SOLVING ALGEBRAIC RICCATI EQUATIONS FOR STABILIZATION OF INCOMPRESSIBLE FLOWS

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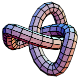

Joint work with Eberhard Bänsch (FAU Erlangen) and Anne Heubner within sub-project *Optimal Control-Based Feedback Stabilization in Multi-Field Flow Problems* of DFG Priority Program Optimization with Partial Differential Equations.

Householder Symposium XVII
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Overview

Motivation

Scientific goals of the project:

- derive and investigate numerical algorithms for optimal control-based boundary feedback stabilization of multi-field flow problems;
- explore the potentials and limitations of feedback-based (Riccati) stabilization techniques;
- extend current methods for flow described by Navier-Stokes equations to flow problems coupled with other field equations of increasing complexity.

$\rightsquigarrow$ Major challenge: solve large-scale algebraic Riccati equations associated to special LQR problem for Oseen-like equations.

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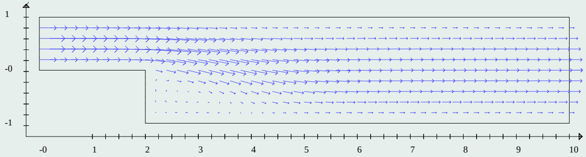

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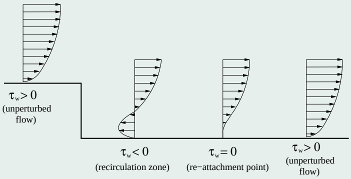

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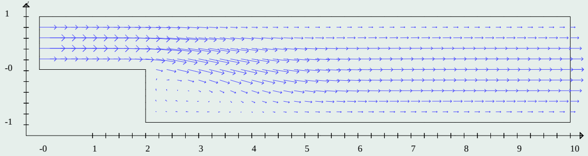

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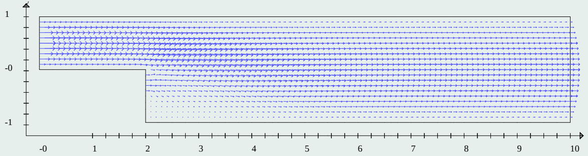

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Motivation

Optimal control-based stabilization for Navier-Stokes equations

- Stabilization to steady-state solutions of flows (with velocity field v and pressure χ), described by Navier-Stokes equations

$$\partial_t v + v \cdot \nabla v - \frac{1}{Re} \Delta v + \nabla \chi = f \tag{1a}$$

$$\operatorname{div} v = 0, \tag{1b}$$

on $Q_\infty := \Omega \times (0, \infty)$, $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$ with smooth boundary $\Gamma := \partial\Omega$, and boundary and initial conditions

$$\begin{aligned} v &= g \quad \text{on } \Sigma_\infty := \Gamma \times (0, \infty), \\ v(0) &= w + z(0) \quad (w \text{ given velocity field}). \end{aligned}$$

- Existence of stabilizing feedback control proved in 2D [Fernández-Cara et al 2004] and 3D [Fursikov 2004].
- Construction of stabilizing feedback control based on associated linear-quadratic optimal control problem:
  - for distributed control, see [Barbu 2003, Barbu/Sritharan 1998, Barbu/Triggiani 2004];
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Assume w solves the stationary Navier-Stokes equations

$$w \cdot \nabla w - \frac{1}{Re}\Delta w + \nabla \chi_s = f, \qquad \operatorname{div} w = 0, \tag{2}$$

with Dirichlet boundary condition $w = g$ on Γ. Furthermore, w is assumed to be *unstable* solution of (1).

If we can determine a Dirichlet boundary control u so that the corresponding controlled system

$$\begin{aligned} \partial_t z + (z\cdot\nabla)w + (w\cdot\nabla)z + (z\cdot\nabla)z - \frac{1}{Re}\Delta z + \nabla p &= 0 \quad \text{in } Q_\infty, \\ \operatorname{div} z &= 0 \quad \text{in } Q_\infty, \\ z &= bu \quad \text{in } \Sigma_\infty, \\ z(0) &= z_0 \quad \text{in } \Omega, \end{aligned}$$

is stable for "small" initial values $z_0 \in X(\Omega) \subset V_n^0(\Omega)$, where

$$V_n^0(\Omega) := L_2 \cap \{\operatorname{div} z = 0\} \cap \{z \cdot n = 0 \text{ on } \Gamma\},$$

then $\exists$ constants $c, \omega > 0$ so that $\|z(t)\|_{X(\Omega)} \le ce^{-\omega t}$.

$\Longrightarrow$ Solution to instationary Navier-Stokes equations with $v = w + z$, $\chi = \chi_s + p$, and $v(0) = w + z_0$ in Ω is controlled to w.

Optimal control-based stabilization for Navier-Stokes equations

Analytical solution [Raymond '05–'08]

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ωz with $\omega > 0$ de-stabilizes the system further, needed to guarantee exponential stabilization of solution of nonlinear system!

Cost functional

$$J(z, u) = \frac{1}{2} \int_0^\infty \langle Pz, Pz \rangle_{L_2(\Omega)} + \rho u(t)^2 \, dt, \tag{4}$$

the linear-quadratic optimal control problem associated to (3) becomes

$$\inf \{ J(z, u) \mid (z, u) \text{ satisfies } (3), \; u \in L_2(0, \infty) \}. \tag{5}$$

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The solution to the instationary Navier-Stokes equations with perturbed initial data is exponentially controlled to the steady-state solution w by the feedback law

$$u = -\rho^{-1} B^* \Pi z_H,$$

where

- $z_H := Pz$, with $P : L_2(\Omega) \mapsto V_n^0(\Omega)$ being the Helmholtz projector ($\rightsquigarrow \operatorname{div} z_H \equiv 0$);
- $\Pi = \Pi^* \in \mathcal{L}(V_n^0(\Omega))$ is the unique nonnegative semidefinite weak solution of the operator Riccati equation

$$0 = I + (A + \omega I)^* \Pi + \Pi(A + \omega I) - \Pi(B_\tau B_\tau^* + \rho^{-1} B_n B_n^*)\Pi,$$

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Solving Large-Scale AREs

Algebraic Riccati Equation (ARE)

General form for $A, G = G^T, W = W^T \in \mathbb{R}^{n \times n}$ given and $X \in \mathbb{R}^{n \times n}$ unknown:

$$0 = \mathcal{R}(X) := A^T X + XA - XGX + W.$$

Large-scale AREs from semi-discretized PDE control problems:

- $n = 10^3 - 10^6$ ($\Longrightarrow 10^6 - 10^{12}$ unknowns!),
- A has sparse representation ($A = -M^{-1}L$ for FEM),
- usually, G, W low-rank with $G, W \in \{BB^T, C^T C\}$, where $B \in \mathbb{R}^{n \times m}$, $m \ll n$, $\quad C \in \mathbb{R}^{p \times n}$, $p \ll n$.
- under the above assumptions, ARE allows for a low-rank approximation

$$X \approx ZZ^T, \quad Z \in \mathbb{R}^{n \times r}.$$

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Low-Rank Newton-ADI for AREs

Re-write Newton's method for AREs

$$A_j^T N_j + N_j A_j = -\mathcal{R}(X_j)$$

$$\Longleftrightarrow$$

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + \underbrace{(X_j + N_j)}_{=X_{j+1}} A_j = \underbrace{-C^T C - X_j B B^T X_j}_{=:-W_j W_j^T}$$

Set $X_j = Z_j Z_j^T$ for $\operatorname{rank}(Z_j) \ll n \Longrightarrow$

$$A_j^T \left(Z_{j+1} Z_{j+1}^T\right) + \left(Z_{j+1} Z_{j+1}^T\right) A_j = -W_j W_j^T$$

Factored Newton Iteration [B./Li/Penzl 1999/2008]

Solve Lyapunov equations for Z_{j+1} directly by factored ADI iteration and use 'sparse + low-rank' structure of A_j.

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Solve Lyapunov equations for Z_{j+1} directly by factored ADI iteration and use 'sparse + low-rank' structure of A_j.

- Convergence for K_0 stabilizing:
  - $A_j = A - BK_j = A - BB^T X_j$ is stable $\forall\ j \geq 0$.
  - $\lim_{j\to\infty} \|\mathcal{R}(X_j)\|_F = 0$ (monotonically).
  - $\lim_{j\to\infty} X_j = X_* \geq 0$ (locally quadratic).
- Need large-scale Lyapunov solver; here, ADI iteration: linear systems with dense, but "sparse+low rank" coefficient matrix A_j:

$$A_j \;=\; A \;-\; B \;\cdot\; K_j$$

= sparse − m ·

- $m \ll n \implies$ efficient "inversion" using Sherman-Morrison-Woodbury formula:

$$(A - BK_j)^{-1} = (I_n + A^{-1}B(I_m - K_jA^{-1}B)^{-1}K_j)A^{-1}.$$

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- Convergence for K_0 stabilizing:
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  - $\lim_{j\to\infty} \|\mathcal{R}(X_j)\|_F = 0$ (monotonically).
  - $\lim_{j\to\infty} X_j = X_* \geq 0$ (locally quadratic).
- Need large-scale Lyapunov solver; here, ADI iteration: linear systems with dense, but "sparse+low rank" coefficient matrix A_j:

$$A_j = A - B \cdot K_j$$

- $m \ll n \Longrightarrow$ efficient "inversion" using Sherman-Morrison-Woodbury formula:

$$(A - BK_j)^{-1} = (I_n + A^{-1}B(I_m - K_jA^{-1}B)^{-1}K_j)A^{-1}.$$

- Need Lyapunov solver that computes low-rank approximation.

ADI with $X_k = Y_k Y_k^T$ yields

Algorithm [Penzl '97, Li/White '02, B./Li/Penzl '99/'08]

$$V_1 \leftarrow \sqrt{-2\mathrm{Re}\,(p_1)}(A + p_1 I)^{-1}B, \qquad Y_1 \leftarrow V_1$$

FOR $j = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{Re}\,(p_k)}{\mathrm{Re}\,(p_{k-1})}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1}V_{k-1}\right),$$

$$Y_k \leftarrow \left[\begin{array}{cc} Y_{k-1} & V_k \end{array}\right]$$

$$Y_k \leftarrow \mathrm{rrqr}(Y_k, \tau) \qquad \text{\% column compression}$$

At convergence, $Y_{k_{\max}} Y_{k_{\max}}^T \approx X$, where

$$\mathrm{range}\,(Y_{k_{\max}}) = \mathrm{range}\,\left(\left[\begin{array}{ccc} V_1 & \ldots & V_{k_{\max}} \end{array}\right]\right), \quad V_k = \Big[\;\Big] \in \mathbb{C}^{n\times m}.$$

Note: Implementation in real arithmetic possible by combining two steps.

Alternatives: K-PIK [Simoncini 06],
low-rank cyclic Smith(ℓ) [Penzl '00, Gugercin/Sorensen/Antoulas '03],

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Low-Rank ADI Method for Lyapunov Equations

Lyapunov equation $0 = AX + XA^T = -BB^T$.

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Performance of Newton's method for accuracy $\sim 1/n$

grid	unknowns	$\frac{\|\mathcal{R}(X)\|_F}{\|X\|_F}$	it. (ADI it.)	CPU (sec.)
8×8	2,080	4.7e-7	2 (8)	0.47
16×16	32,896	1.6e-6	2 (10)	0.49
32×32	524,800	1.8e-5	2 (11)	0.91
64×64	8,390,656	1.8e-5	3 (14)	7.98
128×128	134,225,920	3.7e-6	3 (19)	79.46

Here,

- Convection-diffusion equation,
- $m = 1$ input and $p = 2$ outputs,
- $X = X^T \in \mathbb{R}^{n \times n} \Rightarrow \frac{n(n+1)}{2}$ unknowns.

Confirms mesh independence principle for Newton-Kleinman [Burns/Sachs/Zietsmann 2006].

Problems with Newton-Kleinman

1. Discretization of Helmholtz-projected Oseen equations would need divergence-free finite elements.

   Here, we want to use standard discretization (Taylor-Hood elements available in flow solver NAVIER).

   Explicit projection of ansatz functions possible using application of Helmholtz projection. But: solution of one saddle-point problem per ansatz function.

2. Each step of Newton-Kleinman iteration: solve

   $$A_j^T Z_{j+1} Z_{j+1}^T + Z_{j+1} Z_{j+1}^T A_j = -W_j W_j^T = -M_h - (Z_j Z_j^T B)(Z_j Z_j^T B)^T$$

   $n_v := \operatorname{rank}(M_h) =$ dim of ansatz space for velocities.

   $\rightsquigarrow$ need to solve n_v linear systems of equations in each step of ADI iteration!

3. Linearized system (i.e., $A + \omega I$) is unstable in general.

   Thus, to start the iteration, a stabilizing initial guess is needed!

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Solving the Helmholtz-projected Oseen ARE
$0 = I + (A+\omega I)^T X + X(A+\omega I) - XBB^T X$

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Work with the differential-algebraic equations (DAE)

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\begin{aligned}
\mathbf{E}_{11}\dot{\mathbf{z}}_h(t) &= \mathbf{A}_{11}\mathbf{z}_h(t) + \mathbf{A}_{12}\mathbf{p}_h(t) + \mathbf{B}_1\mathbf{u}(t) \\
\mathbf{0} &= \mathbf{A}_{12}^T\mathbf{z}_h(t) + \mathbf{B}_2\mathbf{u}(t) \\
\mathbf{z}_h(0) &= \mathbf{z}_{h,0}.
\end{aligned}
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obtained from Taylor-Hood FEM applied to Oseen equations.

Solving the Helmholtz-projected Oseen ARE

Solution to 1. Problem/no need for divergence free FE

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Necessary information for low-rank solution of Lyapunov equations can be obtained as in [Heinkenschloss/Sorensen/Sun '07], cf. Dan Sorensen's talk.

Adapted to our situation: need to solve Lyapunov equation

$$
A_j^T Z_{j+1} Z_{j+1}^T P_h \mathbf{E}_{11} P_h^T + P_h \mathbf{E}_{11} P_h^T Z_{j+1} Z_{j+1}^T A_j = -W_j W_j^T,
$$

where

$$
\begin{aligned}
P_h &:= I_{n_v} - \mathbf{A}_{12}(\mathbf{A}_{12}^T\mathbf{E}_{11}^{-1}\mathbf{A}_{12})^{-1}\mathbf{A}_{12}^T\mathbf{E}_{11}^{-1},\\
A_j &:= P_h(\mathbf{A}_{11} - \mathbf{B}_1\mathbf{B}_1^T P_h Z_j Z_j^T P_h \mathbf{E}_{11})P_h,\\
W_j &:= \begin{bmatrix} P_h C^T & P_h \mathbf{E}_{11} P_h Z_j Z_j^T P_h \mathbf{B}_1 \end{bmatrix}.
\end{aligned}
$$

Obtain low-rank factor so that $X_{j+1} \approx Z_{j+1} Z_{j+1}^T$ as

$$Z_{j+1} = \sqrt{\mu} \left[B_{j,\mu}, \; A_{j,\mu} B_{j,\mu}, \; A_{j,\mu}^2 B_{j,\mu}, \; \ldots, \; A_{j,\mu}^j B_{j,\mu} \right],$$

where

- $B_{j,\mu}$ solves the saddle point problem

$$\begin{bmatrix} \mathbf{E}_{11} + \mu(\mathbf{A}_{11} - \mathbf{B}_1 \mathbf{B}_1^T Z_j Z_j^T \mathbf{E}_{11}) & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & 0 \end{bmatrix} \begin{bmatrix} B_{j,\mu} \\ * \end{bmatrix} = \begin{bmatrix} C^T & \mathbf{E}_{11} Z_j Z_j^T \mathbf{B}_1 \\ 0 & 0 \end{bmatrix},$$

- multiplication by $A_{j,\mu}$ is realized by solution of saddle-point problem with the same coefficient matrix,
- and we employ a column compression using RRQR as in [B./Quintana-Ortí '97].

Multishift version also possible, cf. [Heinkenschloss/Sorensen/Sun '07].

Solving the Helmholtz-projected Oseen ARE

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Solving the Helmholtz-projected Oseen ARE

Solution to 2. Problem/remove M_h from r.h.s.

For simplicity, consider standard ARE case:

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + X_{j+1} A_j = -W - X_j B B^T X_j \qquad \text{for } j = 1, 2, \ldots$$

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$\Longleftrightarrow$

$$A_j^T N_j + N_j A_j = -N_{j-1} B B^T N_{j-1} \qquad \text{for } j = 1, 2, \ldots$$

See [Banks/Ito '91, B./Hernández/Pastor '03, Morris/Navasca '05] for details and applications of this variant.

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But: need $N_0 = X_1 - X_0$!

Solution idea:

Compute X_0 and X_1 from full, dense Lyapunov equation on coarse grid, prolongate to fine grid.

Possible refinement: coarse grid corrections using Richardson iteration, nested iteration for ARE [Grasedyck '08].

Solving the Helmholtz-projected Oseen ARE

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AREs for Stabilization of Flow Problems

Peter Benner

Motivation

Solving Large-Scale AREs

Solving the Helmholtz-projected Oseen ARE

Algebraic Bernoulli Equations

Conclusions and Open Problems

Again, consider standard ARE case:

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + X_{j+1} \underbrace{(A - BB^T X_j)}_{=:A_j} = -M_h - X_j BB^T X_j \quad \text{for } j = 1, 2, \ldots$$

Recall: for convergence to stabilizing solution need

$A_0 := A - BB^T X_0$ stable, i.e., all eigenvalues in left half plane.

Solving the Helmholtz-projected Oseen ARE

Solution to 3. Problem/compute stabilizing initial feedback

Again, consider standard ARE case:

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + X_{j+1} \underbrace{(A - BB^T X_j)}_{=:A_j} = -M_h - X_j BB^T X_j \quad \text{for } j = 1, 2, \ldots$$

Recall: for convergence to stabilizing solution need

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Basically, 3 approaches to compute $K_0 := B^T X_0$:

- pole placement (for descriptor systems: [Varga '95]),
- Bass algorithm (based on Lyapunov equation):
  - for standard systems [Armstrong '75],
  - for descriptor systems [Varga '95, B. '08, B./Stykel. '08],
- algebraic Bernoulli equations:
  - for standard systems [B. '06/'07],
    (for discrete-time systems: [Gallivan/Rao/Van Dooren 06]),
  - for descriptor systems [B. '08].

Algebraic Bernoulli Equations

Consider the algebraic Bernoulli equation

$$A^T X + XA - XBB^T X = 0$$

associated to a standard ARE with zero constant term.

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Theorem

a) Let (A, B) be controllable. Then
   - there exist symmetric solutions $X_+ \geq 0$, $X_- \leq 0$, with $X_- \leq X \leq X_+$ for all solutions X of the ABE;
   - X_- is the unique solution satisfying $\Lambda(A - BB^T X_-) \subset \mathbb{C}^+ \cup \imath\mathbb{R}$;
   - X_+ is the unique solution satisfying $\Lambda(A - BB^T X_+) \subset \mathbb{C}^- \cup \imath\mathbb{R}$.
   - If $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, then X_- is the unique anti-stabilizing solution and X_+ is the unique stabilizing solution of the ABE.

b) If (A, B) is stabilizable and $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, then the ABE has a unique stabilizing solution X_+ and $X_+ \geq 0$.

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$$\operatorname{rank}(X_+) = k,$$

where k is the number of eigenvalues of A in $\mathbb{C}^+$.

Hence,

$$X_+ = Y_+ Y_+^T, \quad \text{where} \quad Y_+ \in \mathbb{R}^{n \times k}.$$

Theorem [B. '07]

$$\Lambda(A - BB^T X_+) = (\Lambda(A) \cap \mathbb{C}^-) \cup -(\Lambda(A) \cap \mathbb{C}^+),$$

i.e., unstable eigenvalues are reflected at imaginary axis.

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Computation of X_+

- Solve as ARE (inefficient).
- Sign function method [Barrachina/B./Quintana-Ortí '05].
- Sign function method for Y_+ [B. '06, Barr./B./Q.-Ortí '07].
- Extension to descriptor systems [B. '08].
- For large-scale systems, use partial stabilization idea:
  1. Project onto unstable invariant/deflating subspace of $A/\lambda E - A$,

$$\tilde{Q}^T A \tilde{Q} = \tilde{A} \in \mathbb{R}^{k \times k}, \quad \text{set } \tilde{B} := \tilde{Q}^T B.$$

  2. Solve small-size ABE $\tilde{A}^T \tilde{X} + \tilde{X}\tilde{A} = \tilde{X}\tilde{B}\tilde{B}^T\tilde{X}$ for full-rank $\tilde{X}_+$.
  3. Construct feedback as $F := \tilde{B}^T \tilde{X} \tilde{Q}^T$.

Cf. also related work by [Amodei/Bouchon '08].

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  Cf. also related work by [Amodei/Bouchon '08].

Algebraic Bernoulli Equations

Consider the algebraic Bernoulli equation

$$A^T X + XA - XBB^T X = 0$$

associated to a standard ARE with zero constant term.

Computation of X_+

- Solve as ARE (inefficient).
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$$\begin{aligned} \partial_t v &= \Delta v + \omega v - \nabla \rho + f, \\ 0 &= \operatorname{div} v, \end{aligned} \qquad (\xi, t) \in \Omega \times (0, t_f).$$

Here, $n_v = 480$, $n_p = 255$ with $n_\infty = 510$ and $n_f = 225$.

Volume control

$m = 2$, $\omega = 100$ $\rightsquigarrow$ $k = 3$

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Numerical examples

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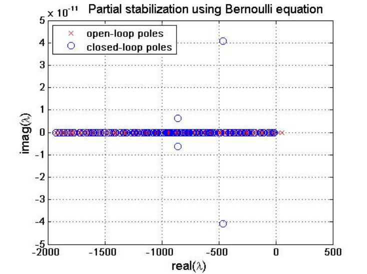

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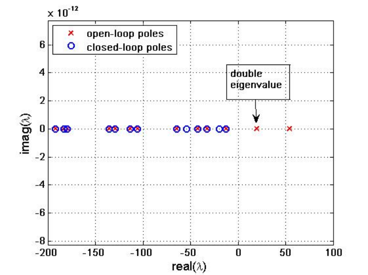

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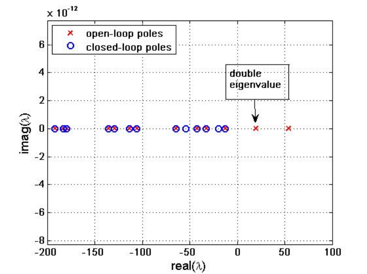

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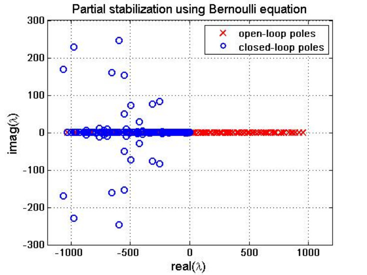

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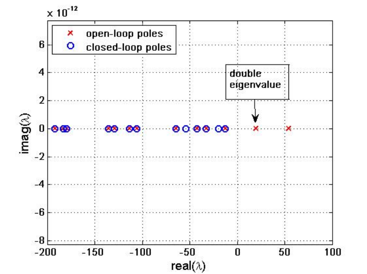

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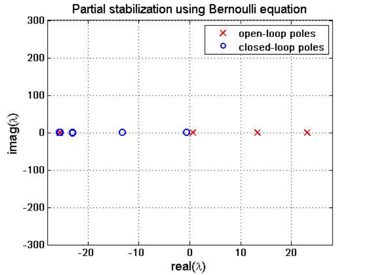

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- Extended and revised version of Lyapack.
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The End.