MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Computing Parametric Reduced-Order Models using Projection and Data

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Workshop on
Data-Driven Methods for
Multi-Scale Physics and Complex Systems

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Joint work with . . .

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PostDoc

Outline

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Introduction to Parametric Models

Parametric Dynamical Systems

$$\Sigma(p) : \left\{ \begin{array}{rcll} E(p)\dot{x}(t;p) & = & f(t, x(t;p), u(t), p), \quad x(t_0) = x_0, & \text{(a)} \\ y(t;p) & = & g(t, x(t;p), u(t), p) & \text{(b)} \end{array} \right.$$

with

- (generalized) states $x(t;p) \in \mathbb{R}^n$ ($E \in \mathbb{R}^{n \times n}$),
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t;p) \in \mathbb{R}^q$, (b) is called output equation,
- $p \in \Omega \subset \mathbb{R}^d$ is a parameter vector, Ω is bounded.

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Applications:

- Repeated simulation for varying material or geometry parameters, boundary conditions;
- control, optimization and design;
- of models, often generated by FE software (e.g., ANSYS, NASTRAN,. . .) or automatic tools (e.g., Modelica).

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Underlying PDE and boundary conditions often not accessible!

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Underlying PDE and boundary conditions often not accessible!

Parametric discretized model often not available,
but matrices for certain parameter values can be extracted,
or output data for given u and p can be generated!

Linear, Time-Invariant (Parametric) Systems

$$\begin{aligned} E(p)\dot{x}(t;p) &= A(p)x(t;p) + B(p)u(t), & A(p), E(p) &\in \mathbb{R}^{n\times n}, \\ y(t;p) &= C(p)x(t;p), & B(p) \in \mathbb{R}^{n\times m}&, C(p) \in \mathbb{R}^{q\times n}. \end{aligned}$$

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Laplace Transformation / Frequency Domain

Application of Laplace transformation $(x(t;p) \mapsto x(s;p), \dot{x}(t;p) \mapsto sx(s;p))$ to linear system with $x(0;p) \equiv 0$:

$$sE(p)x(s;p) = A(p)x(s;p) + B(p)u(s), \quad y(s;p) = C(p)x(s;p),$$

yields I/O-relation in frequency domain:

$$y(s;p) = \Big(\underbrace{C(p)(sE(p) - A(p))^{-1}B(p)}_{=:G(s,p)} \Big) u(s).$$

$G(s,p)$ is the parameter-dependent transfer function of $\Sigma(p)$.

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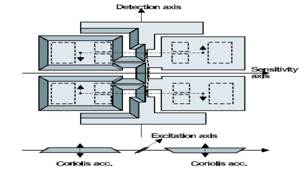
Goal: Fast evaluation of mapping $(u,p) \ \rightarrow\ y(s;p)$.

Motivating Example: Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

- Applications:
  - inertial navigation,
  - electronic stability control (ESP).



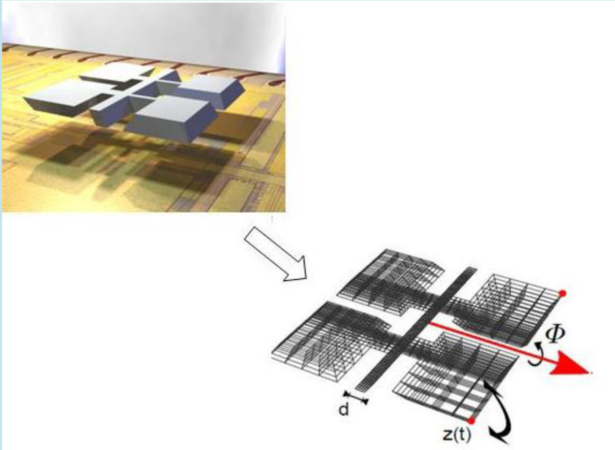
- Voltage applied to electrodes induces vibration of wings, resulting rotation due to Coriolis force yields sensor data.
- FE model of second order: $N = 17.361 \rightsquigarrow n = 34.722,\ m = 1,\ q = 12.$
- Sensor for position control based on acceleration and rotation.

Source: MOR Wiki: http://morwiki.mpi-magdeburg.mpg.de/morwiki/index.php/Gyroscope

Motivating Example: Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Parametric FE model: $M(d)\ddot{x}(t) + D(\theta, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t).$



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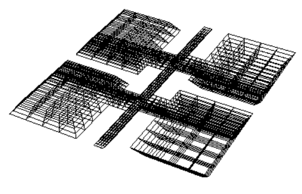
Parametric FE model:

$$M(d)\ddot{x}(t) + D(\theta, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t),$$

where

$$\begin{aligned} M(d) &= M_1 + dM_2, \\ D(\theta, d, \alpha, \beta) &= \theta(D_1 + dD_2) + \alpha M(d) + \beta T(d), \\ T(d) &= T_1 + \frac{1}{d} T_2 + dT_3, \end{aligned}$$

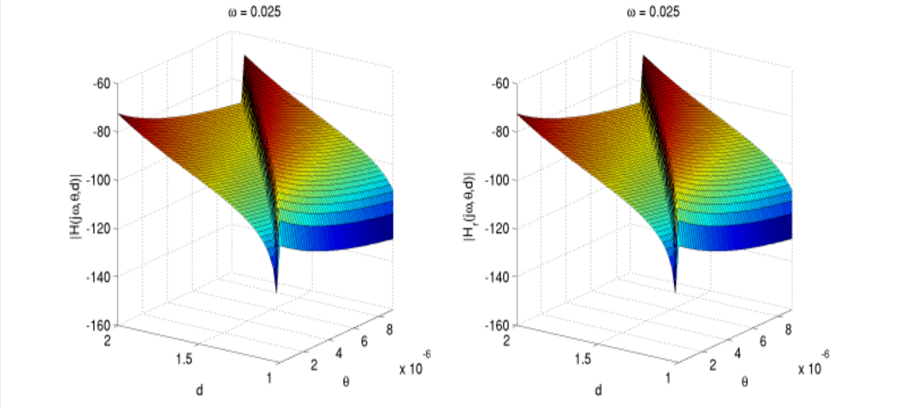


with

- width of bearing: d,
- angular velocity: θ,
- Rayleigh damping parameters: α, β.

Microgyroscope (butterfly gyro)

Original. . . and reduced-order model.



Problem

Approximate the dynamical system

$$\begin{aligned} E(p)\dot{x} &= A(p)x + B(p)u, \qquad && E(p), A(p) \in \mathbb{R}^{n\times n}, \\ y &= C(p)x, && B(p) \in \mathbb{R}^{n\times m}, C(p) \in \mathbb{R}^{q\times n}, \end{aligned}$$

by reduced-order system

$$\begin{aligned} \hat{E}(p)\dot{\hat{x}} &= \hat{A}(p)\hat{x} + \hat{B}(p)u, \quad && \hat{E}(p), \hat{A}(p) \in \mathbb{R}^{r\times r}, \\ \hat{y} &= \hat{C}(p)\hat{x}, && \hat{B}(p) \in \mathbb{R}^{r\times m}, \hat{C}(p) \in \mathbb{R}^{q\times r}, \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\| \quad \forall\ p \in \Omega.$$

Problem

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$\Longrightarrow$ Approximation problem: $\min\limits_{\text{order}\,(\hat{G}) \leq r} \|G - \hat{G}\|.$

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Appropriate parameter-affine representation:

$$\begin{aligned} E(p) &= E_0 + e_1(p)E_1 + \ldots + e_{q_E}(p)E_{q_E}, \\ A(p) &= A_0 + a_1(p)A_1 + \ldots + a_{q_A}(p)A_{q_A}, \\ B(p) &= B_0 + b_1(p)B_1 + \ldots + b_{q_B}(p)B_{q_B}, \\ C(p) &= C_0 + c_1(p)C_1 + \ldots + c_{q_C}(p)C_{q_C}, \end{aligned}$$

allows easy parameter preservation for projection based model reduction.

Parametric System

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Appropriate parameter-affine representation:

$$A(p) \quad = \quad A_0 + a_1(p)A_1 + \ldots + a_{q_A}(p)A_{q_A}, \qquad \ldots$$

allows easy parameter preservation for projection based model reduction.

W.l.o.g. may assume this affine representation:

- Any system can be written in this affine form for some $q_X \leq n^2$, but for efficiency, need $q_X \ll n!$ $(X \in \{E, A, B, C\})$
- Empirical (operator) interpolation yields this structure for "smooth enough" parameter functions. [Barrault/Maday/Nguyen/Patera 2004]

Parametric System

$$\Sigma(p) : \begin{cases} E(p)\dot{x}(t;p) &= A(p)x(t;p) + B(p)u(t), \\ y(t;p) &= C(p)x(t;p). \end{cases}$$

Parametric model reduction goal:

preserve parameters as symbolic quantities in reduced-order model:

$$\hat{\Sigma}(p) : \begin{cases} \hat{E}(p)\dot{\hat{x}}(t;p) &= \hat{A}(p)\hat{x}(t;p) + \hat{B}(p)u(t), \\ \hat{y}(t;p) &= \hat{C}(p)\hat{x}(t;p) \end{cases}$$

with states $\hat{x}(t;p) \in \mathbb{R}^r$ and $r \ll n$.

Outline

Petrov-Galerkin-type projection

For given projection matrices $V, W \in \mathbb{R}^{n\times r}$ with $W^T V = I_r$
($\rightsquigarrow (VW^T)^2 = VW^T$ is projector), compute

$$
\begin{aligned}
\hat{E}(p) &= W^T E_0 V + e_1(p) W^T E_1 V + \ldots + e_{q_E}(p) W^T E_{q_E} V, \\
&= \hat{E}_0 + e_1(p)\hat{E}_1 + \ldots + e_{q_E}(p)\hat{E}_{q_E}, \\
\hat{A}(p) &= W^T A_0 V + a_1(p) W^T A_1 V + \ldots + a_{q_A}(p) W^T A_{q_A} V, \\
&= \hat{A}_0 + a_1(p)\hat{A}_1 + \ldots + a_{q_A}(p)\hat{A}_{q_A}, \\
\hat{B}(p) &= W^T B_0 + b_1(p) W^T B_1 + \ldots + b_{q_B}(p) W^T B_{q_B}, \\
&= \hat{B}_0 + b_1(p)\hat{B}_1 + \ldots + b_{q_B}(p)\hat{B}_{q_B}, \\
\hat{C}(p) &= C_0 V + c_1(p) C_1 V + \ldots + c_{q_C}(p) C_{q_C} V, \\
&= \hat{C}_0 + c_1(p)\hat{C}_1 + \ldots + c_{q_C}(p)\hat{C}_{q_C}.
\end{aligned}
$$

$\rightsquigarrow$ **affine parameter structure is preserved!**

PMOR by Projection

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&= \hat{A}_0 + a_1(p)\hat{A}_1 + \ldots + a_{q_A}(p)\hat{A}_{q_A}, \\
\hat{B}(p) &= W^T B_0 + b_1(p) W^T B_1 + \ldots + b_{q_B}(p) W^T B_{q_B}, \\
&= \hat{B}_0 + b_1(p)\hat{B}_1 + \ldots + b_{q_B}(p)\hat{B}_{q_B}, \\
\hat{C}(p) &= C_0 V + c_1(p) C_1 V + \ldots + c_{q_C}(p) C_{q_C} V, \\
&= \hat{C}_0 + c_1(p)\hat{C}_1 + \ldots + c_{q_C}(p)\hat{C}_{q_C}.
\end{aligned}
$$

$\rightsquigarrow$ **affine parameter structure is preserved!**

PMOR by Projection

Methods

- Rational interpolation, based on
  - multi-moment matching [B., Daniel, Dyczij-Edlinger, Farle, Feng, Gallivan, Gunupudi, Korvink, Nakhla, Rudnyi, Weile, . . .]
  - tangential interpolation and $\mathcal{H}_2$ optimization [Baur, Beattie, B., Breiten, Bruns, Gugercin, . . .]
- Transfer function interpolation (= output interpolation in frequency domain) [Baur, B., . . .]
- Matrix interpolation [Amsallam, Brunsch, Eid, Geuss, Farhat, Lohmann, Mohring, Panzer, Wolf, . . .]
- Manifold interpolation [Amsallam, Bruns, Carlberg, Farhat, Son, Stykel, . . .]
- Snapshot-based methods, like
  - Proper orthogonal/generalized decomposition (POD/PGD) [Carlberg, Chinesta, Cueto, Hinze, Huerta, Kunisch, Nouy, Willcox, Volkwein, . . .]
  - Reduced basis method (RBM) [Grepl, Haasdonk, Hess, Hesthaven, Maday, Quarteroni, Patera, Prud'homme, Ohlberger, Rozza, Stamm, Urban, Veroy, . . .]

$\mathcal{H}_2$-Optimal Model Reduction for Linear Systems

Consider stable (i.e. $\Lambda(A) \subset \mathbb{C}^-$) linear systems Σ,

$$\dot{x}(t) = Ax(t) + Bu(t), \; y(t) = Cx(t) \qquad \simeq \qquad y(s) = \underbrace{C(sI - A)^{-1}B}_{=:G(s)}\, u(s)$$

System norms

Two common system norms for measuring approximation quality:

- $\mathcal{H}_2$-norm, $\|\Sigma\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_0^{2\pi} \operatorname{tr}\left(\left(G^T(-\jmath\omega) G(\jmath\omega) \right) \right) d\omega \right)^{\frac{1}{2}}$,
- $\mathcal{H}_\infty$-norm, $\|\Sigma\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}\left(G(\jmath\omega) \right)$,

where

$$G(s) = C\,(sI - A)^{-1}\,B.$$

Note: $\mathcal{H}_\infty$*-norm approximation* $\rightsquigarrow$ *balanced truncation, Hankel norm approximation.*

Error system and $\mathcal{H}_2$-Optimality [Meier/Luenberger 1967]

In order to find an $\mathcal{H}_2$-optimal reduced system, consider the error system

$$G(s) - \hat{G}(s) = C(sI_n - A)^{-1}B - \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$$

which can be realized by

$$A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.$$

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Assuming a coordinate system in which $\hat{A}$ is diagonal and taking derivatives of

$$\|G(\,.\,) - \hat{G}(\,.\,)\|^2_{\mathcal{H}_2}$$

with respect to free parameters in $\mathrm{diag}(\hat{A}), \hat{B}, \hat{C} \rightsquigarrow$

First-order necessary $\mathcal{H}_2$-optimality conditions (SISO)

$$G(-\hat{\lambda}_i) = \hat{G}(-\hat{\lambda}_i), \qquad G'(-\hat{\lambda}_i) = \hat{G}'(-\hat{\lambda}_i),$$

where $\hat{\lambda}_i$ are the poles of $\hat{\Sigma}$. "*Hermite interpolation at mirror poles*"

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First-order necessary $\mathcal{H}_2$-optimality conditions (MIMO)

$$\begin{aligned} G(-\hat{\lambda}_i)\tilde{B}_i &= \hat{G}(-\hat{\lambda}_i)\tilde{B}_i, && \text{for } i = 1, \ldots, r, \\ \tilde{C}_i^T G(-\hat{\lambda}_i) &= \tilde{C}_i^T \hat{G}(-\hat{\lambda}_i), && \text{for } i = 1, \ldots, r, \\ \tilde{C}_i^T G'(-\hat{\lambda}_i)\tilde{B}_i &= \tilde{C}_i^T \hat{G}'(-\hat{\lambda}_i)\tilde{B}_i && \text{for } i = 1, \ldots, r, \end{aligned}$$

where $\hat{A} = R\hat{\Lambda}R^{-1}$ is the spectral decomposition of the reduced system and $\tilde{B} = \hat{B}^T R^{-1}$, $\tilde{C} = \hat{C}R$. "*Tangential interpolation at mirror poles*"

Interpolation of the Transfer Function [Grimme 1997]

Construct reduced transfer function by Petrov-Galerkin projection $\mathcal{P} = VW^T$, i.e.

$$\hat{G}(s) = CV\left(sI - W^T AV\right)^{-1} W^T B,$$

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where V and W are given as

$$V = \left[(-\mu_1 I - A)^{-1}B, \ldots, (-\mu_r I - A)^{-1}B\right],$$
$$W = \left[(-\mu_1 I - A^T)^{-1}C^T, \ldots, (-\mu_r I - A^T)^{-1}C^T\right].$$

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Then

$$G(-\mu_i) = \hat{G}(-\mu_i) \quad \text{and} \quad G'(-\mu_i) = \hat{G}'(-\mu_i),$$

for $i = 1, \ldots, r$.

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Then

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for $i = 1, \ldots, r$.
Starting with an initial guess for $\hat{\Lambda}$ and setting $\mu_i \equiv \hat{\lambda}_i \rightsquigarrow$ iterative algorithms (IRKA/MIRIAm) that yield $\mathcal{H}_2$-optimal models.

[Gugercin et al. 2006/08], [Bunse-Gerstner et al. 2007], [Van Dooren et al. 2008]

The Basic IRKA Algorithm

Algorithm 1 IRKA (MIMO version/MIRIAm)

Require: A stable, B, C, $\hat{A}$ stable, $\hat{B}$, $\hat{C}$, $\delta > 0$.

Ensure: A^{opt}, B^{opt}, C^{opt}.

while $(\max_{j=1,\ldots,r} \left\{ \frac{|\mu_j - \mu_j^{\text{old}}|}{|\mu_j|} \right\} > \delta)$ **do**

  $\text{diag}\{\mu_1, \ldots, \mu_r\} := T^{-1}\hat{A}T =$ spectral decomposition,
  $\tilde{B} = \hat{B}^H T^{-T}$, $\tilde{C} = \hat{C}T$.

  $V = \left[(-\mu_1 I - A)^{-1} B\tilde{b}_1, \ldots, (-\mu_r I - A)^{-1} B\tilde{b}_r\right]$

  $W = \left[(-\mu_1 I - A^T)^{-1} C^T \tilde{c}_1, \ldots, (-\mu_r I - A^T)^{-1} C^T \tilde{c}_r\right]$

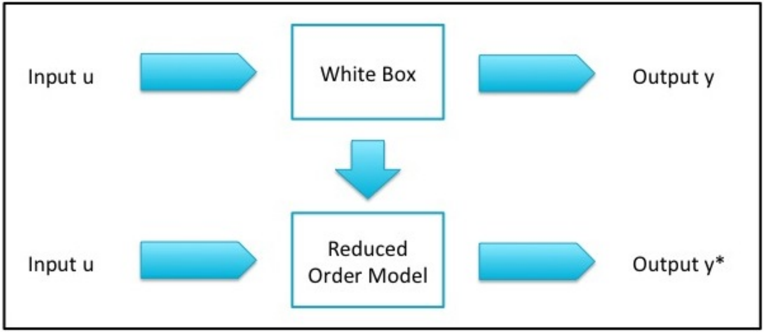
  $V = \text{orth}(V)$, $W = \text{orth}(W)$, $W = W(V^T W)^{-1}$

  $\hat{A} = W^T A V$, $\hat{B} = W^T B$, $\hat{C} = CV$

end while

$A^{opt} = \hat{A}$, $B^{opt} = \hat{B}$, $C^{opt} = \hat{C}$.

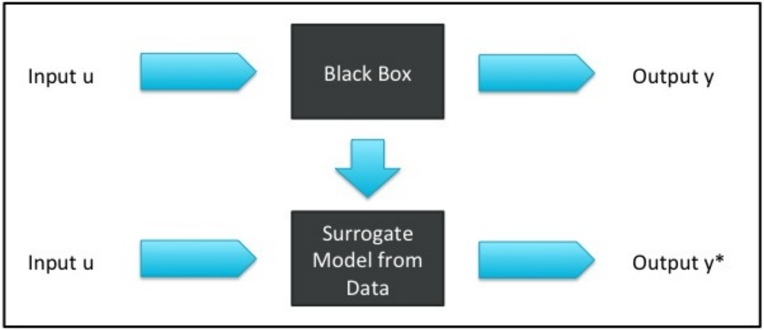
Interpretation as "White Box Model"



- Known behavior from input to output,
- typically given by a (discretized) PDE model, or directly as ODE,
- matrices are available $\rightsquigarrow$ projection-based (P)MOR techniques can be used.

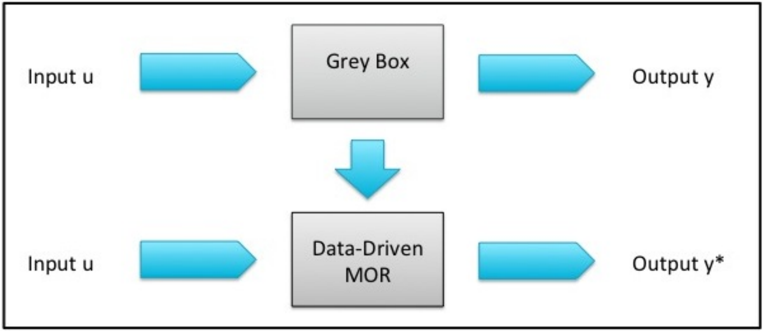
Outline

Black Box / System Identification



- We have input/output data or we can produce input/output data.
- Surrogate model by interpolation (Loewner, Kriging, radial basis functions, kernel methods, . . .) or data fitting (vector fitting, DMD, . . .).

Grey Box



- Approximate model available (e.g., unknown or uncertain parameters, simplified model, . . .)
- Experimental and/or computational data available, as well as model structure.

The Loewner Method [Mayo/Antoulas 2007]

Given **frequencies** together with the **values (measurements)** of the transfer function at those frequencies, the Loewner method is a data driven approach to create a state space system which interpolates there.

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The Basic Method (SISO case, for simplicity of exposition)

Given the interpolation points $(\xi_1, \dots, \xi_N, \sigma_1, \dots, \sigma_r)$, and the associated function values $V^T = [G(\xi_1), \dots, G(\xi_N)]$ and $W = [G(\sigma_1), \dots, G(\sigma_r)]$, we can define the Loewner and shifted Loewner matrices $\mathbb{L}, \mathbb{L}_s$ as the divided differences matrices

$$\mathbb{L}_{ij} := \frac{V_i - W_j}{\xi_i - \sigma_j}, \qquad (\mathbb{L}_s)_{ij} := \frac{\xi_i V_i - \sigma_j W_j}{\xi_i - \sigma_j}.$$

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If $r = N$ and N equals the McMillan degree of the underlying linear system, $(\mathbb{L}_s, \mathbb{L})$ is a regular matrix pair with all its eigenvalues distinct from ξ_i, σ_j, then

$$\tilde{G}(s) = W(\mathbb{L}_s - s\mathbb{L})^{-1} V$$

is a minimal realization of a linear time-invariant system that produces the given data.

(Reduced) Models from Data

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Given **frequencies** together with the **values (measurements)** of the transfer function at those frequencies, the Loewner method is a data driven approach to create a state space system which interpolates there.

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Given the interpolation points $(\xi_1, \ldots, \xi_N, \sigma_1, \ldots, \sigma_r)$, and the associated function values $V^T = [G(\xi_1), \ldots, G(\xi_N)]$ and $W = [G(\sigma_1), \ldots, G(\sigma_r)]$, we can define the Loewner and shifted Loewner matrices $\mathbb{L}, \mathbb{L}_s$ as the divided differences matrices

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In case of *redundant data* $(\operatorname{rank}(\mathbb{L}) < N)$, we obtain an approximate (reduced) model from the (truncated) SVDs

$$[\,\mathbb{L},\, \mathbb{L}_s\,] = Y S_\ell \tilde{X}^H, \qquad \begin{bmatrix} \mathbb{L} \\ \mathbb{L}_s \end{bmatrix} = \tilde{Y} S_r X^H$$

as $\hat{G}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}$, where

$$\hat{E} := -Y^H \mathbb{L} X, \quad \hat{A} := -Y^H \mathbb{L}_s X, \quad \hat{B} := Y^H V, \quad \hat{C} := WX.$$

Outline

A Grey Box Method

Assumptions

- The model —a linear parametric system— is available only at a certain parameter point p^*.
- Moreover, we can collect **frequency response data** $G(\xi_i)$ for the system at certain frequencies $\xi_1, \ldots, \xi_N$ for other parameter configurations $p \neq p^*$.

The set-up could be:

- Black box with these input-output data (or infinite dimensional system), and a particular discretization for $p = p^*$.
- Want to create a reduced parametric linear system that is a good approximation of the true system.

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Idea

[B./Grundel, AML 39:1–6, 2015]

Obtain reduced parametric model from H_2-optimal reduced model at p^* (with optimal interpolation points $\sigma_1, \ldots, \sigma_r$) and Loewner interpolation at frequency response data.

A Grey Box Method

First, consider the case $N = r$ and recall the Loewner pair $(\mathbb{L}, \mathbb{L}_s)$ with the associated transfer function

$$\tilde{G}(s) = W(\mathbb{L}_s - s\mathbb{L})^{-1}V,$$

where V, W contain the interpolation data.

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We will now look at this problem as a rational interpolation problem in the set-up of barycentric interpolation.

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We will now look at this problem as a rational interpolation problem in the set-up of barycentric interpolation.

Here we know that the transfer function

$$\tilde{H}(s) = \frac{\sum_{k=1}^{r} \frac{\alpha_k W_k}{s-\sigma_k}}{\sum_{k=1}^{r} \frac{\alpha_k}{s-\sigma_k} + 1}$$

is a strictly proper rational function that interpolates G at σ_k for all $\alpha_1, \ldots, \alpha_k$, as long as they are not all zero.

Lemma

The two transfer functions $\tilde{G}$ and $\tilde{H}$ are identical exactly when $\mathbb{L}\alpha + V = 0$.

A Grey Box Method

The Barycentric Loewner Method

Consider now the case $N > r$.
The strictly proper rational function in barycentric form, interpolating at σ_k:

$$\tilde{H}(s) = \frac{\sum_{k=1}^{r} \frac{\alpha_k W_k}{s-\sigma_k}}{\sum_{k=1}^{r} \frac{\alpha_k}{s-\sigma_k} + 1}. \tag{1}$$

We want to pick α such that $\tilde{H}(\xi_i) \approx G(\xi_i)$, i.e., such that the difference

$$\tilde{H}(\xi_i) - G(\xi_i) = \frac{\sum_{k=1}^{r} \frac{\alpha_k W_k}{\xi_i-\sigma_k}}{\sum_{k=1}^{r} \frac{\alpha_k}{\xi_i-\sigma_k} + 1} - V_i = \frac{(-\mathbb{L}\alpha - V)_i}{\sum_{k=1}^{r} \frac{\alpha_k}{\xi_i-\sigma_k} + 1} \tag{2}$$

is small (in a least-squares sense).

Lemma

A state space system that has the transfer function $\tilde{H}$ as in (1) with $\mathbb{L}^H\mathbb{L}\alpha + \mathbb{L}^H V = 0$ is given by (Z contains the first r left singular vectors of $\mathbb{L}$):

$$\tilde{E} = -Z^H\mathbb{L}, \quad \tilde{A} = -Z^H\mathbb{L}_s, \quad \tilde{C} = W, \quad \tilde{B} = Z^H V.$$

A Grey Box Method

The IRKA-Loewner Algorithm

(for computing a state-space realization of the system at $p = \bar{p}$.)

Offline Phase

- Compute $\mathcal{H}_2$-optimal interpolation points $\sigma_1, \ldots, \sigma_r$ for the given model at p^*.
  Recommendation: choose center point of parameter domain if possible.

Online Phase

INPUT: reduced order r, parameter $\bar{p}$, interpolation points $\sigma_1, \ldots, \sigma_r$ (from offline phase).

1. Obtain data for $p = \bar{p}$: $\xi_1, \ldots, \xi_N$, $G(\xi_1), \ldots, G(\xi_N)$.
2. Compute $V_i = G(\xi_i)$, $W_j = G(\sigma_j)$, $\mathbb{L}_{ij} = \frac{V_i - W_j}{\xi_i - \sigma_j}$, $(\mathbb{L}_s)_{ij} = \frac{\xi_i V_i - \sigma_j W_j}{\xi_i - \sigma_j}$.
3. Compute the SVD $\mathbb{L} = U \Sigma V^T$.
4. Set $Z = U(:, 1:r)$.
5. $\hat{A} = -Z^H \mathbb{L}$, $\hat{E} = -Z^H \mathbb{L}_s$, $\hat{B} = W$, $\hat{C} = Z^H V$.

A Beam Example

- FEM of a 3D cantilever Timoshenko beam.
- parameter is the length of the beam $p \in [0.8, 1.2]$.
- $n = 240$.

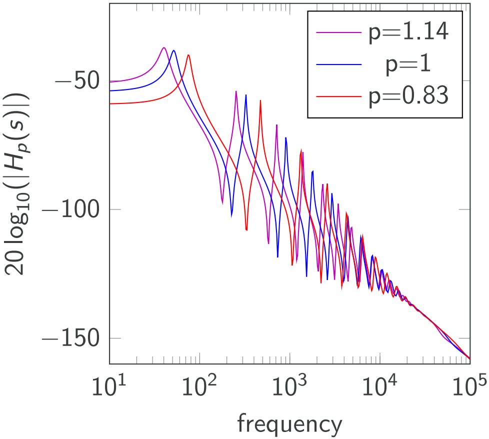
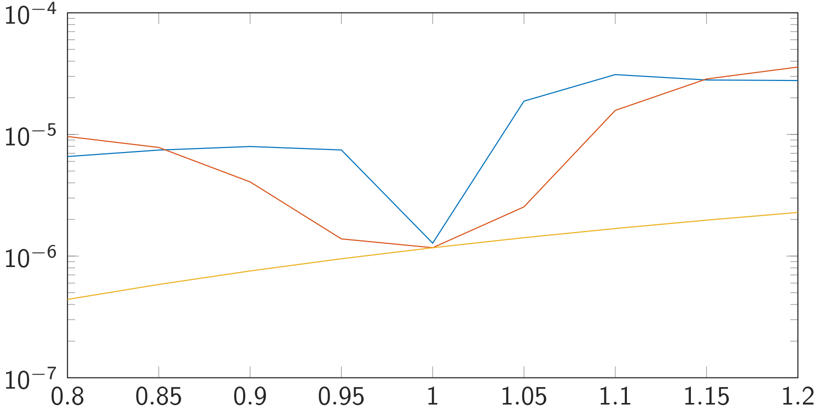


Figure: Bode plots for varying p

A Beam Example

$\mathcal{H}_2$-error for the data driven approach: $r = 4$, 100 data points.



Hermite interpolant at $p = 1$ — IRKA-Loewner model — optimal $\mathcal{H}_2$ at all p

A UQ Example

MATLAB Code

```
function [A,B,C,D,E]=UQexample(Q)
%Q - a fixed random orthogonal matrix
n=100;
A=Q*diag(-10*rand(n,1))*Q';  B=ones(n,1);  C=ones(1,n);  D=0;  E=eye(n);
```

r	N	IRKA		IRKA-Loewner		Hermite		Loewner		
		err	#	err	#	err	#	err	$\#(\mathbb{C}^-)$	#
4	8	0.01	142	0.02	12	0.02	8	10	(5)	8
4	16	0.01	142	0.01	20	0.02	8	0.2	(5)	16
6	12	2E-5	134	5E-5	18	2E-4	12	15	(4)	12
6	24	2E-5	134	3E-5	30	2E-4	12	4E-3	(5)	24
10	20	2E-11	151	2E-10	30	2E-5	20	–	(0)	20
10	40	2E-11	151	1E-10	50	2E-5	20	2E-6	(1)	40

Table: Average over 5 runs.

Outline

Method 1. Projection Using Common Bases

To reduce the parametric full-order model (FOM):

$$E(p)\dot{x} = A(p)x + B(p),$$
$$y = C(p)x$$

where $E(p), A(p) \in \mathbb{R}^{n\times n}$, $B(p) \in \mathbb{R}^{n\times m_i}$, $C(p) \in \mathbb{R}^{m_o\times n}$,

- these methods build common bases $W, V \in \mathbb{R}^{n\times r}$ by assembling global information, e.g., (multi-)moments of the frequency domain system, snapshots, etc., at different values of p;
- approximate $x \approx V\hat{x}$ in the range of V ($\hat{x} \in \mathbb{R}^r$);
- force the residual to be orthogonal to the range of W, i.e.,

$$W^H(E(p)V\dot{\hat{x}} - A(p)V\hat{x} - B(p)) = 0.$$

- obtain the parametric reduced-order model (pROM):

$$\hat{E}(p)\dot{\hat{x}} = \hat{A}(p)x + \hat{B}(p),$$
$$\hat{y} = \hat{C}(p)\hat{x}.$$

where $[\hat{E}(p), \hat{A}(p)] = W^H[E(p), A(p)]V$, $\hat{B}(p) = W^H B(p)$ and $\hat{C}(p) = C(p)V$.

Method 1. Projection Using Common Bases (Continued)

- **Case 1.** Explicit Parameter Dependency:

$$M(p) = \sum_{i=1}^{n_p} M_i \phi_i(p). \qquad (M = E, A, B, \text{ or } C)$$

The reduced operator can be efficiently pre-computed, e.g.,

$$W^H A(p) V = \sum_{i=1}^{n_p} \left(W^H A_i V \right) \phi_i(p).$$

- **Case 2.** Implicit Parameter Dependency:
  - Interpolate the FOM and then Reduce
    is equivalent to
    Reduce and then Interpolate the ROMs because

$$\hat{M}(p) = W^H \Big(\sum_{i=1}^{n_p} f_j(p) M_j \Big) V = \sum_{i=1}^{n_p} f_j(p) \hat{M}_j, \qquad f_j(p_i) = \delta_{ij}, M_j = M(p_j).$$

  - If the parameter dependency of the FOM can be well captured by a certain interpolation method, interpolating the ROMs directly will also work well (thanks to the common bases used).

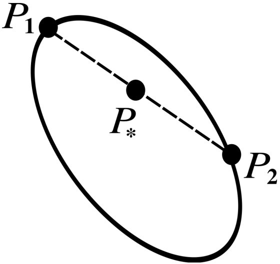
Method 2. Projection Using Individual Bases

- The bases V and W are functions of p:

$$[\hat{E}(p), \hat{A}(p)] = W(p)^H [E(p), A(p)] V(p),$$
$$\hat{B}(p) = W^H(p) B(p), \quad \hat{C}(p) = C(p) V(p).$$

- Question: How to interpolate $V(p)$ and $W(p)$?

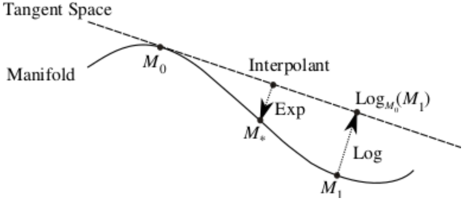
Direct interpolation does not preserve the orthonormality of V and W as in general, the interpolant will not be on the Stiefel manifold of rank-r isometries.



Method 2. Projection Using Individual Bases (Continued)

To preserve a desired property for the interpolated matrix:

1. Find the manifold corresponding to the desired property.
   - Grassmann manifold $\mathcal{G}(k, n)$: set of subspaces of dimension k in $\mathbb{R}^n$.
   - Stiefel manifold $\mathcal{ST}(k, n)$: set of all $n \times k$ orthonormal matrices in $\mathbb{R}^n$.
2. Choose a reference point to determine the tangent space.
3. The Log map: map all points relevant to the interpolation to the tangent space.
4. Interpolate on the tangent space. We must choose a interpolation algorithm so that the interpolant lies also on the tangent space.
5. The Exp map: Map the interpolant back to the manifold.



Alert: This procedure requires "continuity" in $V(p)$ and $W(p)$.

Method 3. Interpolating “Given” ROMs

If we are simply “given” some ROMs valid at different parameter values p:

- Direct interpolation normally does not work:
  - Different realizations (coordinate systems) of ROMs;
  - simply exchanging rows of one ROM breaks “continuity”.
- If we know the bases V_i and W_i, we can build

$$V_{all} = [V_1, V_2, \dots, V_{p_n}], \quad W_{all} = [W_1, W_2, \dots, W_{p_n}],$$

  perform SVD, and use the dominant components to assemble the common bases V and W. Then, the ROMs can be represented using the common realization.
- Sometimes, we do not even know V_i and W_i!
  - For example, ROMs built under the Loewner framework.
  - There also exists methods that interpolate on specific matrix manifold, e.g., the manifold of nonsingular $r \times r$ matrices (the general linear group $GL(r)$), and the manifold of symmetric positive definite matrices.
  - Heuristic methods that attempt to render the bases ”consistent”.

Recall: The Loewner Framework

Dynamical System (Unknown) FRF (Only known as a whole operater)

$$\begin{cases} (sE - A)x &= Bu, \\ y &= Cx. \end{cases} \longrightarrow \qquad H(s) = C(sE - A)^{-1}B.$$

Step 1: Collect data: (V, W are not bases: composed of tangential directions!)

- "Right Data": (λ_i, r_i, w_i) satisfying $H(\lambda_i)r_i = w_i$;
- "Left Data": (μ_j, ℓ_j, v_j) satisfying $\ell_j H(\mu_j) = v_j$.

Step 2: Compute the Loewner matrix $\mathbb{L}$ and the shifted Loewner matrix $\mathbb{L}_s$.

$$(\mathbb{L})_{ij} = \frac{v_i r_j - \ell_i w_j}{\mu_i - \lambda_j}, \qquad (\mathbb{L}_s)_{ij} = \frac{\mu_i v_i r_j - \ell_i w_j \lambda_j}{\mu_i - \lambda_j}.$$

Step 3: Compute the reduced model:

- If the matrix pencil $(\mathbb{L}_s, \mathbb{L})$ is regular, the reduced model is: $\hat{E} = -\mathbb{L}$, $\hat{A} = -\mathbb{L}_s$, $\hat{B} = V$, $\hat{C} = W$.
- If the matrix pencil $(\mathbb{L}_s, \mathbb{L})$ is (numerically) singular:
  1. Compute rank-revealing SVD: $s\mathbb{L} - \mathbb{L}_s = Y\Sigma X^H \approx Y_r \Sigma_r X_r^H$; $(s \in \{\lambda_i\} \cup \{\mu_j\})$
  2. Compute $\hat{E} = -Y_r^H \mathbb{L} X_r$, $\hat{A} = -Y_r^H \mathbb{L}_s X_r$, $\hat{B} = Y_r^H V$, $\hat{C} = W X_r$.

Computing Parametric Models using the Loewner Framework

- In summary, we consider two representations:
  - **The "Original" Representation.**
    - $\hat{E} = -\mathbb{L}$, $\hat{A} = -\mathbb{L}_s$, $\hat{B} = V$, $\hat{C} = W$.
    - Very likely to produce a numerically singular pair $(\hat{A}, \hat{E})$!
  - **The "Compressed" Representation.**
    - $\hat{E} = -Y_r^H \mathbb{L} X_r$, $\hat{A} = -Y_r^H \mathbb{L}_s X_r$, $\hat{B} = Y_r^H V$, $\hat{C} = W X_r$.
    - Yields regular matrix pair $(\hat{A}, \hat{E})$.

How to interpolate the ROMs built by the Loewner Framework?

- There is no "FOM" in the Loewner Framework, and no bases V_i and W_i like in the projection-based methods.
- The Loewner Framework has been extended for PMOR in [Ionita/Antoulas '14] $\rightsquigarrow$ **PMOR-L**.
  This talk explores another possibility: interpolating nonparametric ROMs built under the Loewner Framework.

Interpolating the "Original" Representation

- Assume that the system is parameterized with parameter p.
- Assume that we use the same frequencies and the same left/right input vectors.
- In $H_p(\lambda_i)r_i = w_i(p)$, λ_i and r_i are independent of p, $1 \leq i \leq \mathbb{R}^{n_p}$.
- In $\ell_j H_p(\mu_j) = v_j(p)$, μ_j and ℓ_j are independent of p, $1 \leq j \leq \mathbb{R}^{n_p}$.
- Interpolating $\mathbb{L}(p)$ and $\mathbb{L}_s(p)$

  is equivalent to

  Interpolating $V(p)$ and $W(p)$ and using the Loewner framework:

$$\left(\tilde{\mathbb{L}}(p)\right)_{ij} = \sum_{q=1}^{n_p} \left(\frac{v_i(p_q)r_j - \ell_i w_j(p_q)}{\mu_i - \lambda_j} \right) \phi_q = \frac{\tilde{v}_i(p)r_j - \ell_i \tilde{w}_j(p)}{\mu_i - \lambda_j}$$

$$\left(\tilde{\mathbb{L}}_s(p)\right)_{ij} = \sum_{q=1}^{n_p} \left(\frac{\mu_i v_i(p_q)r_j - \ell_i w_j(p_q)\lambda_j}{\mu_i - \lambda_j} \right) \phi_q = \frac{\mu_i \tilde{v}_i(p_q)r_j - \ell_i \tilde{w}_j(p_q)\lambda_j}{\mu_i - \lambda_j}$$

- In the "Original Representation":
  $\hat{E}(p) = -\mathbb{L}(p)$, $\hat{A}(p) = -\mathbb{L}_s(p)$, $\hat{B}(p) = V(p)$, $\hat{C}(p) = W(p)$.
  - They are all linear functions of $V(p)$ and $W(p)$.
  - Can be directly interpolated: equivalent to interpolating the transfer functions.

Interpolating "Compressed" Representation

- If we store all the "Original" Representations, too much storage are required.
- The ultimate goal is to interpolate the "Compressed" Representation.
- The bases used to "compress" the "Original" Representation vary with parameters (with additional freedom in s_i): $\boxed{s_i\mathbb{L}_i - \mathbb{L}_{s\,i} = Y_i\Sigma_i X_i^H \approx Y_{i,k}\Sigma_{i,k}X_{i,k}^H}$. Here Y_i is a generalized observability matrix and X_i is a generalized controllability matrix.
- **Idea 1**: To preserve the property for interpolation of the original representation $(-\mathbb{L}_i, -\mathbb{L}_{s\,i}, V_i, W_i)$, use common bases Y and X to reduce them.
- **Idea 2a**: Y should contain the dominant components of all observability matrices Y_i. So we compute Y by the following SVD:
$$\left[\, s_1\mathbb{L}_1 - \mathbb{L}_{s\,1} \;\middle|\; s_2\mathbb{L}_2 - \mathbb{L}_{s\,2} \;\middle|\; \dots \;\middle|\; s_{n_q}\mathbb{L}_{n_q} - \mathbb{L}_{s\,n_p} \,\right] = \overline{Y}\overline{\Sigma}_W\overline{X}_W^H \approx Y\Sigma_W X_W^H.$$
- **Idea 2b**: X should contain the dominant components of all controllability matrices X_i. So we compute X by the following SVD:
$$\begin{bmatrix} s_1\mathbb{L}_1 - \mathbb{L}_{s\,1} \\ s_2\mathbb{L}_2 - \mathbb{L}_{s\,2} \\ \dots \\ s_{n_q}\mathbb{L}_{n_q} - \mathbb{L}_{s\,n_p} \end{bmatrix} = \overline{Y}_V\overline{\Sigma}_V\overline{X}^H \approx Y_V\Sigma_V X^H.$$

Interpolating "Compressed" Representation (Continued)

- **Observation**: colspan$\{Y_i\} \subseteq$ colspan$\{\overline{Y}\}$, rowspan$\{X_i\} \subseteq$ rowspan$\{\overline{X}\}$, $\forall\ i$.
- Therefore, X, Y from truncated SVD include the dominant controllable and observable components for all states corresponding to parameter samples.

Algorithm 2 Interpolation of Loewner ROMs in the Compressed Representation

1: Build the common bases Y by computing the truncated SVD of

$$\left[\, s_1\mathbb{L}_1 - \mathbb{L}_{s1} \;\middle|\; s_2\mathbb{L}_2 - \mathbb{L}_{s2} \;\middle|\; \dots \;\middle|\; s_{n_q}\mathbb{L}_{n_q} - \mathbb{L}_{s\,n_p} \,\right] = \overline{Y}\overline{\Sigma}_W \overline{X}_W^H \approx Y\Sigma_W X_W^H.$$

2: Build the common bases X by computing the truncated SVD of

$$\begin{bmatrix} s_1\mathbb{L}_1 - \mathbb{L}_{s1} \\ s_2\mathbb{L}_2 - \mathbb{L}_{s2} \\ \dots \\ s_{n_q}\mathbb{L}_{n_q} - \mathbb{L}_{s\,n_p} \end{bmatrix} = \overline{Y}_V\overline{\Sigma}_V\overline{X}^H \approx Y_V\Sigma_V X^H.$$

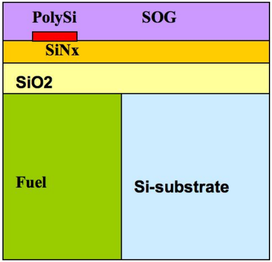
3: Build the "Compressed" Representations using the common bases:

$$\hat{E}_l = -Y^H\mathbb{L}_l X, \quad \hat{A}_l = -Y^H\mathbb{L}_{\sigma,l}X, \quad \hat{B}_l = Y^H V_l, \quad \hat{C}_l = W_l X.$$

4: Given an interpolation operator, the interpolated ROM at p is given by

$$\hat{M}(p) = \sum_{l=1}^{n_p} \hat{M}_l\phi_l(p), \qquad \hat{M} \in \{\hat{E}, \hat{A}, \hat{B}, \hat{C}\}.$$

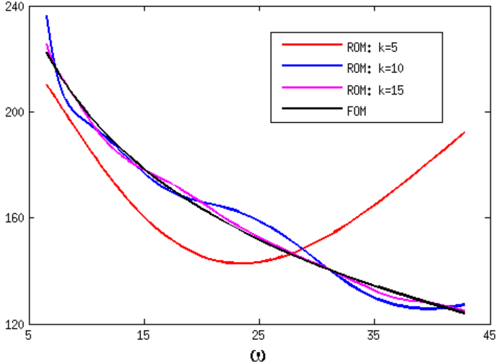
A Microthruster Model



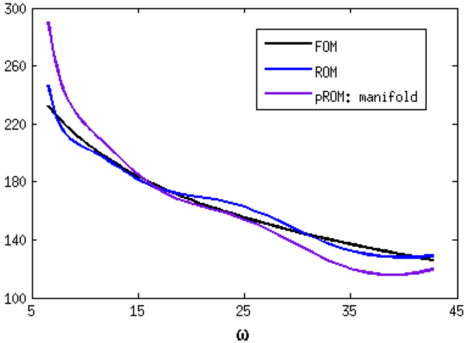
- A propulsive MEMS (micro-electro-mechanical) device.
- We consider a single-input single-output sub-model, $n = 4,257$.
- Pretend that only Input/Output information is available.
- We consider the single parameter: the film coefficient (boundary heat transfer coefficient).

The Convergence of the Loewner Approach

- For the Loewner framework, we take 100 λ and 100 μ samples.
- The magnitude of the transfer function is shown for the parameter value $p = 268.27$.

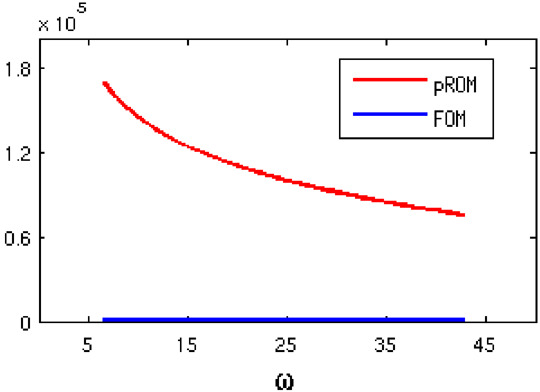


Existing Method: Manifold Interpolation on $GL(r)$ [Amsallem/Farhat '11]



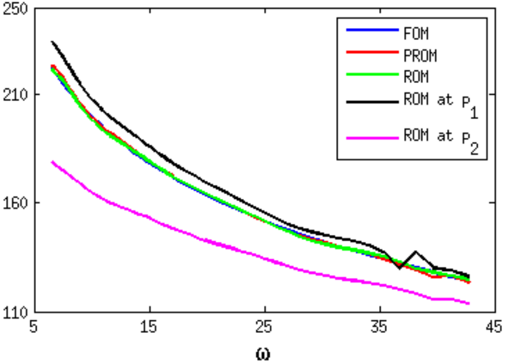
- Values of p used in interpolation: $p_1 = 10$, $p_2 = 268.3$, $p_3 = 7197$.
- To be interpolated at $p = 65.51$. Reference point for tangent space: 268.3.
- Order of the reduced model $r = 10$.

Existing Method: Manifold Interpolation on $GL(r)$ [Amsallem/Farhat '11]



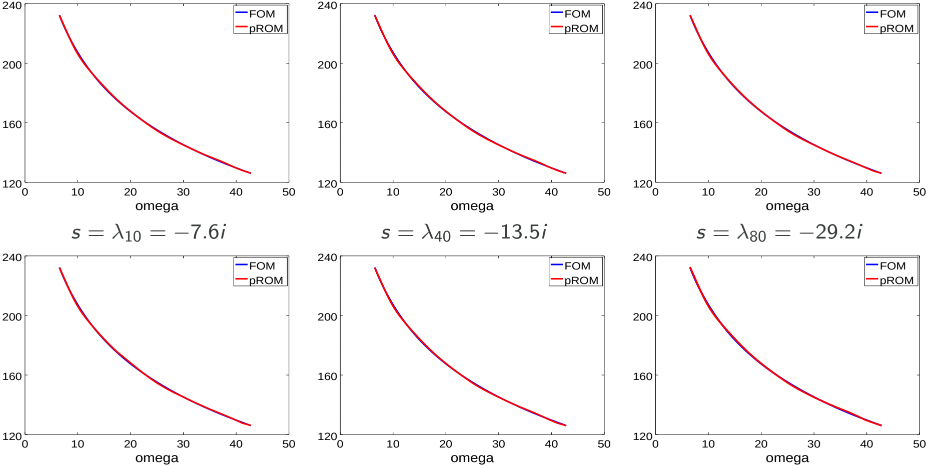
- Order of the reduced model: $r = 11$.
- Does not converge any more.

Method 1: Interpolating the "Original" Representation



- First interpolate (shifted) Loewner matrices, then reduce.
- All reduced models are of order 21.
- The pROM approximates the FOM very well.

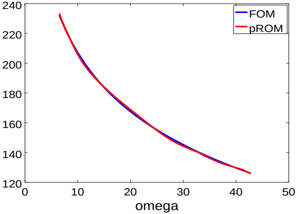
Method 2: Interpolating the "Compressed" Representation



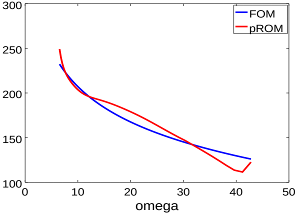
$s = \lambda_{10} = -7.6i$ $\quad s = \lambda_{40} = -13.5i$ $\quad s = \lambda_{80} = -29.2i$

$s = \mu_{10} = -51.9i$ $\quad s = \mu_{40} = -92.2i$ $\quad s = \mu_{80} = -198.7i$

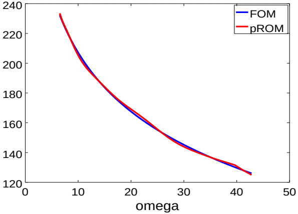
No good: Interpolating “Compressed” ROMs Generated with Individual Bases



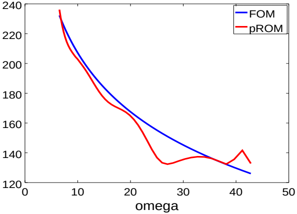
$s = \lambda_{10} = -7.6i$



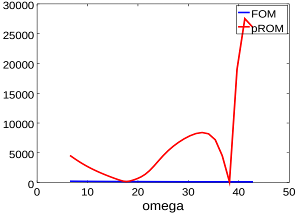
$s = \lambda_{40} = -13.5i$



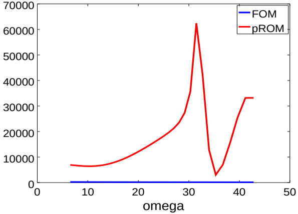
$s = \lambda_{80} = -29.2i$



$s = \mu_{10} = -51.9i$



$s = \mu_{40} = -92.2i$



$s = \mu_{80} = -198.7i$

Method 2: Interpolating the "Compressed" Representation

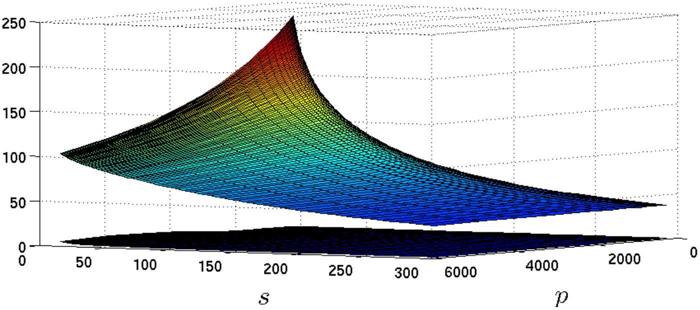


Figure: Response Surface and the Absolute Error

Concluding Remarks

- We have introduced two models to obtain parametric reduced models from data:
  1. **A grey box approach:** the IRKA-Loewner method can efficiently compute an accurate realization of a linear system at any parameter in parameter space, given
     - an H_2-optimal reduced-order model for a reference parameter p^*, and
     - that at other parameters, frequency response data can be collected.
  2. **A black box approach:** obtain a parametric reduced-order model from data without access to the original model matices, but to frequency response data for a number of parameter values. Affine parameter structure not necessary, insensitive to dimension of parameter space. Use of "common basis" approach gives good results for "smooth" transfer functions.
- The parametric Loewner model suffers from the same drawback as transfer function interpolation: spurious peaks in the frequency response may be introduced.
- If applicable, the PMOR-L approach may give slightly more accurate results than the parametric Loewner approach at the same reduced order, but the latter is more efficient in the offline phase and allows for a full parametric ROM (p in all system matrices).
- Need more experiments to identify shortcomings, and test potential remedies for those.