MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Linear Feedback Stabilization of Incompressible Flow Problems

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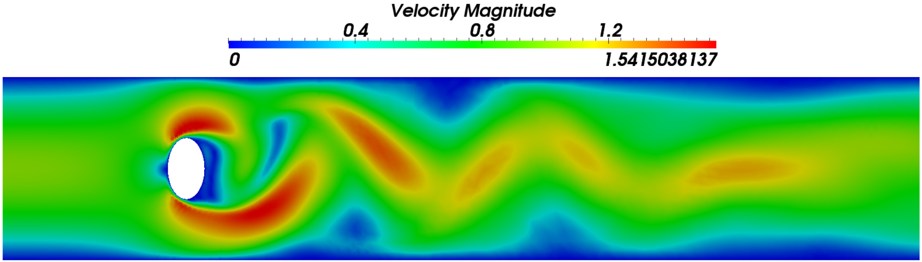
Motivation –Transport Problems as Dynamical Systems–

- **Physical transport processes** are one of the most fundamental dynamical processes in nature.
- **Prediction** and **manipulation** of transport processes are important research topics.
- **Open-loop** controllers are widely used in various engineering fields.
  $\rightarrow$ **Not robust** regarding perturbation
- Dynamical systems are often influenced via so called **distributed control**.
  $\rightarrow$ **Unfeasible** in many real-world areas

$\Rightarrow$ **Boundary feedback stabilization (closed-loop)**
can be used to increase robustness and feasibility.

Overview

- Consider 2D flow problems described by **incompressible Navier–Stokes equations**.
- Riccati feedback approach requires the solution of an **algebraic Riccati equation**.
- Conservation of mass introduces an additional **divergence-free** condition.



Kármán vortex street

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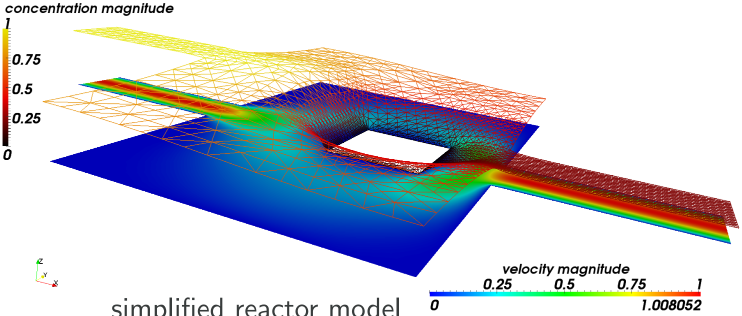
- Riccati feedback approach requires the solution of an **algebraic Riccati equation**.
- Conservation of mass introduces an additional **divergence-free** condition.
- **Coupling** flow problems with another **scalar transport equation**.



simplified reactor model

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   Develop techniques to deal with complex-shifted multi-field flow systems.

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   - implicitly project on "hidden manifold"

   $\Rightarrow$ Nested iteration: solve large-scale sparse saddle point system
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Introduction
–Introduced Framework–

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   - adapt various ideas from [Elman/Silvester/Wathen '05]

   $\Rightarrow$ develop suitable preconditioner to be used with GMRES

   $\Rightarrow$ efficient preconditioner use various approximation methods
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- combine [Kürschner '16], [B./Byers '98], and [Feizinger/Hylla/Sachs '09]

⇒ extend ideas in [B./Heinkenschloss/Saak/Weichelt '16]

⇒ develop a highly compatible method to solve Riccati equations

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- include feedback into forward simulation within NAVIER

$\Rightarrow$ closed-loop forward flow simulation

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1. Introduction
2. Feedback Stabilization for Index-2 DAE Systems
3. Accelerated Solution of Riccati Equations
4. Conclusions

Navier–Stokes Equations

$$\frac{\partial \vec{v}}{\partial t} - \frac{1}{\mathrm{Re}}\Delta \vec{v} + (\vec{v} \cdot \nabla)\vec{v} + \nabla p = \vec{f}$$

$$\operatorname{div} \vec{v} = 0$$

- defined for time $t \in (0, \infty)$ and space $\vec{x} \in \Omega \subset \mathbb{R}^2$ bounded with $\Gamma = \partial\Omega$
- $+$ boundary and initial conditions
- initial boundary value problem with additional algebraic constraints

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$A,\ M \in \mathbb{R}^{n\times n},\ \widehat{G} \in \mathbb{R}^{n\times n_{\mathbf{p}}}$

$B \in \mathbb{R}^{n\times n_r},\ C \in \mathbb{R}^{n_a\times n}$

$\mathbf{u}(t) \in \mathbb{R}^{n_r},\ \mathbf{y}(t) \in \mathbb{R}^{n_a}$

$\operatorname{rank}\left(\widehat{G}\right) = n_{\mathbf{p}}$

Linearize + Discretize → Index-2 DAE

$$M\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = A\mathbf{v}(t) + \widehat{G}\mathbf{p}(t) + B\mathbf{u}(t)$$

$$0 = \widehat{G}^T\mathbf{v}(t)$$

$$\mathbf{y}(t) = C\mathbf{v}(t)$$

$M = M^T \succ 0$

$\mathbf{v}(t) \in \mathbb{R}^{n},\ \mathbf{p}(t) \in \mathbb{R}^{n_{\mathbf{p}}}$

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Showed that projection in [Hei/Sor/Sun '08] is discretized version of Leray projector in [Ray '06].

$$M\Pi^T = \Pi M \quad \wedge \quad \Pi^T\mathbf{v} = \mathbf{v}_{\mathrm{div},0}$$

[**Bänsch**/B./Saak/**Stoll**/Weichelt **'13**,**'15**]

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Concentration Equation

$$\frac{\partial c^{(\vec{v})}}{\partial t} - \frac{1}{\mathrm{Re}\,\mathrm{Sc}}\Delta c^{(\vec{v})} + (\vec{v}\cdot\nabla)c^{(\vec{v})} = 0$$

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Feedback Stabilization for Index-2 DAE Systems –Physics of Multi-Field Flow–

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Extension to coupled flow case, i.e.,

$$\widehat{\Pi} := \begin{bmatrix}\Pi & 0\\ 0 & I\end{bmatrix} \quad \wedge \quad \begin{bmatrix}\Pi^T & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\mathbf{v}\\ \mathbf{c}\end{bmatrix} = \begin{bmatrix}\mathbf{v}_{\mathrm{div},0}\\ \mathbf{c}\end{bmatrix}.$$

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Minimize

$$\mathcal{J}(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty \lambda^2 ||\mathbf{y}||^2 + ||\mathbf{u}||^2 \,\mathrm{d}t$$

subject to

$$\widehat{\Theta}_r^T M \widehat{\Theta}_r \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbf{x}}(t) = \widehat{\Theta}_r^T A \widehat{\Theta}_r \widetilde{\mathbf{x}}(t) + \widehat{\Theta}_r^T B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\widehat{\Theta}_r \widetilde{\mathbf{x}}(t)$$

with $\widehat{\Pi} = \widehat{\Theta}_l \widehat{\Theta}_r^T$ such that $\widehat{\Theta}_r^T \widehat{\Theta}_l = I \in \mathbb{R}^{(n-n_\mathbf{p})\times(n-n_\mathbf{p})}$ and $\widetilde{\mathbf{x}} = \widehat{\Theta}_l^T \mathbf{x}$.

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Riccati Based Feedback Approach

- Optimal control: $\mathbf{u}(t) = -\mathcal{K}\widetilde{\mathbf{x}}(t)$, with feedback: $\mathcal{K} = \mathcal{B}^T\mathcal{X}\mathcal{M}$,

  where $\mathcal{X}$ is the solution of the generalized continuous-time algebraic Riccati equation (GCARE)

$$\mathcal{R}(\mathcal{X}) = \lambda^2\mathcal{C}^T\mathcal{C} + \mathcal{A}^T\mathcal{X}\mathcal{M} + \mathcal{M}\mathcal{X}\mathcal{A} - \mathcal{M}\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}\mathcal{M} = 0.$$

Determine $\mathcal{X} = \mathcal{X}^T \succeq 0$ such that $\mathcal{R}(\mathcal{X}) = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T\mathcal{X}\mathcal{M} + \mathcal{M}\mathcal{X}\mathcal{A} - \mathcal{M}\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}\mathcal{M} = 0.$

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Kleinman–Newton method

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Kleinman–Newton method

Step $\mathbf{m+1}$: Solve the Lyapunov equation

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)})^T\mathcal{X}^{(m+1)}\mathcal{M} + \mathcal{M}\mathcal{X}^{(m+1)}(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)}) = -(\mathcal{W}^{(m)})^T\mathcal{W}^{(m)} \tag{1}$$

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low-rank ADI method

Determine $\mathcal{X} = \mathcal{X}^T \succeq 0$ such that $\mathcal{R}(\mathcal{X}) = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T\mathcal{X}\mathcal{M} + \mathcal{M}\mathcal{X}\mathcal{A} - \mathcal{M}\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}\mathcal{M} = 0.$

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low-rank ADI method

Step ℓ: Solve the projected and shifted linear system

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)} + q_\ell\mathcal{M})^T\mathcal{V}_\ell = \mathcal{Y} \tag{2}$$

Determine $\mathcal{X} = \mathcal{X}^T \succeq 0$ such that $\mathcal{R}(\mathcal{X}) = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T\mathcal{X}\mathcal{M} + \mathcal{M}\mathcal{X}\mathcal{A} - \mathcal{M}\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}\mathcal{M} = 0.$

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Step ℓ: Solve the projected and shifted linear system

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linear solver

Feedback Stabilization for Index-2 DAE Systems –Nested Iteration without Projection–

Determine $\mathcal{X} = \mathcal{X}^T \succeq 0$ such that $\mathcal{R}(\mathcal{X}) = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T\mathcal{X}\mathcal{M} + \mathcal{M}\mathcal{X}\mathcal{A} - \mathcal{M}\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}\mathcal{M} = 0.$

Kleinman–Newton method

Step $\mathbf{m+1}$: Solve the Lyapunov equation

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)})^T\mathcal{X}^{(m+1)}\mathcal{M} + \mathcal{M}\mathcal{X}^{(m+1)}(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)}) = -(\mathcal{W}^{(m)})^T\mathcal{W}^{(m)} \tag{1}$$

low-rank ADI method

Step ℓ: Solve the projected and shifted linear system

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)} + q_\ell\mathcal{M})^T\mathcal{V}_\ell = \mathcal{Y} \tag{2}$$

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Avoid explicit projection using $\widehat{\Theta}_r\mathcal{V}_\ell = V_\ell$, $\mathcal{Y} = \widehat{\Theta}_r^T Y$, and [Hei/Sor/Sun '08]:

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for different ADI shifts $q_\ell \in \mathbb{C}^-$ for a couple of rhs Y.

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$$\begin{bmatrix} A^T + q_\ell M & \widehat{G} \\ \widehat{G}^T & 0 \end{bmatrix} \begin{bmatrix} V_\ell \\ * \end{bmatrix} = \begin{bmatrix} \tilde{Y} \\ 0 \end{bmatrix}$$

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Theorem 4.5 **[B./Heinkenschloss/Saak/Weichelt '16]**

- assume $(\mathbf{A}, \mathbf{B}; \mathbf{M})$ stabilizable, $(\mathbf{C}, \mathbf{A}; \mathbf{M})$ detectable
- $\Rightarrow \exists$ unique, symmetric solution $X^{(*)} = \widehat{\Theta}_r \mathcal{X}^{(*)} \widehat{\Theta}_r^T$ with $\mathcal{R}(\mathcal{X}^{(*)}) = 0$ that stabilizes

$$\left(\begin{bmatrix} A - BB^T X^{(*)} M & \widehat{G} \\ \widehat{G}^T & 0 \end{bmatrix}, \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \right)$$

- for $\{X^{(k)}\}_{k=0}^{\infty}$ defined by $X^{(k)} := \widehat{\Theta}_r \mathcal{X}^{(k)} \widehat{\Theta}_r^T$, (1), and $X^{(0)}$ symmetric with $\left(\mathbf{A} - \mathbf{B}\left(\mathbf{K}^{(0)}\right)^T, \mathbf{M}\right)$ stable, it holds that, for $k \geq 1$,

$$X^{(1)} \succeq X^{(2)} \succeq \cdots \succeq X^{(k)} \succeq 0 \quad \text{and} \quad \lim_{k \to \infty} X^{(k)} = X^{(*)}$$

- $\exists 0 < \widetilde{\kappa} < \infty$ such that, for $k \geq 1$,

$$||X^{(k+1)} - X^{(*)}||_F \leq \widetilde{\kappa} ||X^{(k)} - X^{(*)}||_F^2$$

Additional Contributions [Bänsch/B./Saak/Weichelt '15,'16]

- Suitable approximation framework for Raymond's projected boundary control input.
- Proposed method directly iterates over the feedback matrix $K \in \mathbb{R}^{n \times n_r}$.
- Initial feedback for index-2 DAE systems using a special eigenvalue shifting technique.
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Feedback Stabilization for Index-2 DAE Systems –Remarks/Open Problems–

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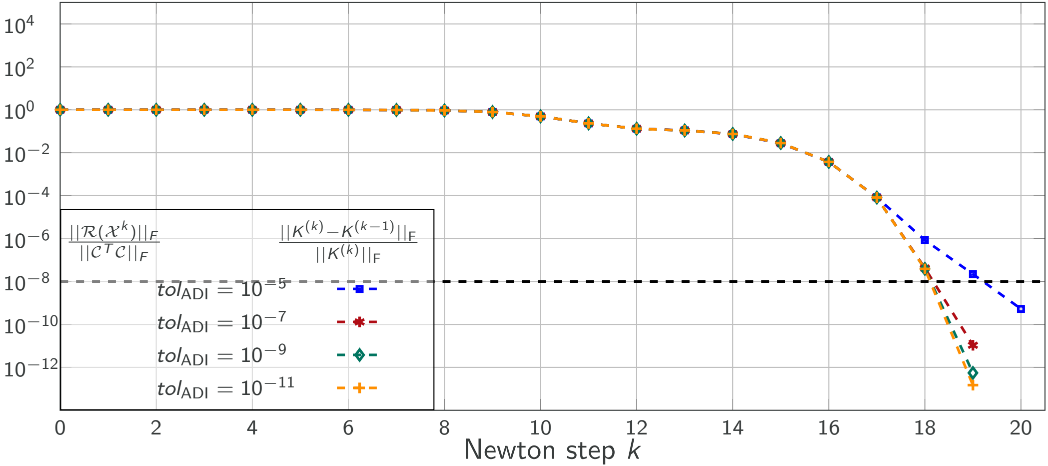
Current Problems

- Determination of suitable stopping criteria/tolerances.
- Computation of projected residuals is very costly ($\approx$ 10x ADI step).
  $\Rightarrow$ use relative change of feedback matrix [B./Li/Penzl '08]

NSE scenario: $\mathrm{Re} = 500$, $n = 5\,468$, $\lambda = 10^2$, $tol_{\text{Newton}} = 10^{-8}$



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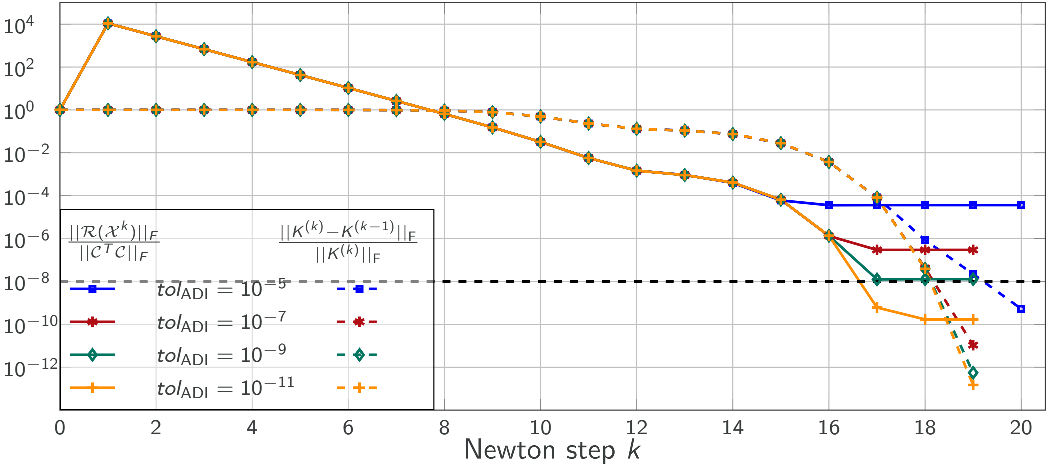

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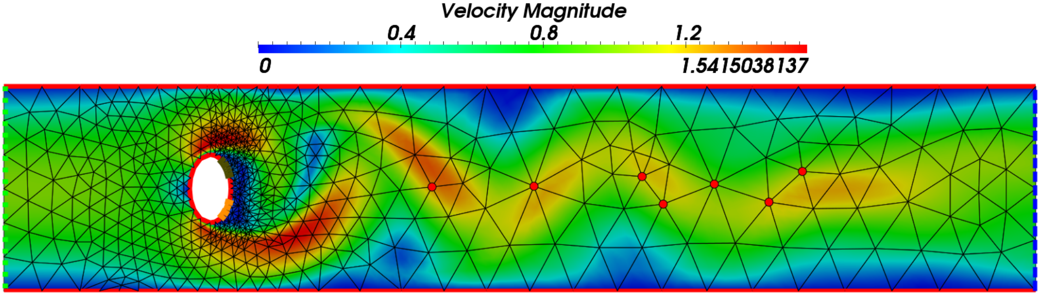

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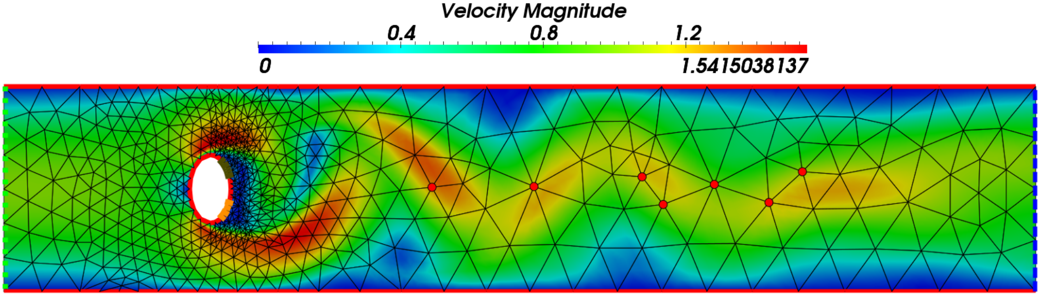

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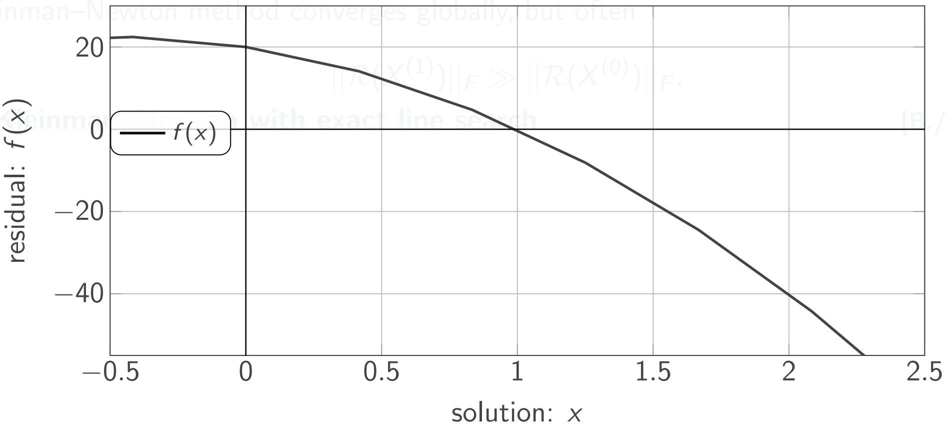

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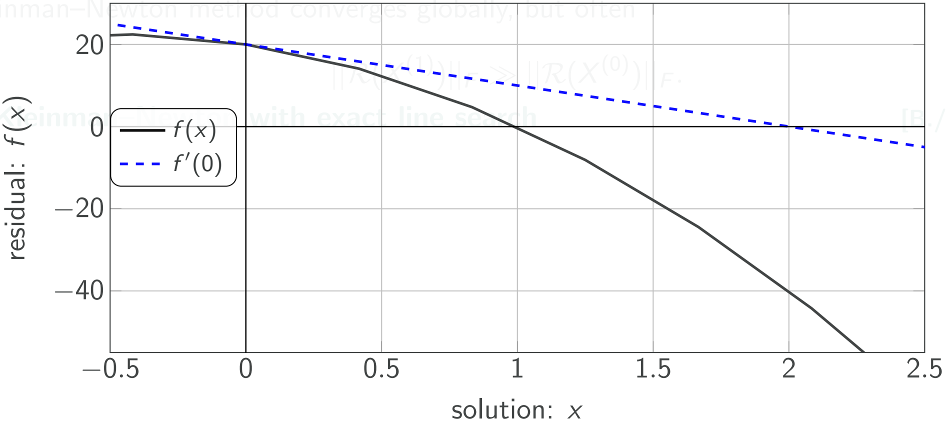

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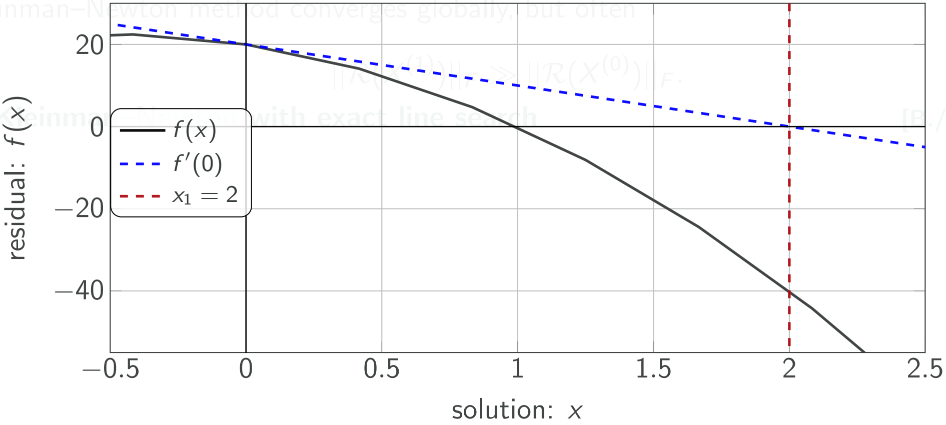

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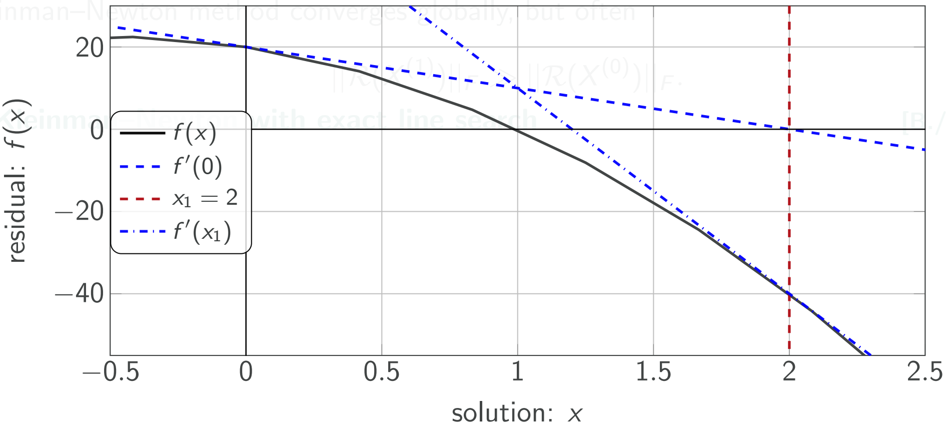

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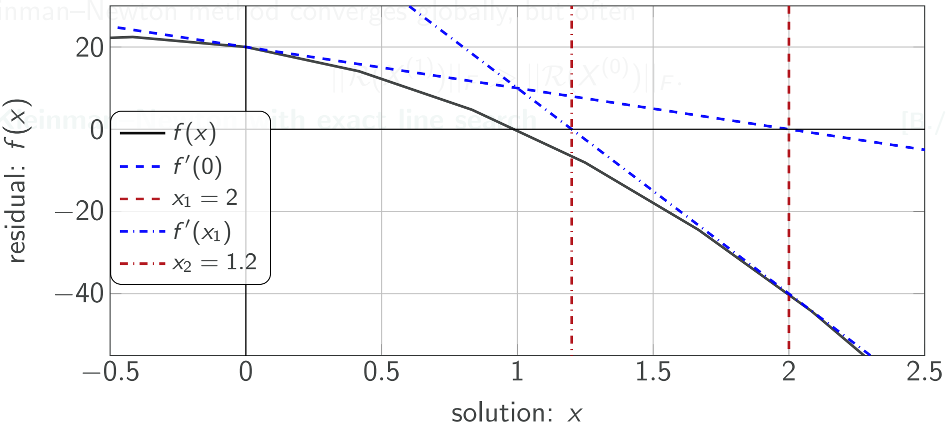

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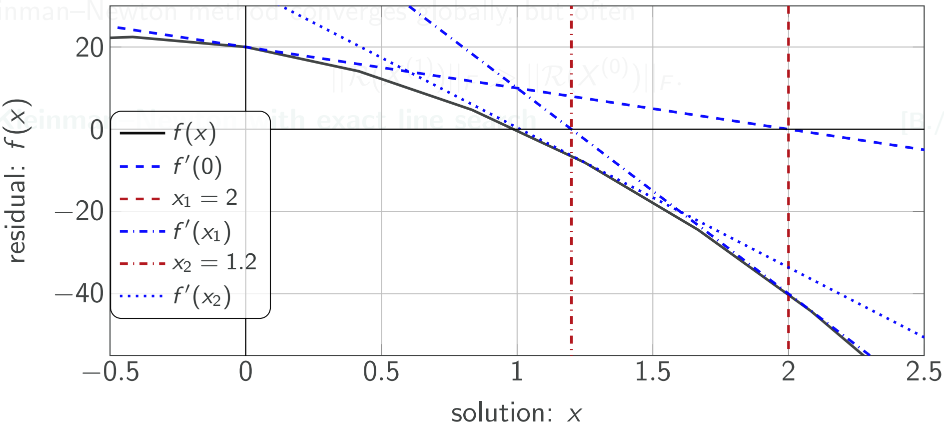

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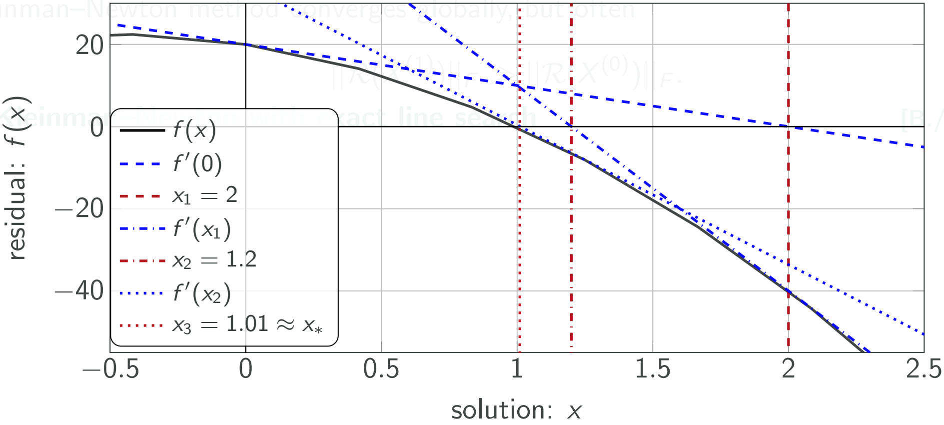

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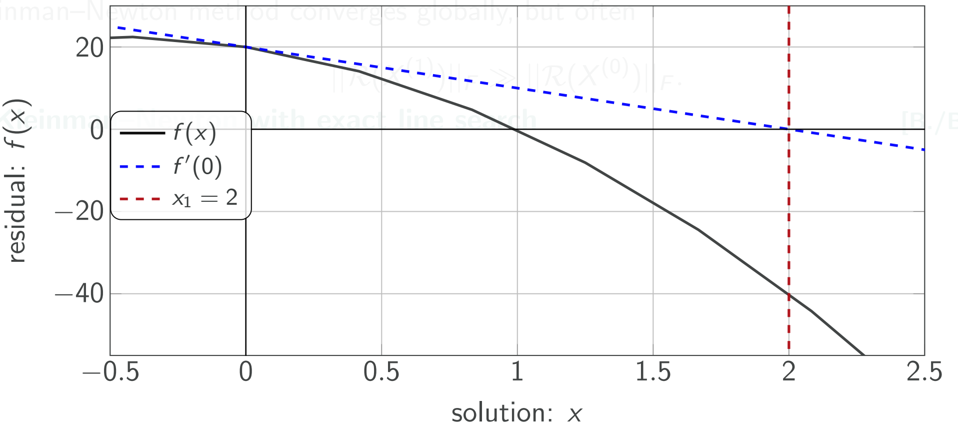

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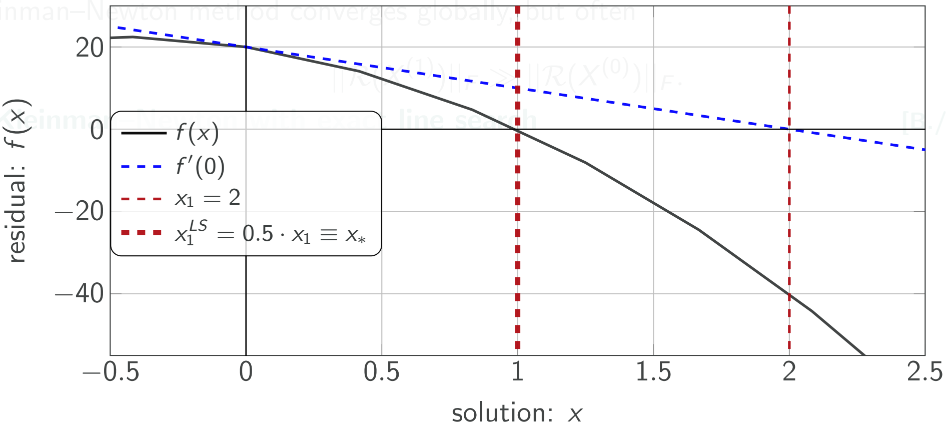

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- **extension to index-2 DAE case "straight forward"**

Theorem **[B./Heinkenschloss/Saak/Weichelt '16]**

Set $\tau_k \in (0,1)$ and assume: $(\mathcal{A}, \mathcal{B}; \mathcal{M})$ stabilizable, $(\mathcal{C}, \mathcal{A}; \mathcal{M})$ detectable, and $\exists\, \widetilde{\mathcal{X}}^{(k+1)} \succeq 0\ \forall k$ that solves

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(k)})^T \widetilde{\mathcal{X}}^{(k+1)} \mathcal{M} + \mathcal{M} \widetilde{\mathcal{X}}^{(k+1)} (\mathcal{A} - \mathcal{B}\mathcal{K}^{(k)}) = -\mathcal{C}^T\mathcal{C} - (\mathcal{K}^{(k)})^T \mathcal{K}^{(k)} + \mathcal{L}^{(k+1)}$$

such that

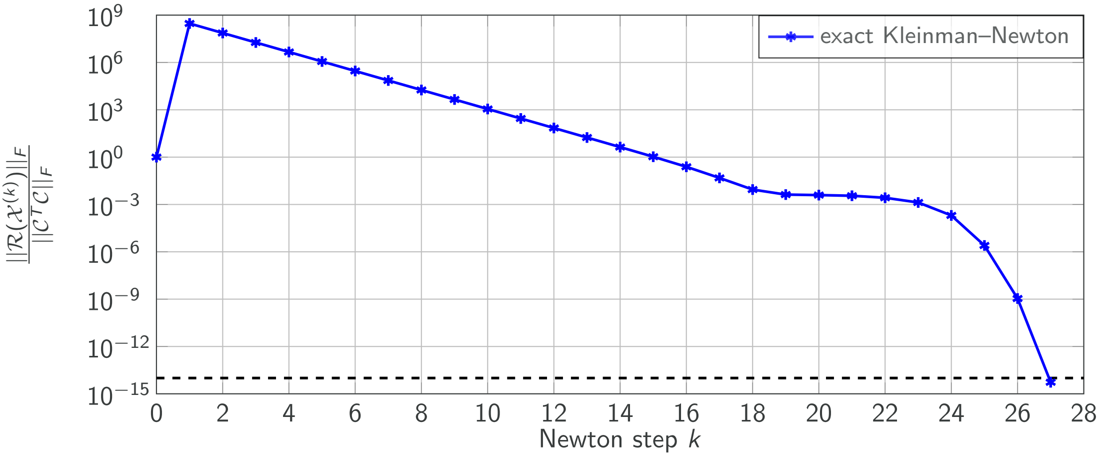
$$||\mathcal{L}^{(k+1)}||_F \le \tau_k ||\mathcal{R}(\mathcal{X}^{(k)})||_F.$$

Find $\xi_k \in (0,1]$ such that $||\mathcal{R}(\mathcal{X}^{(k)} + \xi_k \mathcal{S}^{(k)})||_F \le (1 - \xi_k \alpha)||\mathcal{R}(\mathcal{X}^{(k)})||_F$ and set

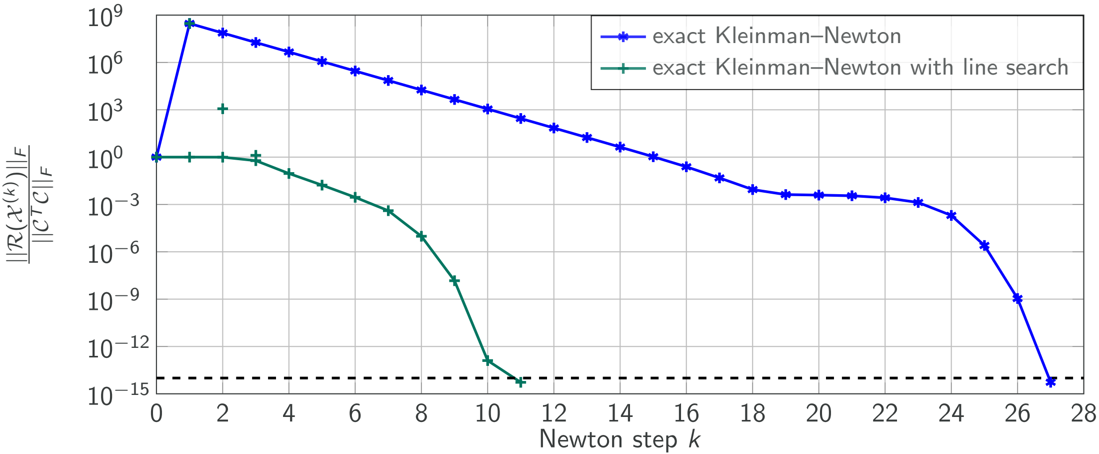
$$\mathcal{X}^{(k+1)} = (1 - \xi_k)\mathcal{X}^{(k)} + \xi_k \widetilde{\mathcal{X}}^{(k+1)}.$$

1. **IF** $\xi_k \ge \xi_{\min} > 0\ \forall k \ \Rightarrow\ \|\mathcal{R}(\mathcal{X}^{(k)})\|_F \to 0$.
2. **IF** $\mathcal{X}^{(k)} \succeq 0$, and $(\mathcal{A} - \mathcal{B}\mathcal{B}^T\mathcal{X}^{(k)}, \mathcal{M})$ stable for $k \ge K > 0 \ \Rightarrow\ \mathcal{X}^{(k)} \to \mathcal{X}^{(*)}$ ($\mathcal{X}^{(*)} \succeq 0$ the unique stabilizing solution).

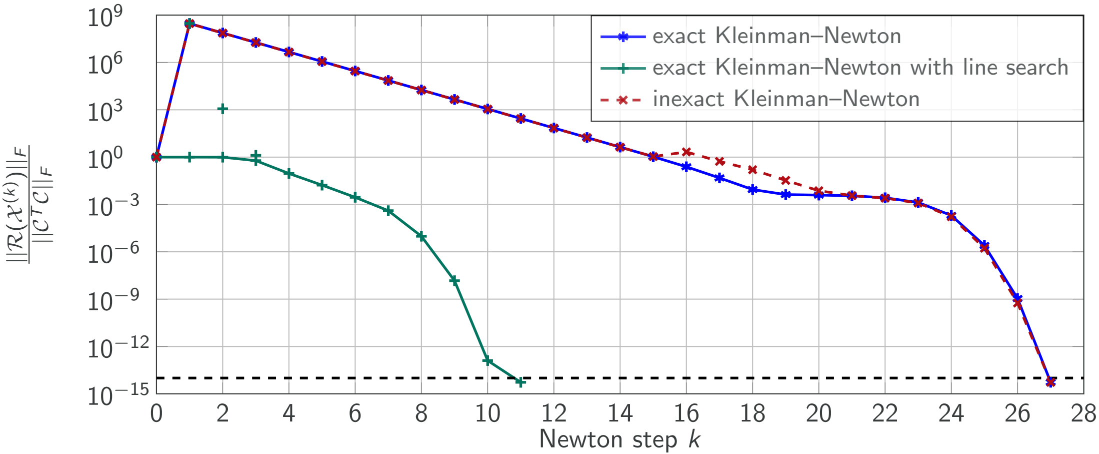
NSE scenario: $\mathrm{Re} = 500$, Level 1, $\lambda = 10^4$, $tol_{\mathrm{Newton}} = 10^{-14}$



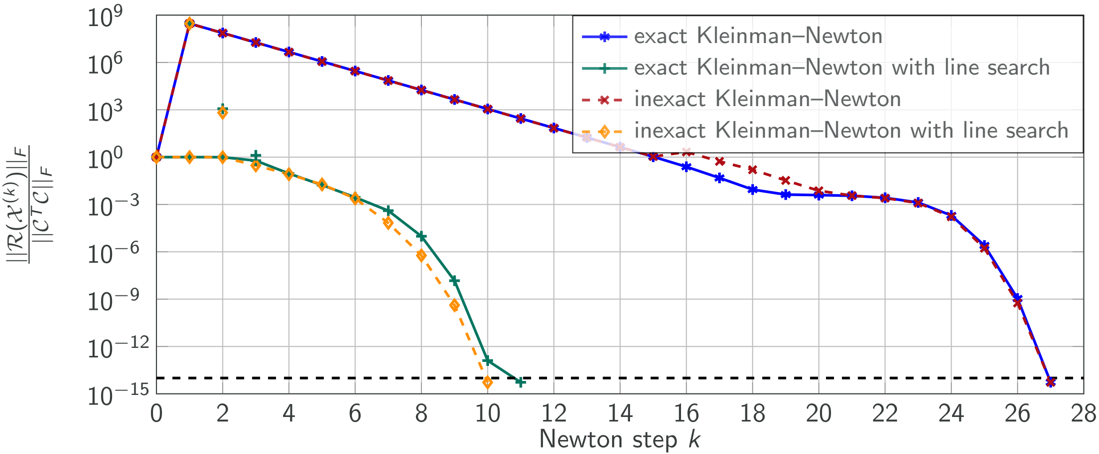
NSE scenario: Re $= 500$, Level 1, $\lambda = 10^4$, $tol_{\text{Newton}} = 10^{-14}$



NSE scenario: Re $= 500$, Level 1, $\lambda = 10^4$, $tol_{\text{Newton}} = 10^{-14}$



NSE scenario: Re $= 500$, Level 1, $\lambda = 10^4$, $tol_{\text{Newton}} = 10^{-14}$



NSE scenario: $\mathrm{Re} = 500$, Level 1, $\lambda = 10^4$, $tol_{\mathrm{Newton}} = 10^{-14}$

	exact KN	exact KN+LS	inexact KN	inexact KN+LS
$\#Newt$	27	11	27	10
$\#ADI$	3185	1351	852	549
$t_{\text{Newt-ADI}}$	1304.769	540.984	331.871	222.295
t_{shift}	29.998	12.568	7.370	5.507
t_{LS}	–		–	
t_{total}	**1334.767**	**553.581**	**339.241**	**227.824**

Table : Numbers of steps and timings in seconds.

NSE scenario: Re $= 500$, Level 1, $\lambda = 10^4$, $tol_{\text{Newton}} = 10^{-14}$

	exact KN	exact KN+LS	inexact KN	inexact KN+LS
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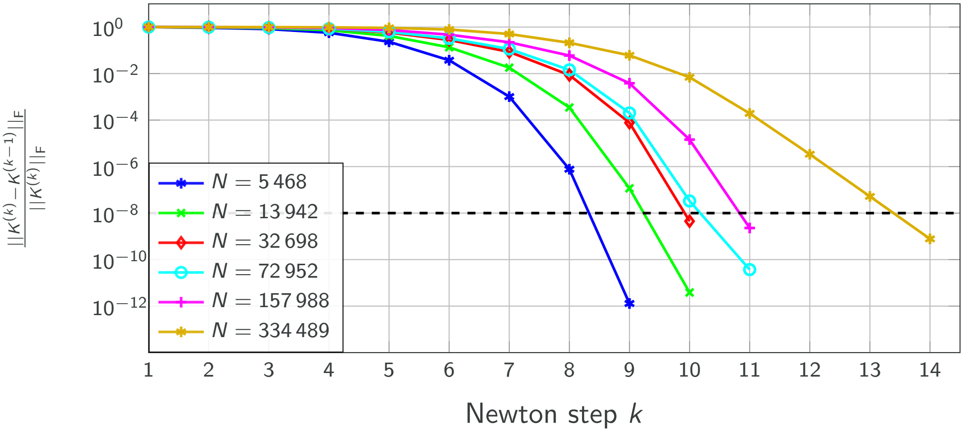
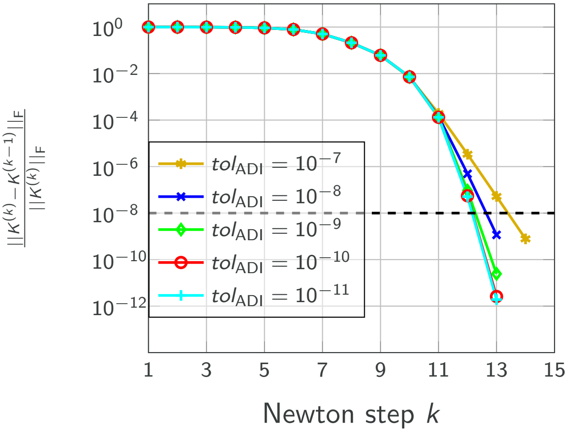
t_{shift}	29.998	12.568	7.370	5.507
t_{LS}	–	**0.029**	–	**0.023**
t_{total}	**1334.767**	**553.581**	**339.241**	**227.824**

Table : Numbers of steps and timings in seconds.

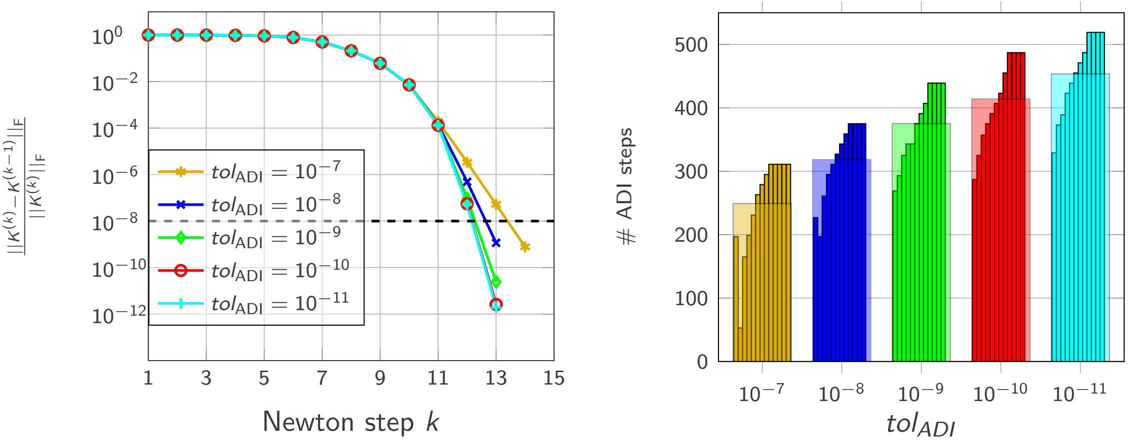
NSE scenario: Re $= 500$, $tol_{\text{ADI}} = 10^{-7}$, $tol_{\text{Newton}} = 10^{-8}$



NSE scenario: NSE scenario: Re $= 500$, $tol_{\text{Newton}} = 10^{-8}$, $N = 334\,489$



NSE scenario: NSE scenario: Re $= 500$, $tol_{\text{Newton}} = 10^{-8}$, $N = 334\,489$



Further solution approaches

- Kleinman–Newton ADI with Galerkin projection [B./Saak '10]
- EKSM [Heyouni/Jbilou '09]
- RKSM [Simoncini/Szyld/Monsalve '14]

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Further test examples

1. 2D diffusion convection reaction problem [B./Heinkenschloss/Saak/Weichelt '15]
2. 3D diffusion convection reaction problem [B./Heinkenschloss/Saak/Weichelt '15]
3. `carex18`: one dimensional heat flow
   SLICOT benchmark collection: Example 4.2.b in [Abels/B. '99]

Main Contributions

- Analyzed **Riccati-based feedback** for **scalar** and **vector-valued transport** problems.
- Wide-spread usability tailored for standard **inf-sup stable finite element** discretizations.
- Established **specially tailored Kleinman–Newton-ADI** that **avoids explicit projections**.
- **Suitable preconditioners** for multi-field flow problems have been developed.
- **Ongoing research** in similar areas has been **incorporated**.
- Major run time improvements due to combination of **inexact Newton** and **line search**.
- Established **new convergence proofs** that were verified by **extensive numerical tests**.

Conclusions

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- Major run time improvements due to combination of **inexact Newton** and **line search**.
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$\Rightarrow$ Showed **overall usability** of new approach by a **closed-loop forward simulation**.

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