MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

System-theoretic Model Order Reduction for Classes of Nonlinear Systems

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MS267 — Model Reduction and Reduced-order Modeling of Dynamical Systems
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Overview

1. Introduction
    - Model Reduction for Control Systems
    - System Classes
    - How general are these system classes?
    - Linear Systems and their Transfer Functions

2. Gramian-based Model Reduction for Linear Systems

3. Balanced Truncation for QB Systems

4. Balanced Truncation for Polynomial Systems

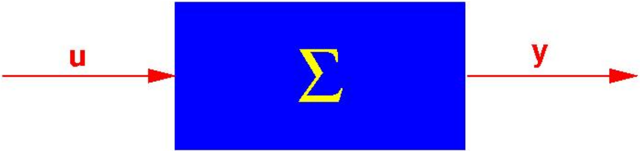
Nonlinear Control Systems

$$\Sigma : \begin{cases} E\dot{x}(t) &= f(t, x(t), u(t)), \quad Ex(t_0) = Ex_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

with

- (generalized) states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^q$.

If E singular $\rightsquigarrow$ descriptor system. Here, $E = I_n$ for simplicity.



Original System ($E = I_n$)

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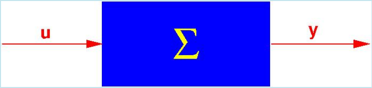

Goals:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Model Reduction for Control Systems

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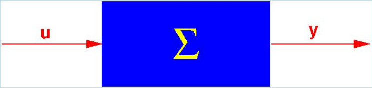

Reduced-Order Model (ROM)

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
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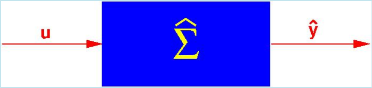

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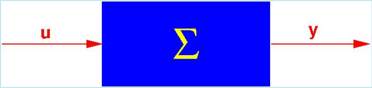

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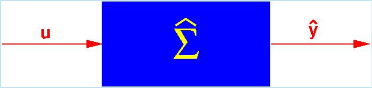

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Model Reduction for Control Systems

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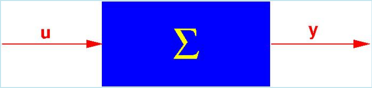

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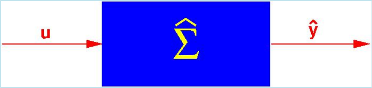

Goals:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Secondary goal: reconstruct approximation of x from $\hat{x}$.

Control-Affine (Autonomous) Systems

$$\begin{aligned}\dot{x}(t) &= f(t,x,u) = \mathcal{A}(x(t)) + \mathcal{B}(x(t))u(t), \quad \mathcal{A}:\mathbb{R}^n \to \mathbb{R}^n,\ \mathcal{B}:\mathbb{R}^n \to \mathbb{R}^{n\times m},\\ y(t) &= g(t,x,u) = \mathcal{C}(x(t)) + \mathcal{D}(x(t))u(t), \quad \mathcal{C}:\mathbb{R}^n \to \mathbb{R}^q,\ \mathcal{D}:\mathbb{R}^n \to \mathbb{R}^{q\times m}.\end{aligned}$$

System Classes

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\end{aligned}
$$

Linear, Time-Invariant (LTI) Systems

$$
\begin{aligned}
\dot{x}(t) &= f(t,x,u) = Ax(t) + Bu(t), \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\\
y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}.
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Bilinear Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = Ax(t) + \sum_{i=1}^m u_i(t)A_i x(t) + Bu(t), \quad A, A_i \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}. \end{aligned}$$

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Quadratic-Bilinear (QB) Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = Ax(t) + H\left(x(t) \otimes x(t)\right) + \sum_{i=1}^{m} u_i(t)A_i x(t) + Bu(t), \\ &\qquad A, A_i \in \mathbb{R}^{n\times n},\ H \in \mathbb{R}^{n\times n^2},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}. \end{aligned}$$

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$$y(t) = g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}.$$

Polynomial Systems

$$\dot{x}(t) = Ax(t) + \sum_{j=2}^{n_p} H_j \left(\otimes^j x(t)\right) + \sum_{j=2}^{n_p}\sum_{k=1}^{m} A_j^k \left(\otimes^j x(t)\right) u_k(t) + Bu(t),$$

H_j, A_j^k of "appropriate size"

$$y(t) = Cx(t), \quad x(0) = 0,$$

System Classes

Control-Affine (Autonomous) Systems

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\end{aligned}
$$

Written in control-affine form:

$$
\begin{aligned}
\mathcal{A}(x) &:= Ax + H\left(x \otimes x\right), & \mathcal{B}(x) &:= [A_1, \ldots, A_m]\left(I_m \otimes x\right) + B\\
\mathcal{C}(x) &:= Cx, & \mathcal{D}(x) &:= D.
\end{aligned}
$$

QB systems can be obtained as approximation (by truncating Taylor expansion) to weakly nonlinear systems [PHILLIPS '03].

C. Gu. QLMOR: A Projection-Based Nonlinear Model Order Reduction Approach Using Quadratic-Linear Representation of Nonlinear Systems. IEEE TRANSACTIONS ON COMPUTER-AIDED DESIGN OF INTEGRATED CIRCUITS AND SYSTEMS, 30(9):1307–1320, 2011.

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QB systems can be obtained as approximation (by truncating Taylor expansion) to weakly nonlinear systems [PHILLIPS '03].

But **exact representation** of smooth nonlinear systems possible:

Theorem [GU '09/'11]

Assume that the state equation of a nonlinear system is given by

$$\dot{x} = a_0 x + a_1 g_1(x) + \ldots + a_k g_k(x) + Bu,$$

where $g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$ are compositions of uni-variable rational, exponential, logarithmic, trigonometric or root functions, respectively. Then, by iteratively taking derivatives and adding algebraic equations, respectively, the nonlinear system can be transformed into a QB(DAE) system.

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Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s), \dot{x}(t) \mapsto sx(s) - x(0))$ to linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

with $x(0) = 0$ yields:

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Model reduction in frequency domain: Fast evaluation of mapping $u \rightarrow y$.

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$\Longrightarrow$ I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)}\Big)u(s).$$

$G(s)$ is the **transfer function** of Σ.

Model reduction in frequency domain: Fast evaluation of mapping $u \rightarrow y$.

Formulating model reduction in frequency domain

Approximate the dynamical system

$$\begin{array}{rcll} \dot{x} & = & Ax + Bu, & A \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times m}, \\ y & = & Cx + Du, & C \in \mathbb{R}^{q \times n},\ D \in \mathbb{R}^{q \times m}, \end{array}$$

by reduced-order system

$$\begin{array}{rcll} \dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}u, & \hat{A} \in \mathbb{R}^{r \times r},\ \hat{B} \in \mathbb{R}^{r \times m}, \\ \hat{y} & = & \hat{C}\hat{x} + \hat{D}u, & \hat{C} \in \mathbb{R}^{q \times r},\ \hat{D} \in \mathbb{R}^{q \times m} \end{array}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.$$

Linear Systems and their Transfer Functions

Formulating model reduction in frequency domain

Approximate the dynamical system

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$\Longrightarrow$ Approximation problem: $\min_{\text{order}\,(\hat{G})\leq r} \|G - \hat{G}\|.$

Outline

Basic concept

- System Σ : $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions P, Q of the Lyapunov equations

  $$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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- Compute balanced realization (needs P, Q!) of the system via state-space transformation

  $$\begin{aligned} \mathcal{T} : (A, B, C) &\mapsto (TAT^{-1}, TB, CT^{-1}) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right). \end{aligned}$$

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}) = (A_{11}, B_1, C_1)$.

Motivation:

HSV are system invariants: they are preserved under $\mathcal{T}$ and determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

"functional analyst's point of view"

Balanced Truncation for Linear Systems

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"functional analyst's point of view"

Minimum energy to reach x_0 in balanced coordinates:

$$\inf_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \int_{-\infty}^{0} u(t)^T u(t)\, dt = x_0^T P^{-1} x_0 = \sum_{j=1}^{n} \frac{1}{\sigma_j} x_{0,j}^2$$

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Energy contained in the system if $x(0) = x_0$ and $u(t) \equiv 0$ in balanced coordinates:

$$\|y\|_2^2 = \int_0^{\infty} y(t)^T y(t)\, dt = x_0^T Q x_0 = \sum_{j=1}^{n} \sigma_j x_{0,j}^2$$

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"functional analyst's point of view"

In balanced coordinates, energy transfer from u_- to y_+ is

$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int\limits_0^\infty y(t)^T y(t)\, dt}{\int\limits_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2.$$

"engineer's point of view"

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$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

"functional analyst's point of view"

In balanced coordinates, energy transfer from u_- to y_+ is

$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int_0^\infty y(t)^T y(t)\, dt}{\int_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2.$$

"engineer's point of view"

$\Longrightarrow$ Truncate states corresponding to "small" HSVs

Properties

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.

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Practical implementation

- Rather than solving Lyapunov equations for P, Q (n^2 unknowns!), find $S, R \in \mathbb{R}^{n \times s}$ with $s \ll n$ such that $P \approx SS^T$, $Q \approx RR^T$.
- Reduced-order model directly obtained via small-scale ($s \times s$) SVD of $R^T S$!
- No $\mathcal{O}(n^3)$ or $\mathcal{O}(n^2)$ computations necessary!

Outline

- Nonlinear balancing based on energy functionals [Scherpen '93, Gray/Mesko '96].

Definition [Scherpen '93, Gray/Mesko '96]

The reachability energy functional, $L_c(x_0)$, and observability energy functional, $L_o(x_0)$ of a system are given as:

$$L_c(x_0) = \inf_{\substack{u \in L_2(-\infty,0] \\ x(-\infty)=0,\ x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 dt, \qquad L_o(x_0) = \frac{1}{2}\int_{0}^{\infty} \|y(t)\|^2 dt.$$

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- Empirical Gramians/frequency-domain POD [Lall et al '99, Willcox/Peraire '02].

Example: controllability Gramian from time domain data (snapshots)

1. Define reachability Gramian of the system

$$P = \int_0^\infty x(t)x(t)^T dt, \quad \text{where } x(t) \text{ solves } \dot{x} = f(x, \delta),\ x(0) = x_0.$$

2. Use time-domain integrator to produce snapshots $x_k \approx x(t_k)$, $k = 1, \ldots, K$.
3. Approximate $P \approx \sum_{k=0}^{K} w_k x_k x_k^T$ with positive weights w_k.
4. Analogously for observability Gramian.
5. Compute balancing transformation and apply it to nonlinear system.

Disadvantage: Depends on chosen training input (e.g., $\delta(t_0)$) like other POD approaches.

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- For recent developments on empirical Gramians, see [Himpe '18].

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$$AP + PA^T + \sum_{i=1}^{m} A_i P A_i^T + BB^T = 0,$$
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- Efficient solution methods for Lyapunov-plus-positive equations are derived in [B./Breiten '13, Shank/Simoncini/Szyld '16].
- **Here we aim at determining algebraic Gramians for QB (and polynomial) systems, which**
  - **provide bounds for the energy functionals of QB systems,**
  - **generalize the Gramians of linear and bilinear systems, and**
  - **allow us to find the states that are hard to control as well as hard to observe in an efficient and reliable way.**

- Consider input → state map of QB system ($m = 1$, $N \equiv A_1$):

$$\dot{x}(t) = Ax(t) + Hx(t) \otimes x(t) + Nx(t)u(t) + Bu(t), \qquad x(0) = 0.$$

- Integration yields

$$x(t) = \int_0^t e^{A\sigma_1} Bu(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Nx(t-\sigma_1)u(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Hx(t-\sigma_1) \otimes x(t-\sigma_1)d\sigma_1$$

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$$+ \int_0^t \int_0^{t-\sigma_1} \int_0^{t-\sigma_1} e^{A\sigma_1} H(e^{A\sigma_2} B \otimes e^{A\sigma_3} B)u(t-\sigma_1-\sigma_2)u(t-\sigma_1-\sigma_3)d\sigma_1 d\sigma_2 d\sigma_3 + \ldots$$

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- By iteratively inserting expressions for $x(t-\bullet)$, we obtain the **Volterra series expansion** for the QB system. [Rugh '81]

Using the *Volterra kernels*, we can define the controllability mappings

$$\Pi_1(t_1) := e^{At_1}B, \qquad \Pi_2(t_1, t_2) := e^{At_1}N\Pi_1(t_2),$$
$$\Pi_3(t_1, t_2, t_3) := e^{At_1}[H(\Pi_1(t_2) \otimes \Pi_1(t_3)), N\Pi_2(t_1, t_2)], \ldots$$

and a candidate for a new Gramian:

$$P := \sum_{k=1}^{\infty} P_k, \qquad \text{where} \quad P_k = \int_0^{\infty} \cdots \int_0^{\infty} \Pi_k(t_1, \ldots, t_k)\Pi_k(t_1, \ldots, t_k)^T \, dt_1 \ldots dt_k.$$

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Theorem [B./Goyal '16]

If it exists, the new controllability Gramian P for QB (MIMO) systems with stable A solves the **quadratic Lyapunov equation**

$$AP + PA^T + \sum_{k=1}^{m} A_k P A_k^T + H(P\otimes P)H^T + BB^T = 0.$$

Note: $H = 0 \rightsquigarrow$ "bilinear reachability Gramian"; if additionally, all $A_k = 0 \rightsquigarrow$ linear one.

- Controllability energy functional (Gramian) of the dual system $\Leftrightarrow$ observability energy functional (Gramian) of the original system.

Gramians for QB Systems

Dual systems and observability Gramians [Fujimoto et al. '02]

- Controllability energy functional (Gramian) of the dual system $\Leftrightarrow$ observability energy functional (Gramian) of the original system.

- Employ close relation between port-Hamiltonian systems and dual systems of nonlinear systems.

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- Controllability energy functional (Gramian) of the dual system ⇔ observability energy functional (Gramian) of the original system.
- Employ close relation between port-Hamiltonian systems and dual systems of nonlinear systems.
- This allows to define dual systems for QB systems:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Hx(t)\otimes x(t) + \sum\nolimits_{k=1}^{m} A_k x(t) u_k(t) + Bu(t), && x(0) = 0,\\
\dot{x_d}(t) &= -A^T x_d(t) - H^{(2)} x(t)\otimes x_d(t) - \sum\nolimits_{k=1}^{m} A_k^T x_d(t) u_k(t) - C^T u_d(t), && x_d(\infty) = 0,\\
y_d(t) &= B^T x_d(t),
\end{aligned}$$

where $H^{(2)}$ is the mode-2 matricization of the QB Hessian.

- Writing down the Volterra series for the dual system $\rightsquigarrow$ observability mapping.

- This provides the observability Gramian Q for the QB system. It solves

$$A^T Q + QA + \sum_{k=1}^{m} A_k^T Q A_k + H^{(2)}(P \otimes Q)\left(H^{(2)}\right)^T + C^T C = 0.$$

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Remarks:

- Observability Gramian depends on controllability Gramian!
- For $H = 0$, obtain "bilinear observability Gramian", and if also all $A_k = 0$, the linear one.

Gramians and Energy Functionals

Bounding the energy functionals:

Lemma [B./Goyal '16]

In a neighborhood of the stable equilibrium, $B_\varepsilon(0)$,

$$L_c(x_0) \geq \frac{1}{2}x_0^T P^{-1} x_0, \qquad L_o(x_0) \leq \frac{1}{2}x_0^T Q x_0, \qquad x_0 \in B_\varepsilon(0),$$

for "small signals" and x_0 pointing in unit directions.

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for "small signals" and x_0 pointing in unit directions.

Another interpretation of Gramians in terms of energy functionals

1. If the system is to be steered from 0 to x_0, where $x_0 \notin \mathrm{range}\,(P)$, then $L_c(x_0) = \infty$ for all feasible input functions u.
2. If the system is (locally) controllable and $x_0 \in \ker\,(Q)$, then $L_o(x_0) = 0$.

Illustration using a scalar system

$$\dot{x}(t) = ax(t) + hx^2(t) + nx(t)u(t) + bu(t), \quad y(t) = cx(t).$$

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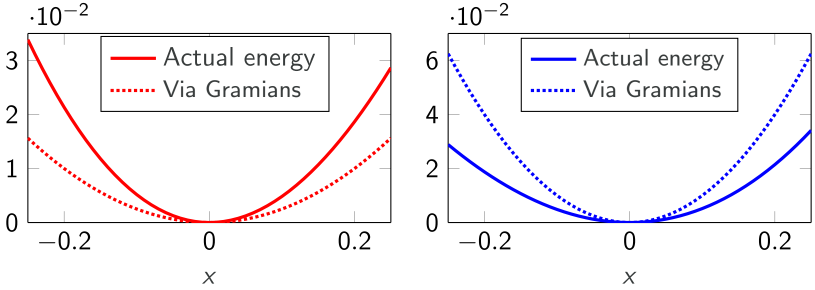

(a) Input energy lower bound.

(b) Output energy upper bound.

Figure: Comparison of energy functionals for $-a = b = c = 2, h = 1, n = 0$.

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- To overcome this issue, we propose truncated Gramians for QB systems.

Definition (Truncated Gramians) [B./Goyal '16]

The truncated Gramians $P_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ for QB systems satisfy

$$AP_{\mathcal{T}} + P_{\mathcal{T}}A^T = -BB^T - \sum\nolimits_{k=1}^{m} A_k P_l A_k^T - H(P_l \otimes P_l)H^T,$$

$$A^T Q_{\mathcal{T}} + Q_{\mathcal{T}}A = -C^T C - \sum\nolimits_{k=1}^{m} A_k^T Q_l A_k - H^{(2)}(P_l \otimes Q_l)(H^{(2)})^T,$$

where

$$AP_l + P_l A^T = -BB^T \quad \text{and} \quad A^T Q_l + Q_l A = -C^T C.$$

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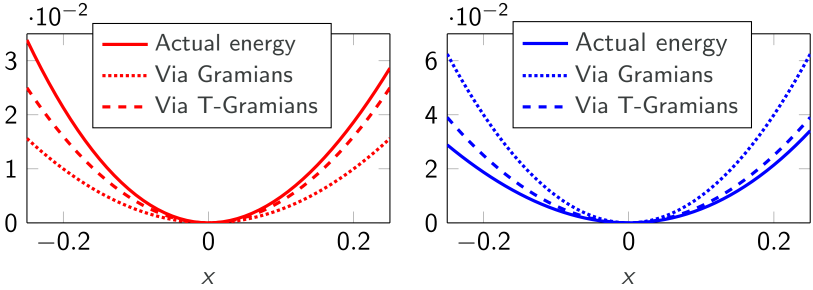

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- Most importantly, we need solutions of only four standard Lyapunov equations.
- Interpretation of controllability/observability of the system via T-Gramians:
  - If the system is to be steered from 0 to x_0, where $x_0 \notin \mathrm{range}\,(P_{\mathcal{T}})$, then $L_c(x_0) = \infty$.
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Algorithm 1 Balanced Truncation MOR for QB Systems (Truncated Gramians).

1: **Input:** A, H, A_k, B, C.

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5: **Output:** reduced-order matrices:

$$\hat{A} = \mathcal{W}^T A\mathcal{V}, \quad \hat{H} = \mathcal{W}^T H(\mathcal{V} \otimes \mathcal{V}), \quad \hat{A}_k = \mathcal{W}^T A_k\mathcal{V},$$
$$\hat{B} = \mathcal{W}^T B, \quad \hat{C} = C\mathcal{V}.$$

Remark: There are efficient ways to compute $\hat{H}$, avoiding the explicit computation of $\mathcal{V} \otimes \mathcal{V}$. [B./Breiten '15, B./Goyal/Gugercin. '16]

$$\begin{aligned} v_t + v^3 &= v_{xx} + v, && (0, L) \times (0, T), \\ v(0, .) &= u(t), && (0, T), \\ v_x(L, .) &= 0, && (0, T), \\ v(x, 0) &= v_0(x), && (0, L). \end{aligned}$$

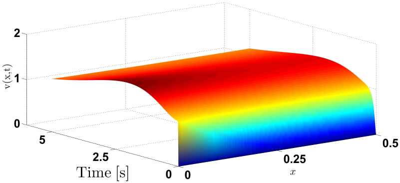

Figure: Chafee-Infante equation.

- Cubic nonlinearity that can be rewritten into QB form. [B./Breiten '15']

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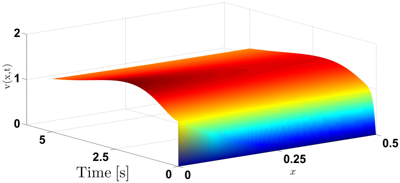

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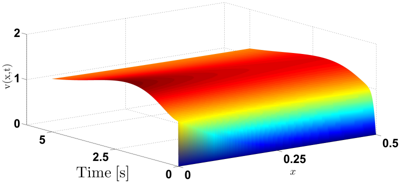

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- We determine the reduced-order system of order $r = 10$.

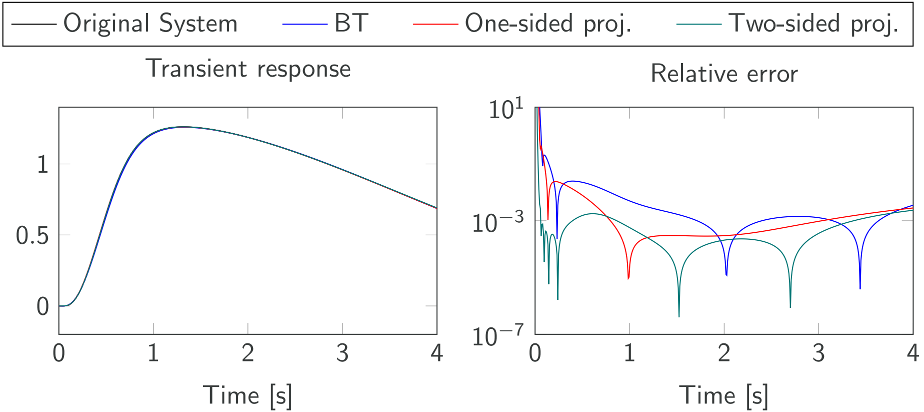


Figure: Boundary control for a control input $u(t) = 5t\ \exp(-t)$.

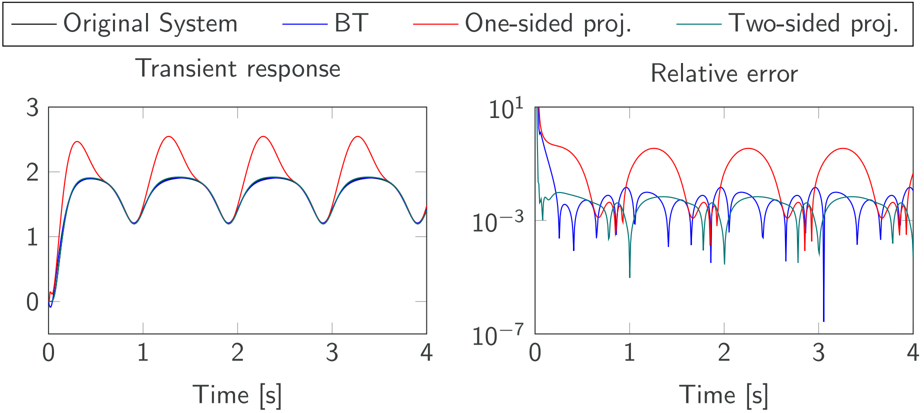


Figure: Boundary control for a control input $u(t) = 25(1 + \sin(2\pi t))/2$.

$$\epsilon v_t(x,t) = \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + q,$$
$$w_t(x,t) = hv(x,t) - \gamma w(x,t) + q,$$

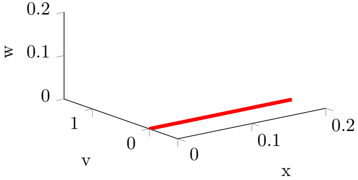
with a nonlinear function

$$f(v(x,t)) = v(v - 0.1)(1 - v).$$

The boundary conditions are as follows:

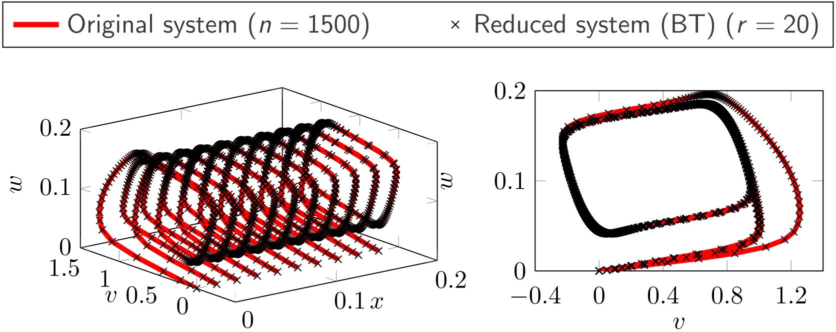
$$v_x(0,t) = i_0(t), \quad v_x(L,t) = 0, \quad t \geq 0,$$

where $\epsilon = 0.015,\ h = 0.5,\ \gamma = 2,\ q = 0.05,$
$L = 0.2.$



- Input $i_0(t) = 5 \cdot 10^4 t^3 \exp(-15t)$ serves as actuator.

Numerical Results
FitzHugh-Nagumo (F-N) model



(a) Limit-cycles at various x.

(b) Projection onto the $v-w$ plane.

Figure: Comparison of the limit-cycles obtained via the original and reduced-order (BT) systems. The reduced-order systems constructed by moment-matching methods were unstable.

Outline

Polynomial Control Systems

Now, consider the class of **polynomial control (PC) Systems**:

$$\dot{x}(t) = Ax(t) + \sum_{j=2}^{n_p} H_j \left(\otimes^j x(t)\right) + \sum_{j=2}^{n_p} \sum_{k=1}^{m} N_j^k \left(\otimes^j x(t)\right) u_k(t) + Bu(t),$$
$$y(t) = Cx(t), \quad x(0) = 0,$$

where

- n_p is the degree of the polynomial part of the system,
- $x(t) \in \mathbb{R}^n$, $\otimes^j x(t) = \underbrace{x(t) \otimes \cdots \otimes x(t)}_{j\text{-times}}$,
- $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$, $n \gg m, p$.
- $A \in \mathbb{R}^{n \times n}$, $H_j, N_j^k \in \mathbb{R}^{n \times n^j}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$.
- Assumption: A is supposed to be Hurwitz $\Rightarrow$ local stability.

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- Assumption: A is supposed to be Hurwitz $\Rightarrow$ local stability.

Examples: FitzHugh-Nagumo and Chafee-Infante equations lead to cubic control systems; cubic-quintic Allen-Cahn equation to quintic control system.

Expanding the response of the PC system into a Volterra series representation and following the same ideas as in the QB case, we define the reachability Gramian as

$$P = \sum_{k=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \bar{P}_k(t_1, \ldots, t_k) \bar{P}_k(t_1, \ldots, t_k)^T dt_1 \ldots dt_k,$$

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where $\bar{P}_1(t_1) = e^{At_1}B, \quad \bar{P}_2(t_1, t_2) = \sum_{k=1}^{m} e^{At_1} N_1^k e^{At_2} B,$

$\bar{P}_3(t_1, t_2, t_3) = e^{At_1} H_2 e^{At_2} B \otimes e^{At_3} B, \ldots$ are the kernels of the Volterra series.

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$\bar{P}_3(t_1, t_2, t_3) = e^{At_1} H_2 e^{At_2} B \otimes e^{At_3} B, \ldots$ are the kernels of the Volterra series.

Theorem

The reachability Gramian **P** of a PC system solves the polynomial Lyapunov equation

$$AP + PA^T + BB^T + \sum_{j=2}^{n_p} H_j \left(\otimes^j P \right) H_j^T + \sum_{j=2}^{n_p} \sum_{k=1}^{m} N_j^k \left(\otimes^j P \right) \left(N_j^k \right)^T = 0.$$

The Observability Gramian is defined as follows

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- First, we write the adjoint system as [FUJIMOTO ET. AL. '02]

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + \sum_{j=2}^{n_p} H_j x_j^{\otimes}(t) + \sum_{j=1}^{n_p}\sum_{k=1}^{m} N_j^k x_j^{\otimes}(t) u_k(t) + Bu(t),\\
\dot{x_d}(t) &= -A^T x_d(t) - \sum_{j=2}^{n_p} H_j^{(2)} x_{d,j}^{\otimes}(t) - \sum_{j=1}^{n_p}\sum_{k=1}^{m} \left(N_j^{k,(2)}\right) x_{d,j}^{\otimes}(t) u_{d,k}(t) - C^T u_d(t), \quad x_d(\infty) = 0,\\
y_d(t) &= B^T x_d(t).
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$$y_d(t) = B^T x_d(t).$$

- Then, by taking the kernel of Volterra series, one has

Theorem

Let **P** be the reachability Gramian. Then, the observability Gramian **Q** of a PC system solves the polynomial Lyapunov equation

$$A^T Q + QA + C^T C + \sum_{j=2}^{n_p} H_j^{(2)}\left(\otimes^{j-1} P \otimes Q\right)\left(H_j^{(2)}\right)^T + \sum_{j=2}^{n_p}\sum_{k=1}^{m} N_j^{k,(2)}\left(\otimes^{j-1} P \otimes Q\right)\left(N_j^{k,(2)}\right)^T = 0.$$

Truncated Gramians

- Polynomial Lyapunov equations are very expensive to solve.
- As for QB systems, we thus propose truncated Gramians that only involve a finite number of kernels.

$$P_{\mathcal{T}} = \sum_{k=1}^{n_p+1} \int_0^\infty \cdots \int_0^\infty \bar{P}_k(t_1, \ldots, t_k) \bar{P}_k(t_1, \ldots, t_k)^T dt_1 \ldots dt_k,$$

Truncated Gramians

The reachability truncated Gramian solves

$$AP_{\mathcal{T}} + P_{\mathcal{T}}A^T + BB^T + \sum_{j=2}^{n_p} H_j \otimes^j P_l H_j^T + \sum_{j=2}^{n_p} \sum_{k=1}^{m} N_j^k \otimes^j P_l \left(N_j^k\right)^T = 0.$$

where $AP_l + P_lA^T + BB^T = 0$

- Advantage: Only need to solve a finite number of (linear) Lyapunov equations.

$$\epsilon v_t(x,t) = \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + q,$$
$$w_t(x,t) = hv(x,t) - \gamma w(x,t) + q,$$

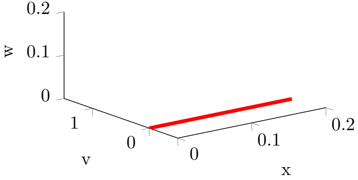
with a nonlinear function

$$f(v(x,t)) = v(v-0.1)(1-v).$$

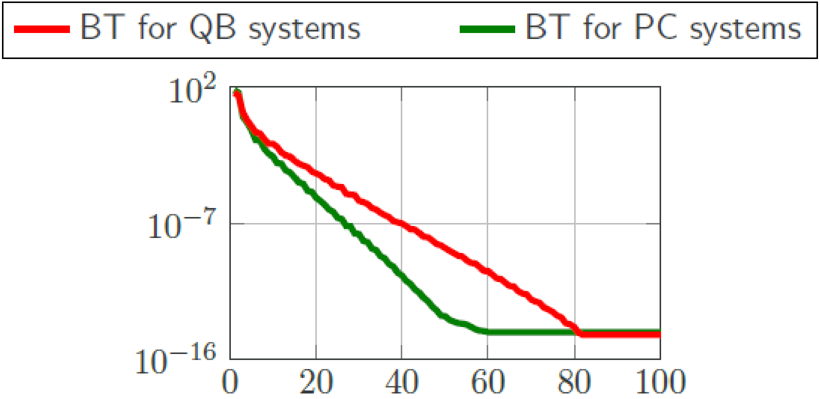
The boundary conditions are as follows:

$$v_x(0,t) = i_0(t), \quad v_x(L,t) = 0, \quad t \geq 0,$$

where $\epsilon = 0.015,\ h = 0.5,\ \gamma = 2,\ q = 0.05,\ L = 0.2.$

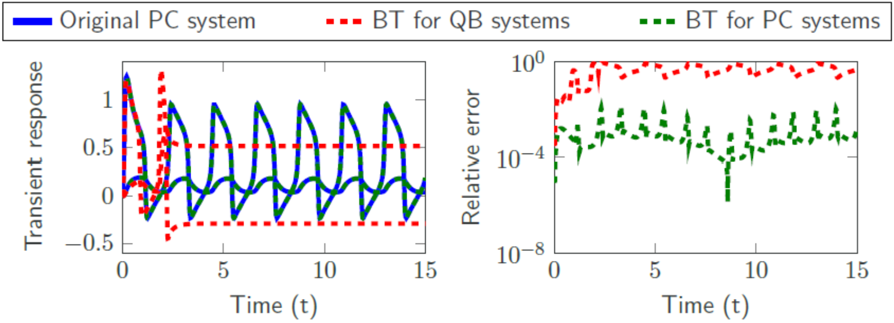


- After discretization we obtain a PC system with cubic nonlinearity of order $n_{pc} = 600$. [B./Breiten '15]
- The transformed quadratic-bilinear (QB) system is of order $n_{qb} = 900$.
- The outputs of interest $v(0,t), w(0,t)$ are the responses at the left boundary at $x = 0$.
- We compare balanced truncation for PC and QB systems.

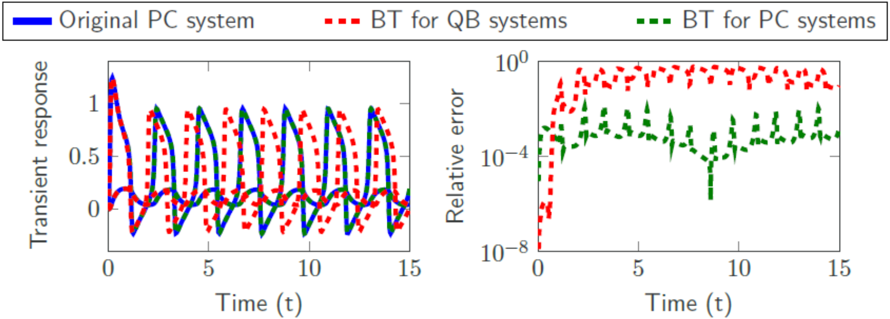


- Decay singular values for PC systems is faster $\Rightarrow$ smaller reduced order model!

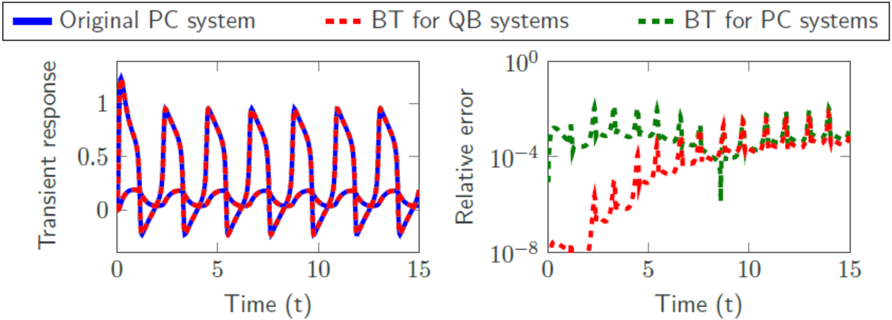
Numerical Example
Time-domain simulations



- Original PC system of order 600. Original QB system of order 900.
- Reduced PC system of order 10. Reduced QB system of order 10.



- Original PC system of order 600. Original QB system of order 900.
- Reduced PC system of order 10. Reduced QB system of order 30.



- Original PC system of order 600. Original QB system of order 900.
- Reduced PC system of order 10. Reduced QB system of order 43.

Conclusions

- BT extended to bilinear, QB, and polynomial systems.
- Local Lyapunov stability is preserved.
- As of yet, only weak motivation by bounding energy functionals.
- No error bounds in terms of "Hankel" singular values.
- Computationally efficient (as compared to nonlinear balancing), and input independent.
- **To do:**
  - improve efficiency of Lyapunov solvers with many right-hand sides further;
  - error bound;
  - conditions for existence of new QB Gramians;
  - extension to descriptor systems;
  - time-limited versions.

For $\mathcal{H}_2$-optimal reduction, extension to bilinear [B./Breiten '12,Flagg/Gugercin '15] and QB [B./Goyal/Gugercin '18] cases, as well as polynomial and parametric systems [B./Goyal '19].

P. Benner and T. Damm.
Lyapunov Equations, Energy Functionals, and Model Order Reduction of Bilinear and Stochastic Systems.
SIAM Journal on Control and Optimiztion, 49(2):686–711, 2011.

P. Benner and T. Breiten.
Low Rank Methods for a Class of Generalized Lyapunov Equations and Related Issues.
Numerische Mathematik, 124(3):441–470, 2013.

P. Benner, P. Goyal, and M. Redmann.
Truncated Gramians for Bilinear Systems and their Advantages in Model Order Reduction.
In P. Benner, M. Ohlberger, T. Patera, G. Rozza, K. Urban (Eds.), Model Reduction of Parametrized Systems, MS & A — Modeling, Simulation and Applications, Vol. 17, pp. 285–300. Springer International Publishing, Cham, 2017.

P. Benner and P. Goyal.
Balanced Truncation Model Order Reduction for Quadratic-Bilinear Control Systems.
arXiv Preprint arXiv:1705.00160, April 2017.

P. Benner, P. Goyal, and Serkan Gugercin.
H_2-Quasi-Optimal Model Order Reduction for Quadratic-Bilinear Control Systems.
SIAM Journal on Matrix Analysis and Applications 39(2):983–1032, 2018.

P. Benner and P. Goyal.
Interpolation-based Model Order Reduction For Polynomial Parametric Systems.
Preprint, February 2019.

P. Benner, P. Goyal, and I. Pontes Duff.
Approximate Balanced Truncation for Polynomial Control Systems.
In preparation.

Announcement

4th Workshop on Model Reduction of Complex Dynamical Systems - MODRED 2019 -

August 28th to 30th, 2019 in Graz

- Overview
- News
- Photos
- Scientific Program
- Participants
- Proceedings
- Location
- Social Activities
- Deadlines
- Abstracts
- Registration
- Accommodation
- Travel
- Useful links
- Contact

The conference starts Wednesday morning and ends on Friday. There will be **plenary talks** by a number of invited speakers. Moreover, there will be several **contributed talks** (20 minutes plus 5 minutes for questions and discussion).

Plenary Speakers

- Serkan Gugercin
- Bernard Haasdonk
- Dirk Hartmann (Siemens)
- Laura Iapichino
- J. Nathan Kutz

Contributed Talks

t.b.a.

Contact: modred2019@uni-graz.at

Last changed: 2018-06-19