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Split-Congruence Transformations Meet Balanced Truncation

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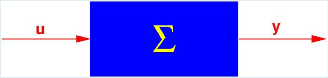
Introduction

Model Reduction of LTI Systems

Original system

$$\Sigma : \begin{cases} E\dot{x}(t) = Ax + Bu, \\ y(t) = Cx + Du. \end{cases}$$

- $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.
- State/descriptor vector $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.



Reduced System

$$\hat{\Sigma} : \begin{cases} \hat{E}\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \\ \hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}u(t). \end{cases}$$

- $\hat{A}, \hat{E} \in \mathbb{R}^{r\times r}$, $\hat{B} \in \mathbb{R}^{r\times m}$, $\hat{C} \in \mathbb{R}^{p\times r}$, $\hat{D} \in \mathbb{R}^{p\times m}$, $r \ll n$.
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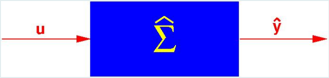

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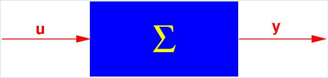

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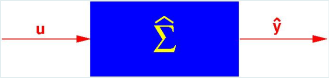

Goal:

$\|y - \hat{y}\| < \text{tol} \cdot \|u\|$ for all admissible input signals.

Model Reduction Based on Balanced Truncation

Linear time-invariant (LTI) systems

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$(A, B, C, D) =$ realization of Σ (non-unique).

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Balancing of LTI systems

Given: Gramians $P, Q \in \mathbb{R}^{n\times n}$ symmetric, positive definite (spd), and contragredient transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$TPT^T = T^{-T}QT^{-1} = \mathrm{diag}(\sigma_1, \ldots, \sigma_n), \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0.$$

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Balancing of Σ w.r.t. P, Q:

$$\Sigma \equiv (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D) \equiv \Sigma.$$

Model Reduction Based on Balanced Truncation

Basis for model reduction method

1. Given $\Sigma \equiv (A, B, C, D)$ and balancing (w.r.t. given P, Q spd) transformation $T \in \mathbb{R}^{n \times n}$, compute (not explicitly!)

$$
\begin{aligned}
(A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\
&= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right)
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2. Truncation ⇝ reduced model:

$$(\hat{A}, \hat{B}, \hat{C}, \hat{D}) = (A_{11}, B_1, C_1, D).$$

Model Reduction Based on Balanced Truncation

Classical Balanced Truncation (BT)

Mullis/Roberts '76, Moore '81

- $P/Q =$ controllability/observability Gramians of $\Sigma \equiv (A, B, C, D)$.
- For asymptotically stable systems, P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0.$$

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- $\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1^{\mathrm{BT}}, \ldots, \sigma_n^{\mathrm{BT}}\}$ are Hankel singular values (HSVs) of Σ.
  HSVs are system invariants ($\rightsquigarrow$ "energy preservation" motivation).

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- Asymptotic stability is preserved!
- Computable error bound is by-product of computations:

$$\|y - y^{\mathrm{BT}}\|_2 \leq 2 \sum_{j=r+1}^{n} \sigma_j^{\mathrm{BT}} \|u\|_2,$$

allows adaptive choice of r!

Model Reduction Based on Balanced Truncation

- Choice of other Gramians yields preservation of structural properties (z.B. minimum phase, passivity, bounded realness, ...).
- Variants for unstable systems exist.
- Application to systems with mass matrix ($E\dot{x} = Ax + Bu$) possible without forming $E^{-1}A, E^{-1}B$!
  Variants for E singular exist.
- Applications to second order systems (mechanical systems) $\rightsquigarrow$ later.
- Classical implementations require $\mathcal{O}(n^3)$ operations and $\mathcal{O}(n^2)$ memory $\rightsquigarrow$ way too expensive for systems with $n \gg 1000$!

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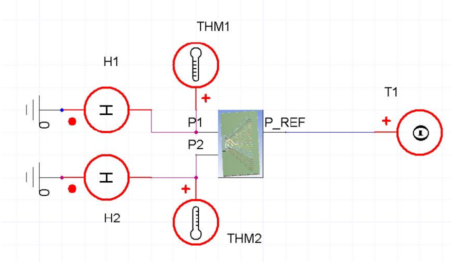
- Classical implementations require $\mathcal{O}(n^3)$ operations and $\mathcal{O}(n^2)$ memory $\rightsquigarrow$ way too expensive for systems with $n \gg 1000$!
- But: new numerical techniques developed since 1996 $\Rightarrow$
  $n = 1.000.000$ nowadays computable in MATLAB®!
  (Computing times $< 1h$ on quadcore.)

Model Reduction Based on Balanced Truncation

Numerical Example: Electro-Thermic Simulation of Integrated Circuit (IC)

[Source: Evgenii Rudnyi, CADFEM GmbH]

- SIMPLORER® test circuit with 2 transistors.



- Conservative thermic sub-system in SIMPLORER:
  voltage ⇝ temperature, current ⇝ heat flow.
- Original model: $n = 270.593,\ m = p = 2 \Rightarrow$
  Computing time (on Intel Xeon dualcore 3GHz, 1 Thread):
  - Solution of Lyapunov equations: $\approx 22min$.
  - Computation of reduced models: 44sec. ($r = 20$) – 49sec. ($r = 70$).
  - Bode plot (MATLAB on Intel Core i7, 2,67GHz, 12GB):
    7.5h for original system , $<$ 1min for reduced system.

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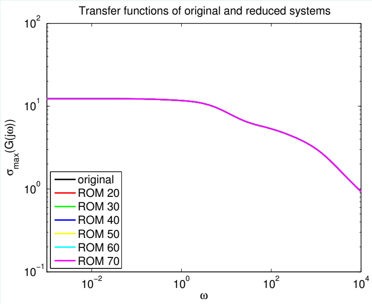
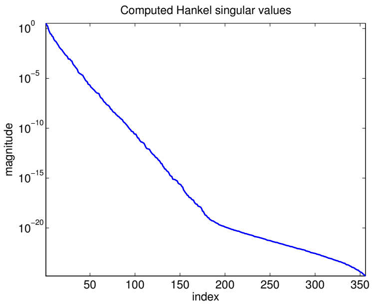
Bode Plot (Amplitude)



Hankel Singular Values



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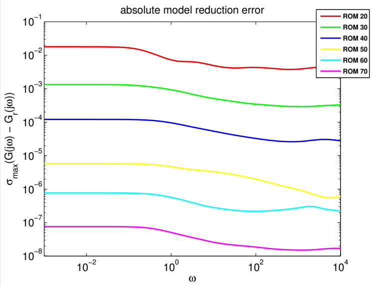
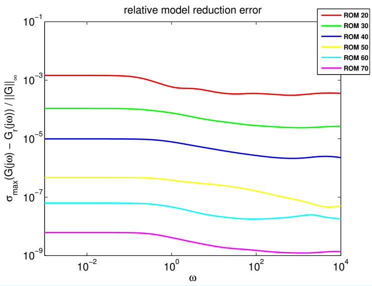
Absolute Error



Relative Error



Split-Congruence Model Reduction

Split-Congruence Transformations

Split-congruence model reduction [Kerns/Yang '98]

Given descriptor system $(E; A, B, C)$ and orthogonal projection VV^T, $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \in \mathbb{R}^{n\times r}$, compute reduced-order model

$$(\hat{E}, \hat{A}, \hat{B}, \hat{C}) = (\mathcal{V}^T E\mathcal{V}, \mathcal{V}^T A\mathcal{V}, \mathcal{V}^T B, C\mathcal{V}), \text{ where } \mathcal{V} = \begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix}. \quad (1)$$

$\Longrightarrow$ 2×2 block structure of realization is preserved, e.g.,

$$\hat{A} = \mathcal{V}^T A\mathcal{V} = \begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix},$$

so that physically motivated partitioning of state vector can be preserved.

Applications:

$\rightarrow$ RLC network equations,

$\rightarrow$ mechanical systems.

K. Kerns, A. Yang.
Preservation of passivity during RLC network reduction via split congruence transformations.
IEEE Trans. CAD Integr. Circuits Syst. 17(7):582-–591, 1998.

Split-Congruence Model Reduction

Split-Congruence Transformations

RLC network equations

System structure of RLC networks w/o voltage sources (MNA form):

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \quad A = \begin{bmatrix} -A_1 & -A_2^T \\ A_2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = C^T,$$

where $A_1, E_1 \geq 0$, $E_2 > 0$.

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where $\hat{A}_1, \hat{E}_1 \geq 0$, and, if $\operatorname{rank}(V_1) = r$, $\hat{E}_2 > 0$. $\Longrightarrow$ Preservation of

1. stability,
2. structure of transfer function:

$$\hat{G}(s) = \hat{B}_1^T \left(s\hat{E}_1 + \hat{A}_1 + \frac{1}{s}\hat{A}_2^T \hat{E}_2^{-1} \hat{A}_2\right)\hat{B}_1,$$

and, hence, of passivity and reciprocity ($\Rightarrow$ reduced-order model can be synthesized as circuit, e.g., [Reis '10]).

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Note: used, e.g., in

- PRIMA [Odabasioglu/Celik/Pileggi '97],
- SPRIM [Freund '04–'08].

Split-Congruence Model Reduction

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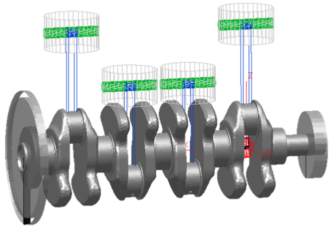
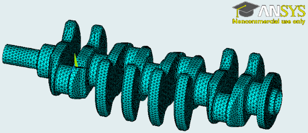
Mechanical Systems

Structural dynamics/vibration analysis, e.g.,

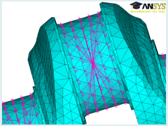




⇒ systems of second-order differential equations:

$$M\ddot{q} + D\dot{z} + Kq = Bu, \quad y = C_p q + C_v \dot{q}.$$

Inputs are e.g., forces acting on crankshaft (piston/con-rod):



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Mechanical Systems

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Linearization ($x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$):

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix} x + \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad y = [\, C_p,\ C_v \,] x.$$

Compute projection subspace $\mathrm{range}\left(\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\right)$ using standard (one-sided) model reduction method applied to linearization and split-congruence model reduction $\Longrightarrow$ second-order structure is preserved:

$$\hat{M} = V_2^T M V_2,\ \hat{D} = V_2^T D V_2,\ \hat{K} = V_2^T K V_1,\ \hat{B} = V_2^T B,\ \hat{C}_p = C_p V_1,\ \hat{C}_v = C_v V_2.$$

Note: (implicitly) used, e.g., in

- SOAR [Bai/Su '05, Salimbahrami/Lohmann '06],
- Krylov subspace methods for higher-order dynamical systems [Freund '08], ...

Split-Congruence Balanced Truncation (scBT)

(Very) basic idea: let $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \in \mathbb{R}^{n \times r}$ be projection matrix computed by BT, then use split-congruence model reduction.

Notes:

- $\mathrm{range}\,(V) \subset \mathrm{range}\,(\mathcal{V})$.
- For standard systems with $E = I_n$, $\hat{E} = \begin{bmatrix} V_1^T V_1 & 0 \\ 0 & V_2^T V_2 \end{bmatrix}$.
- But: in general, $W \neq V$, so split congruence cannot be expected to yield reasonable approximation.

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$\Longrightarrow$ Consider $G(s)$ symmetric ($A = A^T < 0,\ C = B^T$):

$\Rightarrow$ *Gramians coincide, $P = Q$.*

$\Rightarrow$ *BT needs only one Lyapunov equation, $W \equiv V$*

$\Rightarrow$ *(sc)BT automatically preserves stability and passivity.*

Possible advantage of scBT: preservation of block structure, no mixing of physically unrelated variables.

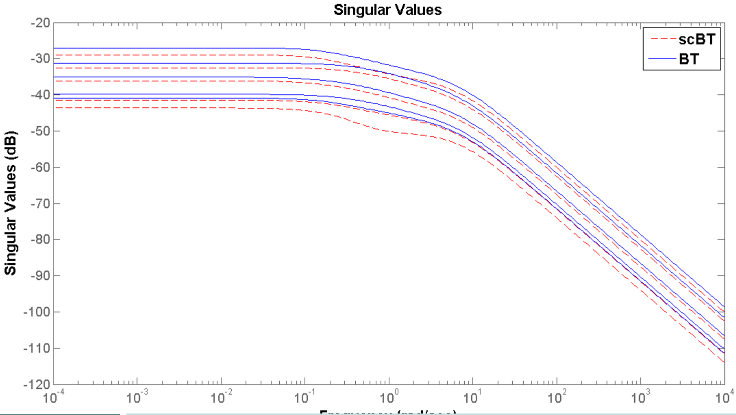
Clear disadvantage: doubling of reduced order.

Split-Congruence Balanced Truncation

Numerical Example

- Heat equation on unit square, $n = 400, m = 5$,
- reduced-order $r = 20$ / 40, relative $\mathcal{H}_\infty$ errors $5 \cdot 10^{-6}$ / $4 \cdot 10^{-6}$ ($\delta_{rel} = 1.25 \cdot 10^{-5}$).

Singular values of error systems



Split-Congruence Balanced Truncation for Hamiltonian Transfer Functions

Hamiltonian Transfer Function [Fuhrmann '83]

Let $H = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, then the transfer function $G(s) = C(sE - A)^{-1}B$ is called Hamiltonian if

$$A^T = HAH, \quad E^T = HEH, \quad C^T = HB.$$

Example:

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_2 \\ -A_2^T & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = C^T,$$

where $E_1 = E_1^T$, $E_2 = E_2^T$ and $A_1 = A_1^T$.

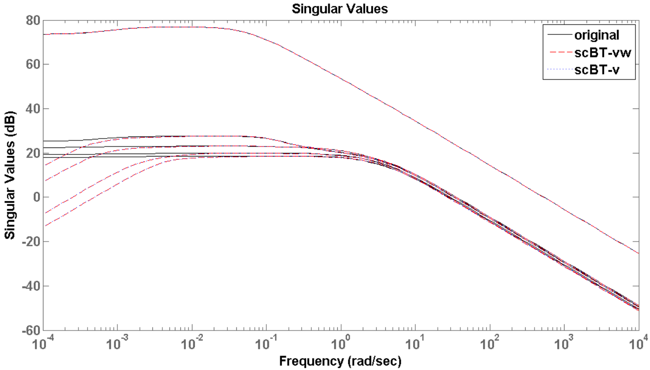
Note: for Hamiltonian systems, the controllability and observability Gramians satisfy $Q = HPH = \begin{bmatrix} P_{11} & -P_{12} \\ -P_{12}^T & P_{22} \end{bmatrix}$ $\Rightarrow$ only one Gramian computation required.

Generalization: J-Hermitian transfer functions: H nonsingular, $HB = C^T F$ for nonsingular F.

Split-Congruence Balanced Truncation for Hamiltonian Transfer Function

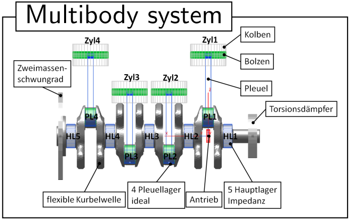
Numerical Example

- Hamiltonian transfer function with $E = I_n$, $A_1 = -\Delta_h$, $A_2 = \nabla_h$ on unit square, $n_1 = 400, n_2 = 100$, i.e., $n = 500$, $m = 5$.
- reduced-order $r = 20$ / 40.
- Variants: scBT-v with $\mathcal{V} = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}$, scBT-w with $\mathcal{W} = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}$, scBT-vw with two-sided projection with $\mathcal{V}, \mathcal{W}$.



Symmetric Second-Order Systems

FVV Project 1029, partners P. Eberhard (U. Stuttgart), G. Knoll (U Kassel)



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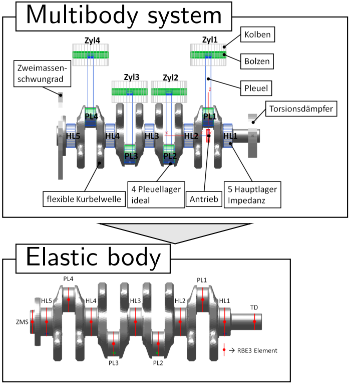

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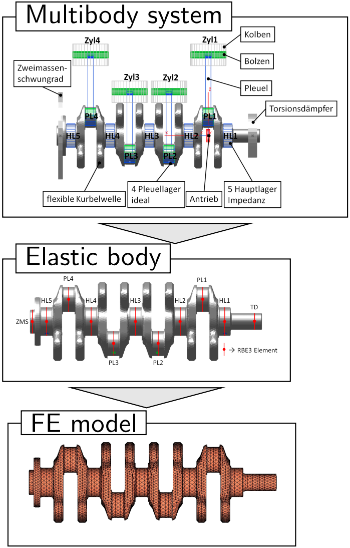

Symmetric Second-Order Systems

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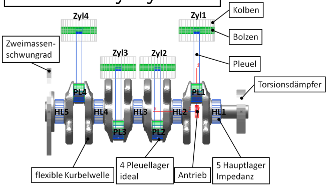
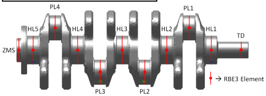
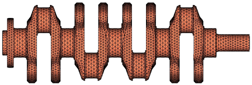
Multibody system



Elastic body



FE model



Symmetric second-order system

$$M\ddot{x} + D\dot{x} + Kx = Bu$$

$$y = B^T x$$

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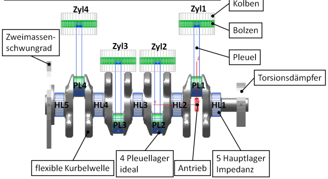

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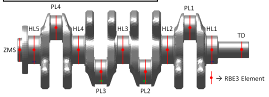

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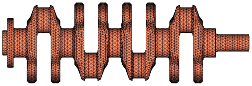

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$$M\ddot{x} + D\dot{x} + Kx = Bu$$
$$y = B^T x$$

Reduced second-order model

$$\tilde{M}\ddot{\tilde{x}} + \tilde{D}\dot{\tilde{x}} + \tilde{K}\tilde{x} = \tilde{B}u$$
$$\tilde{y} = \tilde{B}^T \tilde{x}$$

Symmetric Second-Order Systems

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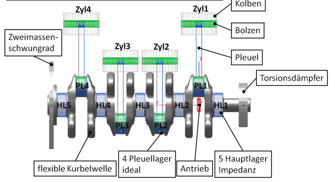

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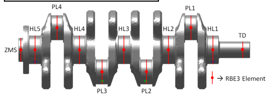

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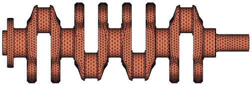

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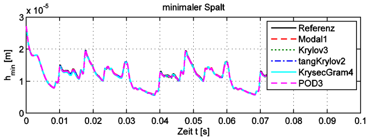
Simulation



Symmetric Second-Order Systems

Symmetric Linerizations

Linearizations

Standard linearization

$$\underbrace{\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}}_{:=\mathcal{E}} \dot{x} = \underbrace{\begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}}_{:=\mathcal{A}} x + \underbrace{\begin{bmatrix} 0 \\ B \end{bmatrix}}_{:=\mathcal{B}} u, \quad y = \underbrace{[B^T, 0]}_{:=\mathcal{C}} x.$$

does not exhibit desired symmetry $(\mathcal{A} \neq \mathcal{A}^T, \mathcal{C} \neq \mathcal{B}^T)$.

Recall: for symmetric transfer function, solutions of corresponding generalized Lyapunov equations

$$\mathcal{A}\mathcal{P}E^T + \mathcal{A}^T\mathcal{P}\mathcal{E} = -\mathcal{B}\mathcal{B}^T, \quad \mathcal{A}^T\mathcal{Q}E + \mathcal{A}\mathcal{Q}\mathcal{E}^T = -\mathcal{C}^T\mathcal{C}.$$

coincide $\Rightarrow$ only 1 Lyapunov solve, i.e., only 1 ADI iteration.

Note: $\mathcal{E}^{-1}\mathcal{A} + \mathcal{A}^{-1}\mathcal{E} \not< 0$ in any case!

Symmetric Second-Order Systems

Symmetric Linerizations

Linearizations

Next idea: "symmetric" linearization

$$\underbrace{\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}}_{:=\mathcal{E}} \dot{x} = \underbrace{\begin{bmatrix} 0 & -K \\ -K & -D \end{bmatrix}}_{:=\mathcal{A}} x + \begin{bmatrix} 0 \\ B \end{bmatrix} u, \quad y = [\, B^T, 0 \,] x,$$

now has $\mathcal{A} = \mathcal{A}^T$, but $C \neq B^T$!

Recall: for symmetric transfer function, solutions of corresponding generalized Lyapunov equations

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Symmetric Second-Order Systems

Symmetric Linerizations

Linearizations

Non-standard symmetric linearization

$$\underbrace{\begin{bmatrix} D & M \\ M & 0 \end{bmatrix}}_{:=\mathcal{E}} \dot{x} = \underbrace{\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}}_{:=\mathcal{A}} x + \begin{bmatrix} B \\ 0 \end{bmatrix} u, \quad y = [B^T, 0]x,$$

is what we need: $\mathcal{A} = \mathcal{A}^T$, $C = B^T$!

Recall: for symmetric transfer function, solutions of corresponding generalized Lyapunov equations

$$\mathcal{A}\mathcal{P}E^T + \mathcal{A}^T\mathcal{P}\mathcal{E} = -\mathcal{B}\mathcal{B}^T, \quad \mathcal{A}^T\mathcal{Q}E + \mathcal{A}\mathcal{Q}\mathcal{E}^T = -\mathcal{C}^T\mathcal{C}.$$

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Symmetric Second-Order Systems

Computation of Gramians

Problem: Exact computation of $\mathcal{P}$ (and $\mathcal{Q}$) too expensive ($\mathcal{O}(n^3)$ flops, $\mathcal{O}(n^2)$ memory)

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$$\tilde{\mathcal{S}}\,\tilde{\mathcal{S}}^T \approx \mathcal{P} \quad \text{und} \quad \tilde{\mathcal{R}}\,\tilde{\mathcal{R}}^T \approx \mathcal{Q}$$

with ADI iteration.

Further gimmicks:

- Re-write ADI iteration in terms of M, D, K, e.g., only solves of the form $(\mu_j^2 M + \mu_j D + K)v = w$ [B./Saak '09];
- Purely real arithmetic with only 1 linear system per pair of complex conjugate shifts $\rightsquigarrow$ acceleration by factor 2–4 [B./Saak/Kürschner '11].

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SVD computation for balanced truncation:

$$\tilde{\mathcal{S}}^T\,\mathcal{E}\,\tilde{\mathcal{R}} = \blacksquare = X\,\Sigma\,Y^T = [X_1,\ X_2]\begin{bmatrix}\Sigma_1 & \\ & \Sigma_2\end{bmatrix}[Y_1,\ Y_2]^T$$

computation of truncation matrices

$$W := \tilde{\mathcal{R}}^T Y_1 \Sigma_1^{-\frac{1}{2}} \equiv V := \tilde{\mathcal{S}}^T X_1 \Sigma_1^{-\frac{1}{2}}.$$

Symmetric Second-Order Systems

Second-Order Balanced Truncation

Second-Order Balanced Truncation [Meyer/Srinivasan '96]

Partitioning of Gramians $\mathcal{P}$, $\mathcal{Q}$ (or corresponding factors $\mathcal{S}$, $\mathcal{R}$) of linearization:

$$\mathcal{P} = \underbrace{\begin{bmatrix} \mathcal{S}_p \\ \mathcal{S}_v \end{bmatrix}}_{=\mathcal{S}} \underbrace{\begin{bmatrix} \mathcal{S}_p^T & \mathcal{S}_v^T \end{bmatrix}}_{=\mathcal{S}^T} = \begin{bmatrix} \mathcal{P}_p & \mathcal{P}_o \\ \mathcal{P}_o & \mathcal{P}_v \end{bmatrix}, \quad \mathcal{Q} = \underbrace{\begin{bmatrix} \mathcal{R}_p \\ \mathcal{R}_v \end{bmatrix}}_{=\mathcal{R}} \underbrace{\begin{bmatrix} \mathcal{R}_p^T & \mathcal{R}_v^T \end{bmatrix}}_{=\mathcal{R}^T} = \begin{bmatrix} \mathcal{Q}_p & \mathcal{Q}_o \\ \mathcal{Q}_o & \mathcal{Q}_v \end{bmatrix}.$$

Second-order Gramians:

$$\mathcal{P}_p = \mathcal{S}_p \mathcal{S}_p^T \quad - \quad \text{position controllability Gramian},$$

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Second-order Gramians:

$$\mathcal{P}_p = \mathcal{S}_p \mathcal{S}_p^T \quad - \quad \text{position controllability Gramian},$$
$$\mathcal{P}_v = \mathcal{S}_v \mathcal{S}_v^T \quad - \quad \text{velocity controllability Gramian},$$

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Second-order Gramians:

$$\begin{aligned} \mathcal{P}_p &= \mathcal{S}_p\mathcal{S}_p^T && - \text{ position controllability Gramian,} \\ \mathcal{P}_v &= \mathcal{S}_v\mathcal{S}_v^T && - \text{ velocity controllability Gramian,} \\ \mathcal{Q}_p &= \mathcal{R}_p\mathcal{R}_p^T && - \text{ position observability Gramian,} \end{aligned}$$

Symmetric Second-Order Systems

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Symmetric Second-Order Systems

Second-Order Balanced Truncation

Second-Order Balanced Truncation [Meyer/Srinivasan '96]

Pairwise contragredient diagonalization of two of the second-order Gramians yields **4** possible balancing schemes:

Typ	Balancing	right proj.	left proj.
position-position	$\mathcal{P}_p = \mathcal{Q}_p = \Sigma_{pp}$	$V = \mathcal{S}_p X_p \Sigma_{pp}^{-\frac{1}{2}}$	$W = \mathcal{R}_p Y_p \Sigma_{pp}^{-\frac{1}{2}}$
position-velocity	$\mathcal{P}_p = \mathcal{Q}_v = \Sigma_{pv}$	$V = \mathcal{S}_p X_p \Sigma_{pv}^{-\frac{1}{2}}$	$W = \mathcal{R}_v Y_v \Sigma_{pv}^{-\frac{1}{2}}$
velocity-position	$\mathcal{P}_v = \mathcal{Q}_p = \Sigma_{vp}$	$V = \mathcal{S}_v X_v \Sigma_{vp}^{-\frac{1}{2}}$	$W = \mathcal{R}_p Y_p \Sigma_{vp}^{-\frac{1}{2}}$
velocity-velocity	$\mathcal{P}_v = \mathcal{Q}_v = \Sigma_{vv}$	$V = \mathcal{S}_v X_v \Sigma_{vv}^{-\frac{1}{2}}$	$W = \mathcal{R}_v Y_v \Sigma_{vv}^{-\frac{1}{2}}$

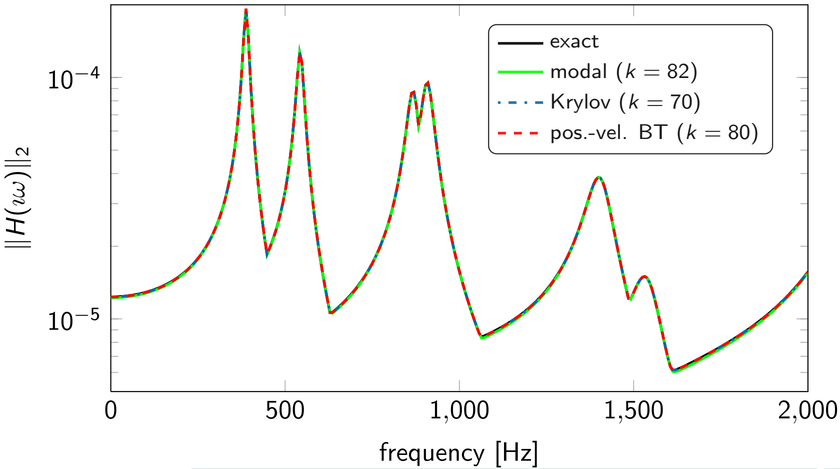
where, e.g.,

$$X_p \Sigma_{pp} Y_p^T = \mathcal{S}_p^T M \mathcal{R}_p.$$

Symmetric Second-Order Systems

Numerical Examples: Crank Shaft

Order $n = 46.860$, 35 inputs/outputs



Symmetric Second-Order Systems
Numerical Examples: Crank Shaft

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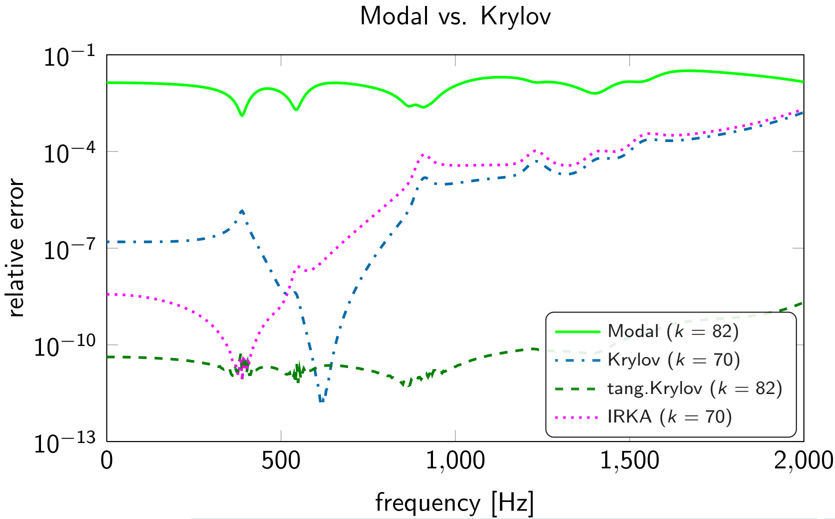

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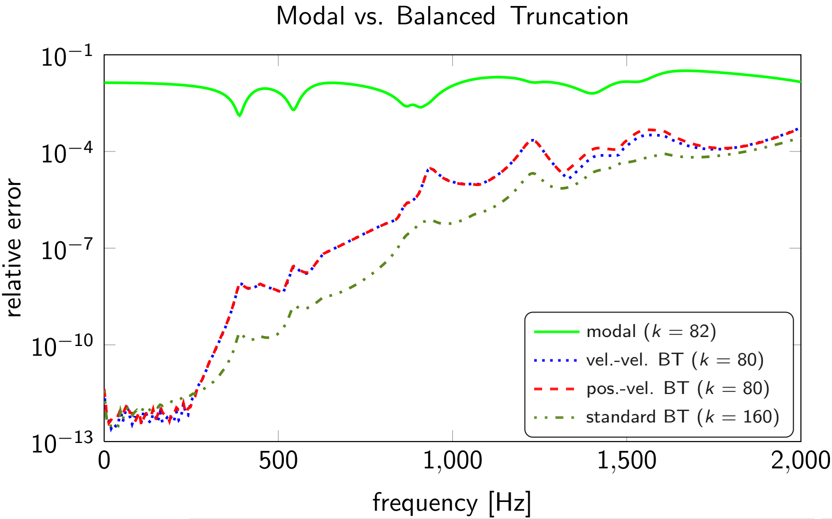
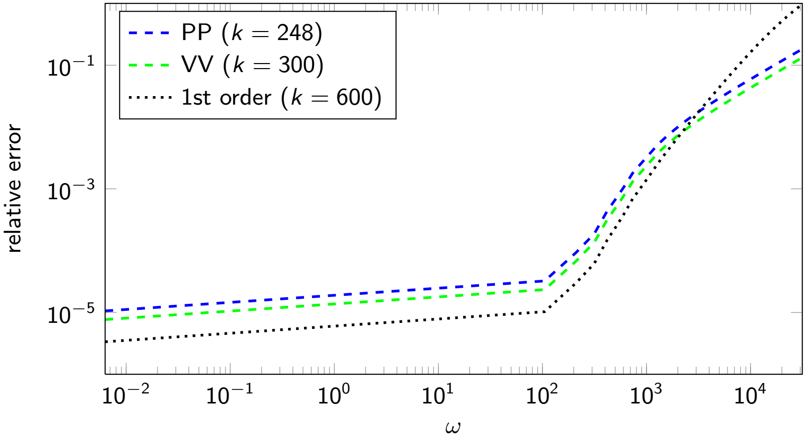


Symmetric Second-Order Systems

Numerical Examples: Control of Continuous Faceplate Deformable Mirrors

Symmetric second-order system, $n = 83,508$, $m = p = 672$, $\mathrm{tol}_{\mathrm{BT}} = 10^{-10}$.



[Source: T. Ruppel, ISYS, U Stuttgart]

Conclusions

- Split-congruence model reduction is an easy tool to preserve block-structures in linear systems, avoids mixing of physically unrelated variables in reduced-order models.
- Split-congruence balanced truncation seems to work well for symmetric transfer functions.
- Symmetric transfer functions often arise from second-order systems in elastic multibody simulation.
- Symmetric ADI iteration for these systems is very efficient (compared to ADI applied to standard linearization).
- **Future work:**
  - error bound for scBT applied to symmetric transfer functions;
  - analyze symmetric ADI iteration w.r.t. stability/robustness.