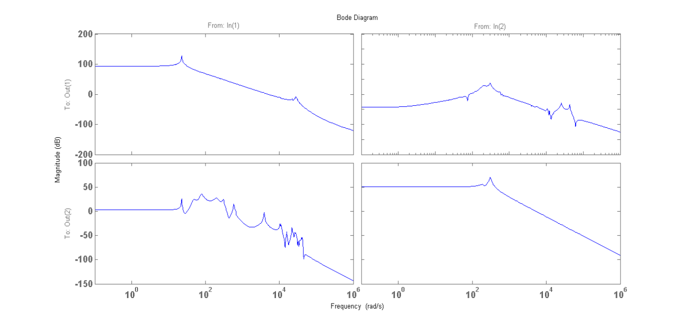

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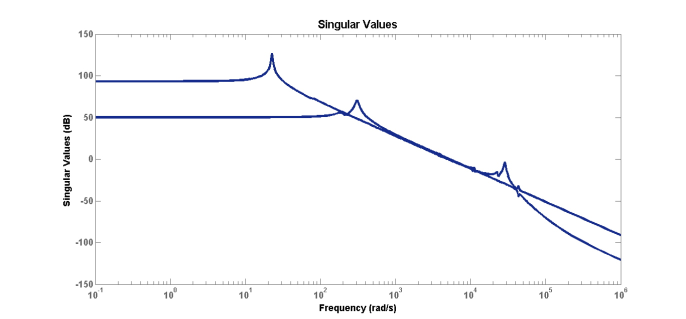

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Accelerating Frequency response calculations

- Use your Numerical Analysis. . .

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Example (ISS Module 12A, structure model)

- $n = 1412$, $m = 2$, $q = 3$
- MATLAB built-in command `freqresp`:

```
>> tic, h=freqresp(sys,[100]); toc
Elapsed time is 9.135650 seconds.
```

- Use sparse arithmetic:

```
>> tic, hs=C*((100*i*speye(1412)-As)\B); toc
Elapsed time is 0.007246 seconds.
```

- Note: solve $(\jmath\omega I - A)X = B$, rather than $(\jmath\omega I - A)^T Y^T = C^T$ as $m < q$.

Accelerating Frequency response calculations

- Use your Numerical Analysis...
- Intelligent use of iterative methods, e.g., block-Krylov methods, recycling Krylov subspaces, shift-invariance of Krylov subspaces, ...
  [Freund, Frommer, de Sturler, Meerbergen, Morgan, Nabben, Parks, Simoncini, Soodhalter, Szyld, Vuik, ...]

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- Here: employ model (order) reduction techniques!
  $\rightsquigarrow$ Replace A, B, C, D by $(\hat{A}, \hat{B}, \hat{C}, \hat{D}) \in \mathbb{R}^{r\times r} \times \mathbb{R}^{r\times m} \times \mathbb{R}^{q\times r} \times \mathbb{R}^{q\times m}$ with

$$r \ll n$$

so that

$$\|G(\jmath\omega) - \hat{G}(\jmath\omega)\|$$

is small in desired frequency range!

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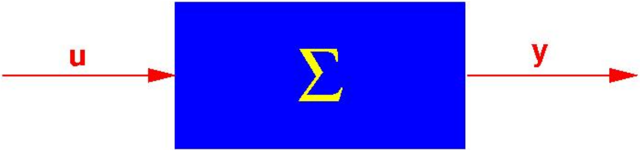
- $\|\,.\,\| \in \{\|\,.\,\|_{\mathcal{H}_2}, \|\,.\,\|_{\mathcal{H}_\infty}\}$.

Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)), \quad x(t_0) = x_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^q$.



Original System

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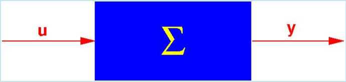

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Model Reduction for Dynamical Systems

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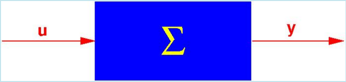

Reduced-Order Model — ROM

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- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
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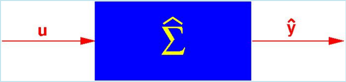

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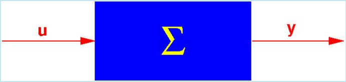

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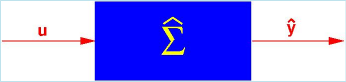

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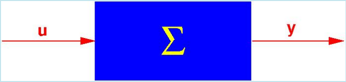

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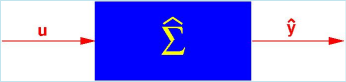

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Secondary goal: reconstruct approximation of x from $\hat{x}$.

Introduction

Linear Systems

Linear, Time-Invariant (LTI) Systems

$$
\begin{aligned}
\dot{x} &= f(t,x,u) = Ax + Bu, \quad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\
y &= g(t,x,u) = Cx + Du, \quad C \in \mathbb{R}^{q\times n}, \quad D \in \mathbb{R}^{q\times m}.
\end{aligned}
$$

Linear Systems

Formulating model reduction in frequency domain

Approximate the dynamical system

$$\begin{array}{rcll} \dot{x} &=& Ax + Bu, & A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y &=& Cx + Du, & C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}, \end{array}$$

by reduced-order system

$$\begin{array}{rcll} \dot{\hat{x}} &=& \hat{A}\hat{x} + \hat{B}u, & \hat{A} \in \mathbb{R}^{r\times r},\ \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &=& \hat{C}\hat{x} + \hat{D}u, & \hat{C} \in \mathbb{R}^{q\times r},\ \hat{D} \in \mathbb{R}^{q\times m} \end{array}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.$$

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$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.$$

$\Longrightarrow$ Approximation problem: $\min\limits_{\text{order}\,(\hat{G}) \leq r} \|G - \hat{G}\|.$

Application Areas

- structural mechanics / (elastic) multibody simulation
- systems and control theory
- micro-electronics / circuit simulation / VLSI design
- computational electromagnetics,
- design of MEMS/NEMS (micro/nano-electrical-mechanical systems),
- computational acoustics,
- ...

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- ...
- **Current trend:** more and more multi-physics problems, i.e., coupling of several field equations, e.g.,
  - electro-thermal (e.g., bondwire heating in chip design),
  - fluid-structure-interaction,
  - ...

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Basic idea

- $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians = solutions P, Q of Lyapunov equations

  $$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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- Compute balanced realization (needs P, Q!) of the system via state-space transformation

  $$\begin{aligned} \mathcal{T} : (A, B, C) &\mapsto (TAT^{-1}, TB, CT^{-1}) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right). \end{aligned}$$

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}) = (A_{11}, B_1, C_1)$.

Properties

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$$\|y - \hat{y}\|_2 \leq \left(2 \sum_{k=\hat{n}+1}^{n} \sigma_k\right) \|u\|_2.$$

Balanced Truncation for Linear Systems

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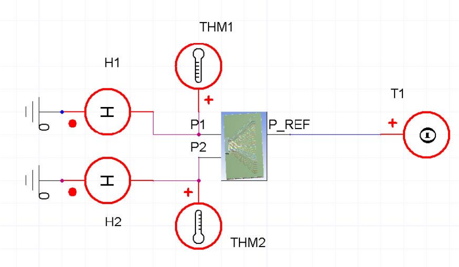
$$\|y - \hat{y}\|_2 \leq \left(2 \sum_{k=\hat{n}+1}^{n} \sigma_k\right) \|u\|_2.$$

Practical implementation

- Rather than solving Lyapunov equations for P, Q (n^2 unknowns!), find $S, R \in \mathbb{R}^{n \times s}$ with $s \ll n$ such that $P \approx SS^T$, $Q \approx RR^T$.
- Reduced-order model directly obtained via small-scale ($s \times s$) SVD of $R^T S$!
- No $\mathcal{O}(n^3)$ or $\mathcal{O}(n^2)$ computations necessary!

Electro-Thermal Simulation of Integrated Circuit (IC) [Source: Evgenii Rudnyi, CADFEM GmbH]

- SIMPLORER® test circuit with 2 transistors.



- Conservative thermal sub-system in SIMPLORER: voltage ⇝ temperature, current ⇝ heat flow.
- Original model: $n = 270.593$, $m = q = 2 \Rightarrow$ Computing time (on Intel Xeon dualcore 3GHz, 1 Thread):
  - Main computational cost for set-up data $\approx 22min$.
  - Computation of reduced models from set-up data: 44–49sec. ($r = 20$–70).
  - Bode plot (MATLAB on Intel Core i7, 2,67GHz, 12GB): 7.5h for original system , $<$ 1min for reduced system.

Electro-Thermal Simulation of Integrated Circuit (IC) [Source: Evgenii Rudnyi, CADFEM GmbH]

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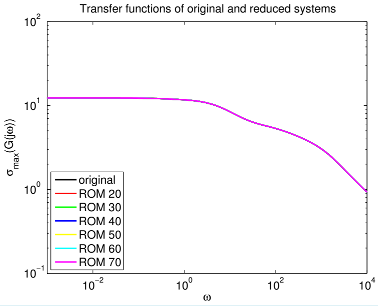
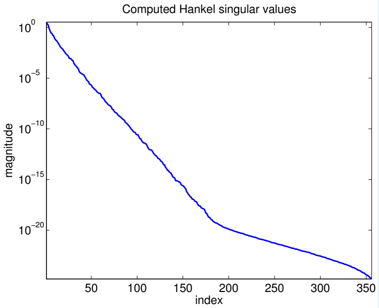
Bode magnitude plot



Hankel Singular Values



Computational Examples

Electro-Thermal Simulation of Integrated Circuit (IC) [Source: Evgenii Rudnyi, CADFEM GmbH]

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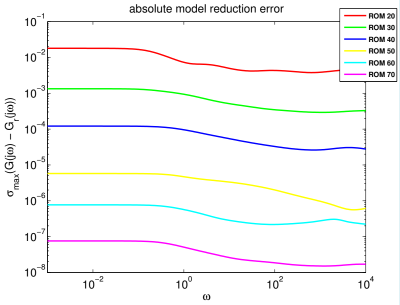
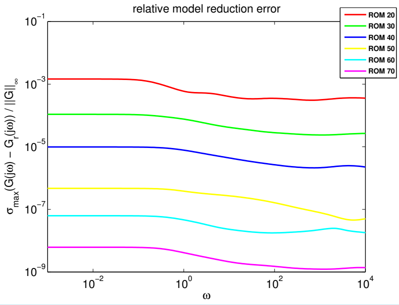
Absolute Error

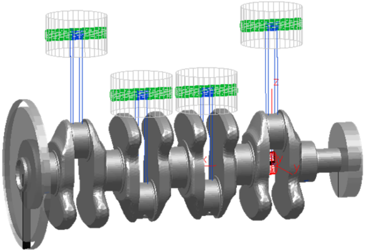


Relative Error

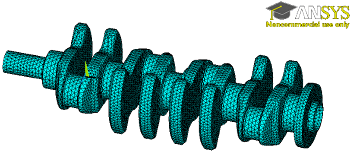


Computational Examples

Elastic Multi-Body Simulation (EMBS)



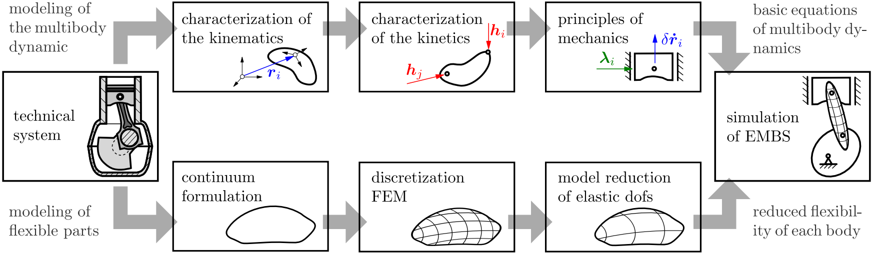
FEM ⇝



- Resolving complex 3D geometries ⇒ can involve millions of degrees of freedom.
- EMBS: ROM is used as surrogate in simulation runs with varying forcing terms.

Source: ITM, U Stuttgart

MOR in EMBS

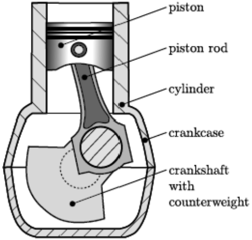


Christine Nowakowski, Patrick Kürschner, Peter Eberhard, Peter Benner.
Model Reduction of an Elastic Crankshaft for Elastic Multibody Simulations
ZAMM, 93(4):198–216, 2013.

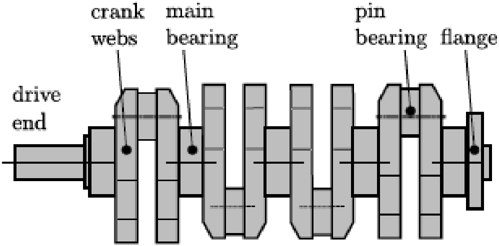
EMBS in Tribological Study of Combustion Engine

- Consider coupling laws between elements of the combustion engine; tribological contacts describe the relative motion between solids separated by fluid film lubrication.
- Need to compute hydrodynamic pressure distribution.
- Crankshaft modeled as elastic body, all other parts rigid.
- LTI system with $n = 84,252$, $m = q = 35$.

Structure of crank drive:



Crankshaft of a four-cylinder engine:



EMBS in Tribological Study of Combustion Engine

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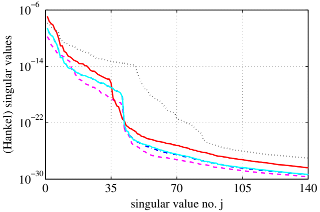
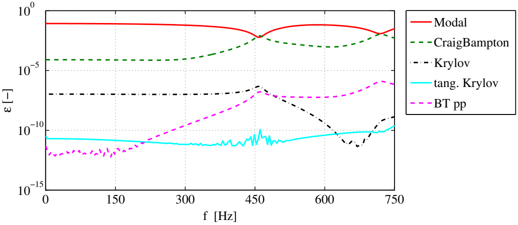
- LTI system with $n = 84,252$, $m = q = 35$.

ROM of order $r = 70$ computed by the different methods, including second-order variant of balanced truncation ($<$ 2min to compute ROM):





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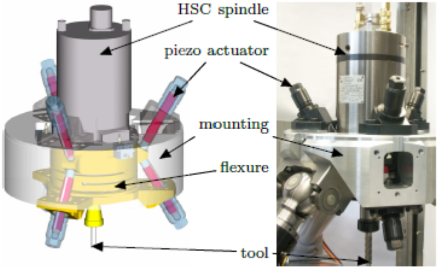
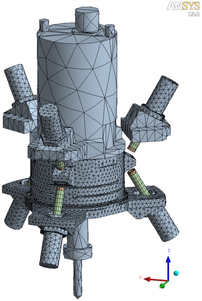
Mechatronics / Piezo-Actuated Spindle Head

- Used for localized actuation to superimpose micro motions of machine tool.
- Descriptor LTI system with $n = 290,137$, $m = q = 9$.

Piezo-actuated structure (CAD):



FEM model:

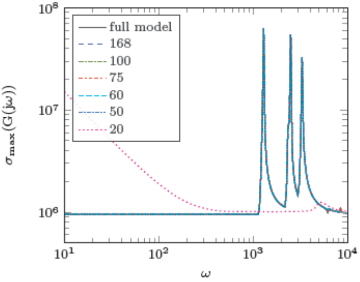


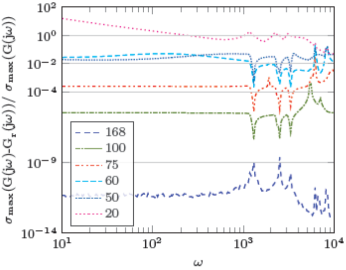
Source: Fraunhofer IWU Chemnitz/Dresden

Mechatronics / Piezo-Actuated Spindle Head

- Used for localized actuation to superimpose micro motions of machine tool.
- Descriptor LTI system with $n = 290,137$, $m = q = 9$.

ROM of orders $r = 20, \ldots, 168$ computed by variant of balanced truncation for descriptor systems, sigma plot (left) and relative errors (right):





Mohammad Monir Uddin, Jens Saak, Burkhard Kranz, Peter Benner.

Computation of a Compact State Space Model for an Adaptive Spindle Head Configuration with Piezo Actuators using Balanced Truncation. *Production Engineering*, 6(6):577–586, 2012.

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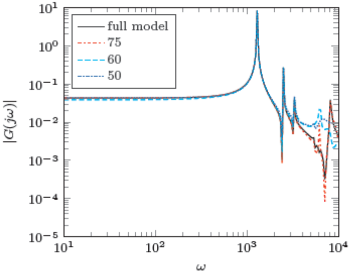
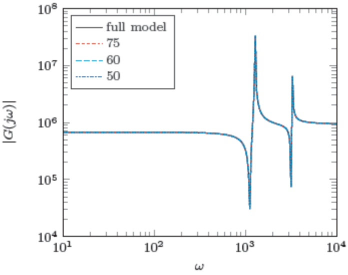
ROM of orders $r = 20 \ldots 168$ computed by variant of balanced truncation for descriptor systems, Bode magnitude plots for $1 \to 9$ (left), $9 \to 9$ (right):





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ROM of orders $r = 20 \ldots 168$ computed by variant of balanced truncation.

system dimension	execution time (sec)	speedup
290,137	90.00	
168	0.029	3,103
75	0.019	4,737
60	0.017	5,294
50	0.014	6,429
20	0.013	6,923

Peter Benner, Jens Saak, Mohammad Monir Uddin.
Structure preserving model order reduction of large sparse second-order index-1 systems and application to a mechatronics model. *Mathematical and Computer Modelling of Dynamical Systems,* 22(6):509–523, 2016.

Mohammad Monir Uddin, Jens Saak, Burkhard Kranz, Peter Benner.
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Computation of reduced-order model by projection

Given linear (descriptor) system $E\dot{x} = Ax + Bu,\ y = Cx$ with transfer function $G(s) = C(sE - A)^{-1}B$, a ROM is obtained using truncation matrices $V, W \in \mathbb{R}^{n \times r}$ with $W^T V = I_r$ ($\rightsquigarrow (VW^T)^2 = VW^T$ is projector) by computing

$$\hat{E} = W^T EV,\ \hat{A} = W^T AV,\ \hat{B} = W^T B,\ \hat{C} = CV.$$

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

Interpolatory Model Reduction

Computation of reduced-order model by projection

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Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

Rational Interpolation/Moment-Matching

Choose V, W such that

$$G(s_j) = \hat{G}(s_j), \quad j = 1, \ldots, k,$$

and

$$\frac{d^i}{ds^i} G(s_j) = \frac{d^i}{ds^i} \hat{G}(s_j), \quad i = 1, \ldots, K_j, \quad j = 1, \ldots, k.$$

Theorem (simplified) [Grimme '97, Villemagne/Skelton '87]

If

$$\begin{aligned} \mathrm{span}\left\{(s_1E-A)^{-1}B,\ldots,(s_kE-A)^{-1}B\right\} &\subset \mathrm{Ran}(V), \\ \mathrm{span}\left\{(s_1E-A)^{-T}C^T,\ldots,(s_kE-A)^{-T}C^T\right\} &\subset \mathrm{Ran}(W), \end{aligned}$$

then

$$G(s_j)=\hat{G}(s_j), \quad \frac{d}{ds}G(s_j)=\frac{d}{ds}\hat{G}(s_j), \quad \text{for } j=1,\ldots,k.$$

Interpolatory Model Reduction

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then

$$G(s_j)=\hat{G}(s_j),\quad \frac{d}{ds}G(s_j)=\frac{d}{ds}\hat{G}(s_j),\quad \text{for } j=1,\ldots,k.$$

Remarks:

computation of V, W from rational Krylov subspaces, e.g.,

- dual rational Arnoldi/Lanczos [Grimme '97],
- Iter. Rational Krylov-Alg. (IRKA) [Antoulas/Beattie/Gugercin '06/'08].

Interpolatory Model Reduction

Theorem (simplified) [Grimme '97, Villemagne/Skelton '87]

If

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then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds}G(s_j) = \frac{d}{ds}\hat{G}(s_j), \quad \text{for } j = 1, \ldots, k.$$

Remarks:

using Galerkin/one-sided projection ($W \equiv V$) yields $G(s_j) = \hat{G}(s_j)$, but in general

$$\frac{d}{ds}G(s_j) \neq \frac{d}{ds}\hat{G}(s_j).$$

Theorem (simplified) [Grimme '97, Villemagne/Skelton '87]

If

$$\begin{aligned}\operatorname{span}\left\{(s_1E-A)^{-1}B,\ldots,(s_kE-A)^{-1}B\right\} &\subset \operatorname{Ran}(V),\\ \operatorname{span}\left\{(s_1E-A)^{-T}C^T,\ldots,(s_kE-A)^{-T}C^T\right\} &\subset \operatorname{Ran}(W),\end{aligned}$$

then

$$G(s_j)=\hat{G}(s_j),\quad \frac{d}{ds}G(s_j)=\frac{d}{ds}\hat{G}(s_j),\quad \text{for } j=1,\ldots,k.$$

Remarks:

$k=1$, standard Krylov subspace(s) of dimension K:

$$\operatorname{range}(V)=\mathcal{K}_K((s_1I-A)^{-1},(s_1I-A)^{-1}B).$$

$\rightsquigarrow$ moment-matching methods/Padé approximation [Freund/Feldmann '95],

$$\frac{d^i}{ds^i}G(s_1)=\frac{d^i}{ds^i}\hat{G}(s_1),\quad i=0,\ldots,K-1(+K).$$

Interpolatory Model Reduction

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$k = 1$, standard Krylov subspace(s) of dimension K:

$$\text{range}\,(V) = \mathcal{K}_K((s_1 I - A)^{-1}, (s_1 I - A)^{-1} B).$$

⇝ moment-matching methods/Padé approximation [Freund/Feldmann '95],

$$\frac{d^i}{ds^i} G(s_1) = \frac{d^i}{ds^i} \hat{G}(s_1), \quad i = 0, \ldots, K-1 (+K).$$

News:

Adaptive choice of interpolation points and number of moments to be matched based on dual-weighted residual based error estimate!

Lihong Feng, Jan G. Korvink, Peter Benner.

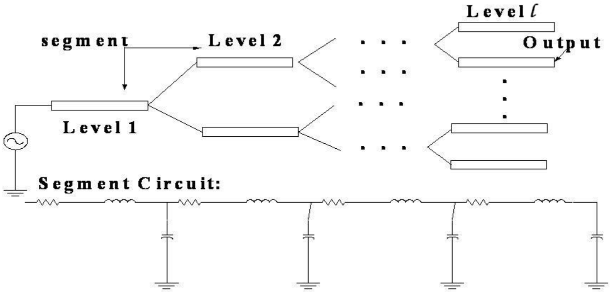
A Fully Adaptive Scheme for Model Order Reduction Based on Moment-Matching. *IEEE Transactions on Components, Packaging, and Manufacturing Technology,* 5(12):1872–1884, 2015.

Lihong Feng, Athanasios C. Antoulas, Peter Benner

Some a posteriori error bounds for reduced order modelling of (non-)parametrized linear systems. *MPI Magdeburg Preprints* MPIMD/15-17, October 2015.

Micro-electronics: clock tree

- Each segment has 4 RL pairs in series, representing the wiring on a chip, with four capacitors to ground, representing the wire-substrate interaction,
- $n = 6,134$, $m = q = 1$ (SISO).



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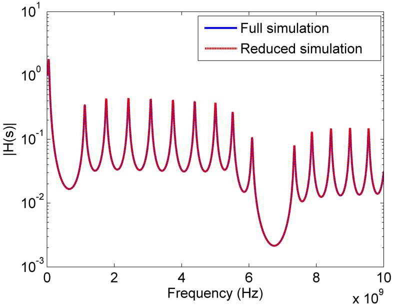
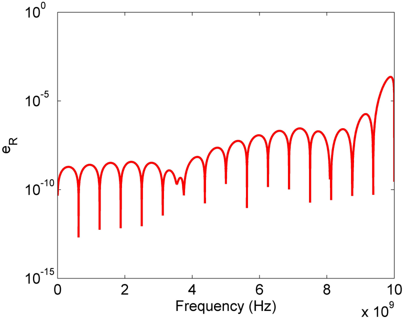
Sigma plot ($r = 18$):



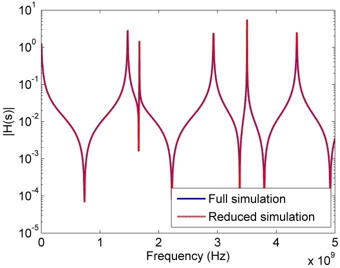
Relative error ($r = 18$):



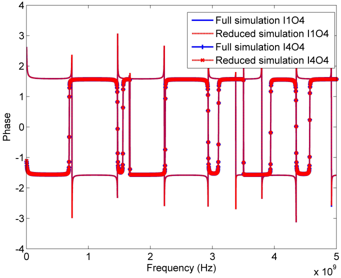
Micro-electronics: SPICE MNA model

- Model of a CMOS-inverter driven two-bit bus determined, using modified modal analysis, by SPICE.
- $n = 980$, $m = q = 4$ (MIMO), ROM of order $r = 48$.

Bode plot (magnitude, $1 \to 4$):



Bode plot (phase, $1, 4 \to 4$):



Source: *The SLICOT Benchmark Collection for Model Reduction,*
`http://slicot.org/20-site/126-benchmark-examples-for-model-reduction`

The PMOR Problem

Approximate the dynamical system

$$\begin{array}{rcll} E(p)\dot{x} & = & A(p)x + B(p)u, & E(p), A(p) \in \mathbb{R}^{n\times n}, \\ y & = & C(p)x, & B(p) \in \mathbb{R}^{n\times m}, C(p) \in \mathbb{R}^{q\times n}, \end{array}$$

by reduced-order system

$$\begin{array}{rcll} \hat{E}(p)\dot{\hat{x}} & = & \hat{A}(p)\hat{x} + \hat{B}(p)u, & \hat{E}(p), \hat{A}(p) \in \mathbb{R}^{r\times r}, \\ \hat{y} & = & \hat{C}(p)\hat{x}, & \hat{B}(p) \in \mathbb{R}^{r\times m}, \hat{C}(p) \in \mathbb{R}^{q\times r}, \end{array}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\| \quad \forall\, p \in \Omega \subset \mathbb{R}^d.$$

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Approximate the dynamical system

$$\begin{aligned} E(p)\dot{x} &= A(p)x + B(p)u, && E(p), A(p) \in \mathbb{R}^{n\times n}, \\ y &= C(p)x, && B(p) \in \mathbb{R}^{n\times m}, C(p) \in \mathbb{R}^{q\times n}, \end{aligned}$$

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of order $r \ll n$, such that

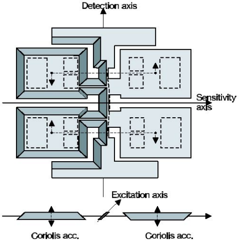
$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\| \quad \forall\ p \in \Omega \subset \mathbb{R}^d.$$

$\Longrightarrow$ Approximation problem: $\min_{\text{order}\,(\hat{G})\leq r} \|G - \hat{G}\|.$

Example: Microsystems/MEMS Design (butterfly gyro)

- Applications:
  - inertial navigation,
  - electronic stability control (ESP).

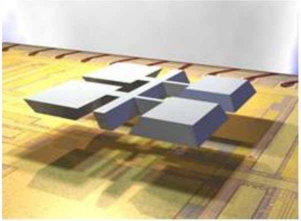
- Voltage applied to electrodes induces vibration of wings, resulting rotation due to Coriolis force yields sensor data.
- FE model of second order: $N = 17.361 \rightsquigarrow n = 34.722,\ m = 1,\ q = 12.$
- Sensor for position control based on acceleration and rotation.

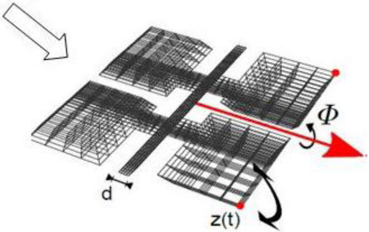


Source: MOR Wiki http://morwiki.mpi-magdeburg.mpg.de/morwiki/index.php/Gyroscope

Example: Microsystems/MEMS Design (butterfly gyro)

Parametric FE model: $M(d)\ddot{x}(t) + D(\theta, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t)$.





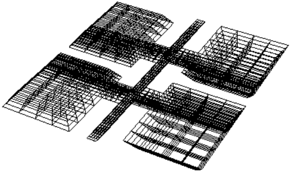
Example: Microsystems/MEMS Design (butterfly gyro)

Parametric FE model:

$$M(d)\ddot{x}(t) + D(\theta, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t),$$

where

$$\begin{aligned} M(d) &= M_1 + dM_2, \\ D(\theta, d, \alpha, \beta) &= \theta(D_1 + dD_2) + \alpha M(d) + \beta T(d), \\ T(d) &= T_1 + \frac{1}{d}T_2 + dT_3, \end{aligned}$$



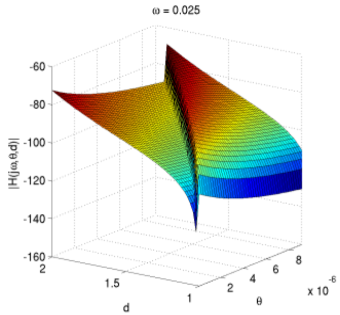
with

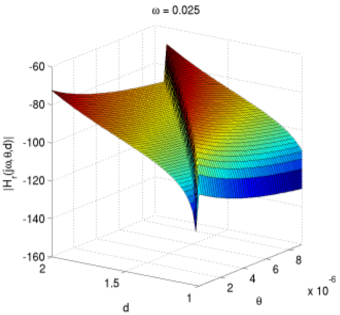
- width of bearing: d,
- angular velocity: θ,
- Rayleigh damping parameters: α, β.

Example: Microsystems/MEMS Design (butterfly gyro)

Response surfaces: $\sigma_{\max}(G(\jmath\omega, p))$ vs. p at ω,

original. . . and reduced-order model.





Computation times:

ca. 1 week ca. 1.5 hours

Conclusions

- A huge savings in computational time and energy can be obtained using reduced-order models in frequency response analysis.

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- How do we get MOR into the pro software packages in CAE / CSE ?

Further Reading

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