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Parametric Model Order Reduction using Interpolation on Sparse Grids

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Outline

Parametric Model Order Reduction
using Interpolation on Sparse Grids

- Balanced truncation/interpolatory MOR
- Examples with 1 parameter
  - Lagrange interpolation
  - Hermite interpolation
  - Rational interpolation
- Multidimensional interpolation
  - Use of sparse grids
  - Numerical results
- Conclusions/Outlook

Parametric model order reduction

Parametric system

$$\begin{aligned} \tfrac{d}{dt}x(t,p) &= A(p)\,x(t,p) + B(p)\,u(t) \\ y(t,p) &= C(p)^T x(t,p) \end{aligned}$$

with

- parameter vector $p \in \Omega \subset \mathbb{R}^d$, Ω compact
- $x(t,p) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$, output $y(t,p) \in \mathbb{R}^q$, $m, q \ll n$
- stability: $\lambda(A(p)) \subset \mathbb{C}^-$ for all p

Reduced-order parametric system

$$\begin{aligned} \tfrac{d}{dt}\hat{x}(t,p) &= \hat{A}(p)\,\hat{x}(t,p) + \hat{B}(p)\,u(t) \\ \hat{y}(t,p) &= \hat{C}(p)^T \hat{x}(t,p) \end{aligned}$$

- $x(t,p) \in \mathbb{R}^r$ of reduced dimension $r \ll n$
- $\|y - \hat{y}\|$ bounded

Balanced truncation/interpolatory MOR for $d = 1$

1. Choose $k + 1$ interpolation points $p_0, \ldots, p_k \in [a,\ b]$.

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2. Apply balanced truncation to $G_j(s) := G(s, p_j)$:

$$\hat{G}_j(s) = \hat{C}_j^T (sI_{r_j} - \hat{A}_j)^{-1} \hat{B}_j, \qquad \textit{for } j = 0, \ldots, k.$$

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3. Parametric reduced-order system by interpolation:

$$\hat{G}_I(s,p) = \sum_{j=0}^{k} \varphi_j(p) \hat{G}_j(s) = \sum_{j=0}^{k} \varphi_j(p)\ \hat{C}_j^T (sI_{r_j} - \hat{A}_j)^{-1} \hat{B}_j$$

with interpolation conditions:

$$\hat{G}_I(s,p_j) = \hat{G}_j(s) \approx G_j(s) = G(s,p_j), \quad \textit{for } j = 0, \ldots, k.$$

$$\hat{G}_I(s,p) = \sum_{j=0}^{k} \varphi_j(p)\hat{G}_j(s) = \sum_{j=0}^{k} \varphi_j(p)\ \hat{C}_j^T(sI_{r_j} - \hat{A}_j)^{-1}\hat{B}_j$$

$$= \begin{bmatrix} \hat{C}_0(p) \\ \vdots \\ \hat{C}_k(p) \end{bmatrix}^T \begin{bmatrix} (sI_{r_0} - \hat{A}_0)^{-1} & & \\ & \ddots & \\ & & (sI_{r_k} - \hat{A}_k)^{-1} \end{bmatrix} \begin{bmatrix} \hat{B}_0 \\ \vdots \\ \hat{B}_k \end{bmatrix}$$

+ reduced complexity in numerical simulation:

  costs for evaluation of transfer function reduced from $\mathcal{O}(n^3)$ for $G(s,p)$ to $\mathcal{O}(k \max(r_j)^3)$ for $\hat{G}_I(s,p)$

+ reduced storage requirements from $\mathcal{O}(n^2)$ for original system to $\mathcal{O}((k+1)\max(r_j)^2)$ for reduced-order system global error bound by combination of BT error bound, i.e.

$$\|G_j(\cdot) - \hat{G}_j(\cdot)\|_{\mathcal{H}_\infty} \leq 2\left(\sum_{i=r_j+1}^{n} \sigma_i\right) < \mathrm{tol} \qquad (1)$$

and error estimates for interpolation:

$$\sup_{\substack{s\in\mathbb{C}^+\\ p\in[a,b]}} \|G(s,p) - \hat{G}_I(s,p)\| = \sup_{\substack{s\in\mathbb{C}^+\\ p\in[a,b]}} \|G(s,p) - \sum_{j=0}^{k} \varphi_j(p)\hat{G}_j(s)\|$$

$$\leq \sup_{\substack{s\in\mathbb{C}^+\\ p\in[a,b]}} \|G(s,p) - G_I(s,p)\| + \sup_{\substack{s\in\mathbb{C}^+\\ p\in[a,b]}} \|\sum_{j=0}^{k} \varphi_j(p)(G_j(s) - \hat{G}_j(s))\|$$

$$\overset{(1)}{\leq} \underbrace{\sup_{\substack{s\in\mathbb{C}^+\\ p\in[a,b]}} \|\sum_{j=0}^{k} \varphi_j(p)(G_j(s) - \hat{G}_j(s))\|}_{interpolation\ \ error} + \underbrace{\mathrm{tol} \cdot \sup_{p\in[a,b]} |\sum_{j=0}^{k} \varphi_j(p)|}_{BT\ \ error}$$

Flow meter (anemometer)

- Sensor to measure flow rate of liquids and gases.
- An engineering requirement in this case is a compact flow meter model that allows us to use fluid properties (flow velocity) as parameters.
- Mathematical model: Linear convection-diffusion equation.

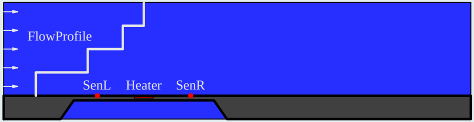

Figure: Flow meter model generated with ANSYS

Anemometer discretization

Convection-diffusion equation with fluid properties
fluid velocity $\vec{v}$, heat capacity c, thermal conductivity κ:

$$c\frac{\partial T}{\partial t}(t,\xi) = \nabla \cdot (\kappa \nabla T(t,\xi)) - c\vec{v}\cdot\nabla T + \dot{q},\ \xi \in (0,1)^2$$

$$\Downarrow \quad \text{space discretization } n = 29008$$

$$\begin{aligned} E\frac{d}{dt}T(t) &= \underbrace{(-K_d - pK_c)}_{A(p)} T(t) + b\,u(t) \\ y(t) &= c^\top T(t) \end{aligned}$$

First: preserve flow velocity as single parameter p.

Numerical results - Lagrange interpolation

1. choose $p_0, \cdots, p_{11} \in [0,\ 1]$ as Chebyshev points (second kind)

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3. Lagrange interpolation using barycentric formula:

$$\hat{G}_I(s,p) = \frac{\sum\limits_{j=0}^{k} \frac{w_j}{p-p_j}\ \hat{G}_j(s)}{\sum\limits_{j=0}^{k} \frac{w_j}{p-p_j}}, \quad w_j = \frac{1}{\prod_{i \neq j}(p_j - p_i)}$$

$$w_j = (-1)^j \delta_j, \quad \delta_j = \begin{cases} 1/2 & j = 0 \text{ or } k \\ 1 & \text{otherwise} \end{cases}$$

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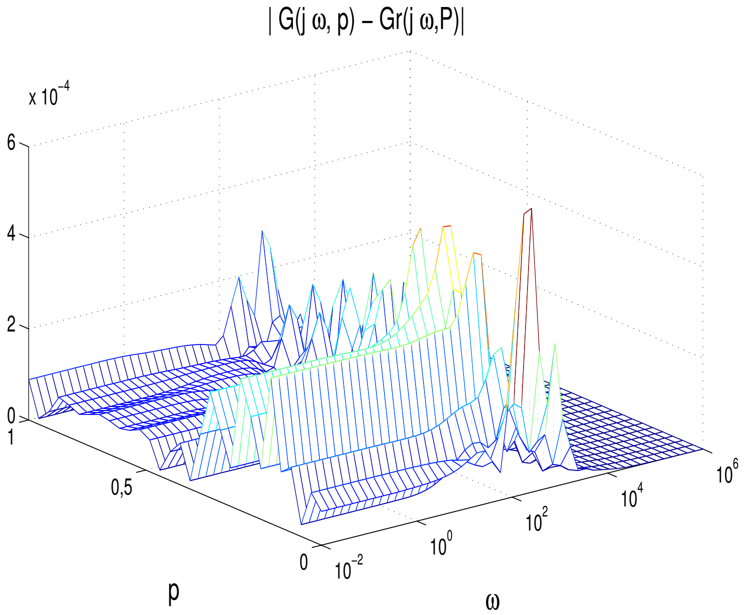
$$w_j = (-1)^j \delta_j, \quad \delta_j = \begin{cases} 1/2 & j = 0 \text{ or } k \\ 1 & \text{otherwise} \end{cases}$$

4. $\sup_{\substack{s \in [\jmath 10^{-2}, \jmath 10^{6}] \\ p \in [0,1]}} \|G(s,p) - \hat{G}_I(s,p)\| \leq 6.2 \times 10^{-4}$

Numerical results - Lagrange interpolation



Numerical results - Hermite interpolation

Anemometer example with $n = 29008$

$$G(s,p) = c^T(sE - A(p))^{-1}b, \qquad p \in [0,\, 1],$$

and 12 Chebyshev points, BT tolerance $10^{-4} \Rightarrow r = 75$

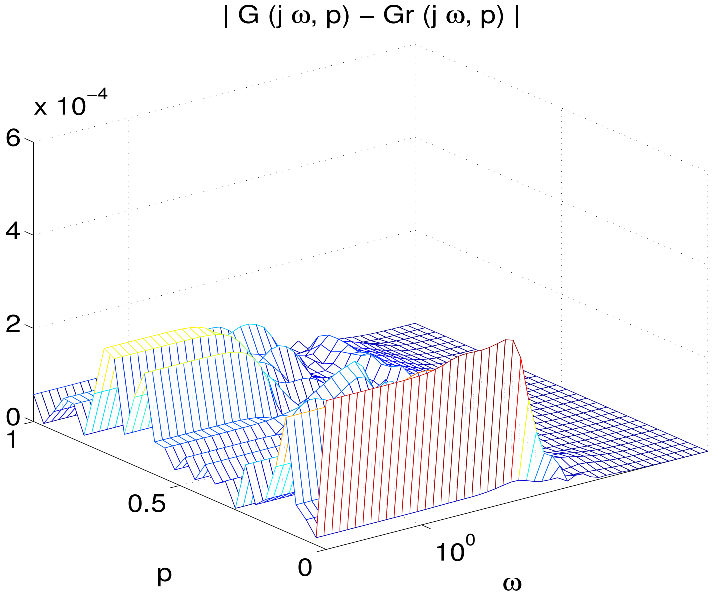
1. At $[p_0, \ldots, p_{11}]$:

$$\begin{aligned} \hat{G}_I(s,p_j) &= \hat{G}_j(s) \approx G(s,p_j), \\ \frac{\partial \hat{G}_I(s,p_j)}{\partial p} &= \hat{c}^T(sE_{r_j} - \hat{A}_j)^{-1}\hat{A}_1(sE_{r_j} - \hat{A}_j)^{-1}\hat{b} \approx \frac{\partial G(s,p_j)}{\partial p} \end{aligned}$$

2. $\hat{G}_I(s,p)$ by Hermite interpolation with error estimate:

$$\sup_{\substack{s \in [\jmath 10^{-2}, \jmath 10^{6}] \\ p \in [0,1]}} \|G(s,p) - \hat{G}_I(s,p)\| \leq 3.5 \times 10^{-4}$$

Numerical results - Hermite interpolation



Numerical results - rational interpolation

Anemometer example with $n = 29008$

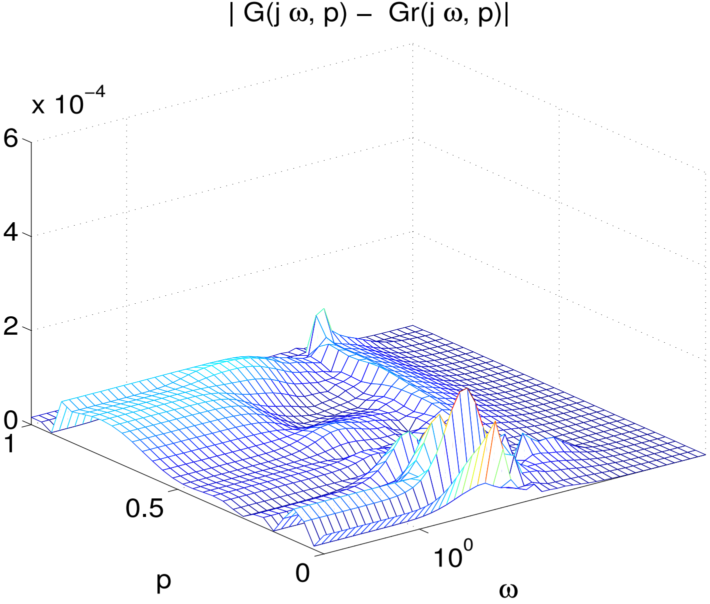
$$G(s,p) = c^T(sE - A(p))^{-1}b, \qquad p \in [0,\,1],$$

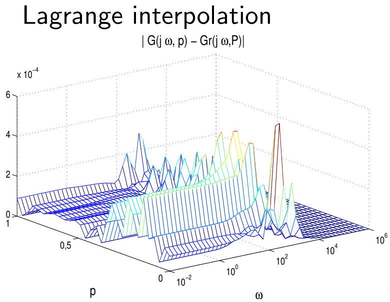
and 12 Chebyshev points, BT tolerance $10^{-4} \Rightarrow r = 75$

1. $\hat{G}_I(s,p) = \frac{num(s,p)}{den(s,p)}$ with $num \in \Pi_{\lfloor \frac{k+1}{2} \rfloor}$ and $den \in \Pi_{\lfloor \frac{k}{2} \rfloor}$
   representation by generalized continued fraction,
   computation by divided inverse differences
2. $\hat{G}_I(s,p)$ by rational interpolation with error estimate:

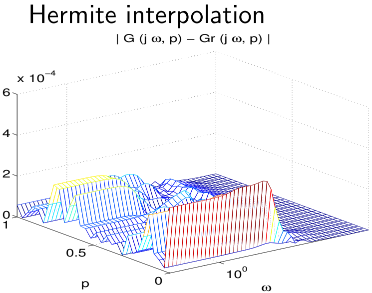
$$\sup_{\substack{s \in [\jmath 10^{-2}, \jmath 10^{6}] \\ p \in [0,1]}} \|G(s,p) - \hat{G}_I(s,p)\| \leq 2.4 \times 10^{-4}$$

Numerical results - rational interpolation

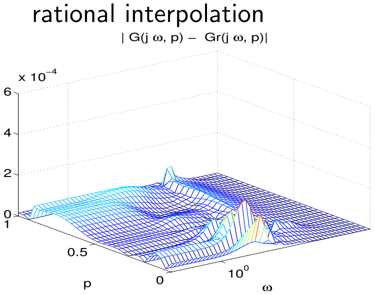




Lagrange interpolation
| G(j ω, p) – Gr(j ω,P)|
x 10^-4
6
4
2
0
1
0,5
0
p
10^-2
10^0
10^2
10^4
10^6
ω



Hermite interpolation
| G (j ω, p) – Gr (j ω, p) |
x 10^-4
6
4
2
0
1
0.5
0
p
10^0
ω



rational interpolation
| G(j ω, p) – Gr(j ω, p)|
x 10^-4
6
4
2
0
1
0.5
0
p
10^0
ω

But:

for higher dimensional parameter spaces $p \in [0,1]^d$ with $d \geq 3$ we need many interpolation points $\Rightarrow$ many times BT,

i.e. very high complexity!

Thus:

employ sparse grid interpolation [*Smolyak 63, Zenger 91, Griebel 91, Bungartz 92*]

main advantages:

- requires significantly fewer grid points
- preserves asymptotic error decay with increasing grid resolution (up to logarithmic factor)

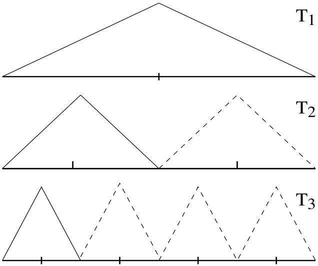
Hierarchical basis decomposition in $d = 1$

On $[0,\ 1]$ construct (equidistant) grid with mesh size $h_\ell = 2^{-\ell}$ and associated $(2^\ell - 1)$-dim. space of piecewise linear functions S_ℓ

hierarchical basis decomposition:

$$S_\ell = T_1 \oplus \cdots \oplus T_\ell$$

subspaces of S_3:



basis of $T_1,\ T_2,\ T_3$

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hierarchical basis decomposition:

$$S_\ell = T_1 \oplus \cdots \oplus T_\ell$$

For $G(s, \cdot) \in C^2([0, 1])$ and interpolant

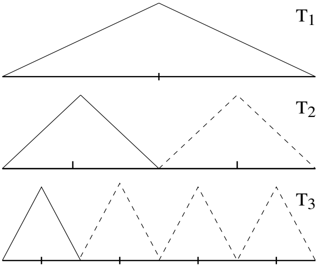
$G_I \in S_\ell$

$$G_I = \sum_{i=1}^{\ell} g_i, \qquad g_i \in T_i$$

the interpolation error is bounded

- $\|G(s, \cdot) - G_I\|_\infty \leq \mathcal{O}(h_\ell^2)$
- $\|g_i\|_\infty \leq \frac{1}{2} 4^{-i} \|\frac{\partial^2 G(s,\cdot)}{\partial p^2}\|_\infty$

subspaces of S_3:



basis of T_1, T_2, T_3

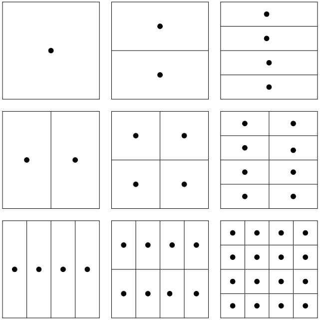
Hierarchical basis decomposition in $d = 2$

On $[0,\ 1]^2$ construct rectangular grid with mesh size $h_{\ell_1} = 2^{-\ell_1}, h_{\ell_2} = 2^{-\ell_2}$ and $(2^\ell - 1)^2$-dim. space of piecewise bilinear functions $S_{\underline{\ell}}$ $\quad (\underline{\ell} := (\ell, \ell))$

hierarchical basis decomposition:

$$S_{\underline{\ell}} = \bigoplus_{i_1=1}^{\ell} \bigoplus_{i_2=1}^{\ell} T_{\underline{i}}, \quad \underline{i} = (i_1, i_2)$$

subspaces of S_{33}:



supports of basis of $T_{11} \cdots$

Hierarchical basis decomposition in $d = 2$

On $[0,\ 1]^2$ construct rectangular grid with mesh size $h_{\ell_1} = 2^{-\ell_1}, h_{\ell_2} = 2^{-\ell_2}$ and $(2^\ell - 1)^2$-dim. space of piecewise bilinear functions $S_{\underline{\ell}}$ $\quad (\underline{\ell} := (\ell, \ell))$

hierarchical basis decomposition:

$$S_{\underline{\ell}} = \bigoplus_{i_1=1}^{\ell} \bigoplus_{i_2=1}^{\ell} T_{\underline{i}}, \qquad \underline{i} = (i_1, i_2)$$

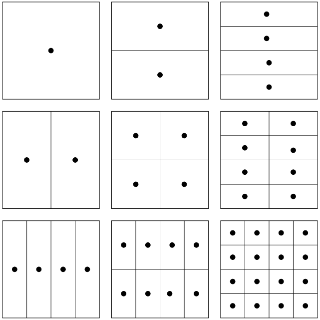
For $G : [0,\ 1]^2 \to \mathbb{R}$, $\frac{\partial^4 G}{\partial p_1^2 \partial p_2^2} \in C^0([0,\ 1]^2)$

$$G_I = \sum_{i_1=1}^{\ell} \sum_{i_2=1}^{\ell} g_{\underline{i}}, \qquad g_{\underline{i}} \in T_{\underline{i}}$$

the interpolation error is bounded

- $\|G(s,\cdot) - G_I\|_\infty \leq \mathcal{O}(h_\ell^2)$
- $\|g_{\underline{i}}\|_\infty \leq \frac{1}{4} 4^{-i_1-i_2} \| \frac{\partial^4 G(s,\cdot)}{\partial x_1^2 \partial x_2^2} \|_\infty$

subspaces of S_{33}:



supports of basis of $T_{11} \cdots$

Sparse grids in $d = 2$

sparse decomposition: $\tilde{S}_{\underline{\ell}} = \bigoplus_{i_1+i_2 \leq \ell+1} T_{\underline{i}}, \quad \underline{i} = (i_1, i_2)$

with reduced dimension:

$$\dim \tilde{S}_{\underline{\ell}} = 2^{\ell}(\ell - 1) + 1$$

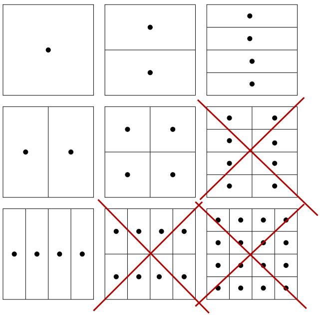
For $G : [0,\ 1]^2 \to \mathbb{R},\ \frac{\partial^4 G}{\partial p_1^2 \partial p_2^2} \in C^0([0,\ 1]^2)$

$$\tilde{G}_I = \sum_{i_1+i_2 \leq \ell+1} g_{\underline{i}}, \qquad g_{\underline{i}} \in T_{\underline{i}}$$

the interpolation error is bounded

$$\|G(s,\cdot) - \tilde{G}_I\|_\infty \leq \mathcal{O}(h_\ell^2 log(h_\ell^{-1}))$$

subspaces of S_{33}:



supports of basis of $T_{11} \cdots$

Sparse grids [Smolyak 63, Zenger 91, Griebel 91, Bungartz 92]

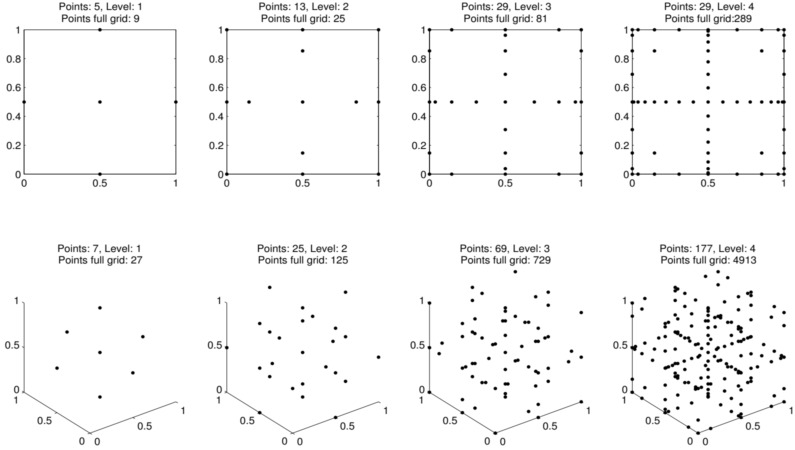
On $[0,\ 1]^d$ construct rectangular grid with mesh size $h_{\underline{\ell}}$.

For $G(s,\cdot):[0,\ 1]^d \to \mathbb{R}$, $\frac{\partial^{2d} G(s,\cdot)}{\partial p_1^2 \ldots \partial p_d^2} \in C^0([0,\ 1]^d)$ search

interpolant G_I in space of piecewise d-linear functions:

	full grid space	sparse grid space
	$S_{\underline{\ell}} = \bigoplus_{i_1=1}^{\ell} \cdots \bigoplus_{i_d=1}^{\ell} T_{\underline{i}}$	$\tilde{S}_{\underline{\ell}} = \bigoplus_{\vert\underline{i}\vert_1 \leq \ell+d-1} T_{\underline{i}}$
dimension	$\mathcal{O}(h_{\ell}^{-d})$	$\mathcal{O}(h_{\ell}^{-1}(log(h_{\ell}^{-1}))^{d-1})$
$\Vert G(s,\cdot) - G_I \Vert_\infty$	$\mathcal{O}(h_{\ell}^{2})$	$\mathcal{O}(h_{\ell}^{2}(log(h_{\ell}^{-1}))^{d-1})$

MATLAB Sparse Grid Interpolation Toolbox:



We employ sparse grids for high-dim. parameter space $p \in \mathcal{I}^d$.

Consider again

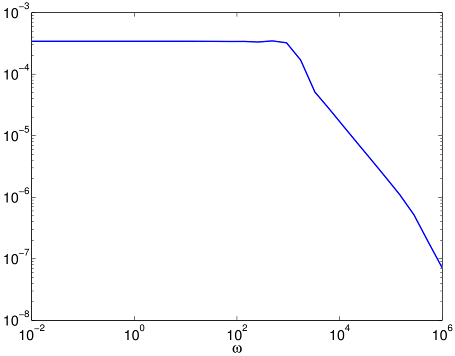
$$c\frac{\partial T}{\partial t}(t,\xi) = \nabla \cdot (\kappa \nabla T(t,\xi)) - c\vec{v}\cdot\nabla T + \dot{q}$$

with fluid properties:
fluid velocity $\vec{v}$, heat capacity c, thermal conductivity κ

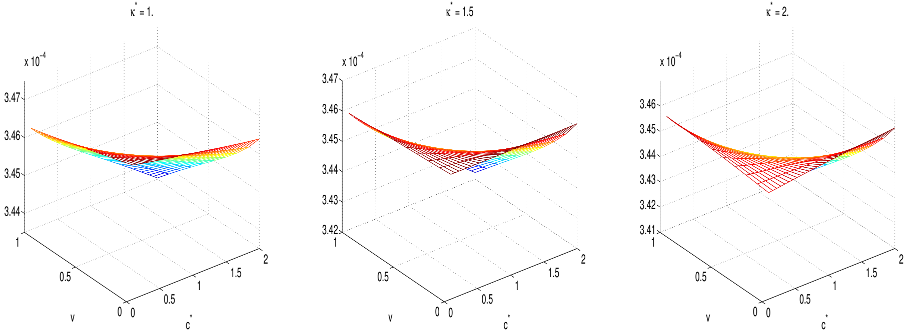
$$\begin{aligned}\underbrace{(M_s + p_0 M_f)}_{E(p_0)} \tfrac{d}{dt} T(t) &= \underbrace{(-K_{d,s} - p_1 K_{d,f} - p_2 K_c)}_{A(p_1,p_2)} T(t) + b\, u(t) \\ y(t) &= c^\top T(t)\end{aligned}$$

- Now: parameter space $[0,\ 1] \times [0.1,\ 2] \times [1,\ 2]$
- MATLAB Sparse Grid Interpolation Toolbox [*Klimke/Wohlmuth 05, Klimke 07*]
- Chebyshev-Gauss-Lobatto grid with polynomial interpolation

Numerical results - Anemometer $d = 3$



- we choose level for grid refinement: $\ell = 2$
  $\Rightarrow$ 25 sparse grid points
- error tol for BT applied to $G(s, p^j)$: 10^{-3}
  $r_j = 2 \Rightarrow$ system of reduced order $r = 50$
- estimated interpolation error (from Toolbox): 4.2×10^{-4}


κ* = 1.
x 10^-4
3.47
3.46
3.45
3.44
v
c*
κ* = 1.5
x 10^-4
3.47
3.46
3.45
3.44
3.43
3.42
v
c*
κ* = 2.
x 10^-4
3.46
3.45
3.44
3.43
3.42
3.41
v
c*

Conclusions/Outlook

- We have developed a balanced truncation/interpolatory method for parametric model reduction with reduced complexity in numerical simulations, error estimates.
- The method can be applied to higher ($d \leq 10$) dimensional parameter spaces [*Baur/B., at-Automatisierungstechnik, 2009*].

Next steps:

- Only function for evaluation of reduced-order system, search for explicit description of TFM, state-space model.
- Which interpolation method fits best to which problem?
- Look at other interpolation based techniques:
  - weighted interpolation in time domain [*Panzer/Mohring/Eid/Lohmann, at-Automatisierungstech. 2010*].
  - high-order multivariate interpolation on nonlinear matrix manifolds [*Amsallem, 2010*].
- Adaptive sparse grids [*Gerstner/Griebel 2003*].
- Sparse grid discretization directly on PDE level.

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