SYSTEM-THEORETIC METHODS FOR MODEL REDUCTION OF LARGE-SCALE SYSTEMS: SIMULATION, CONTROL, AND INVERSE PROBLEMS

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Overview

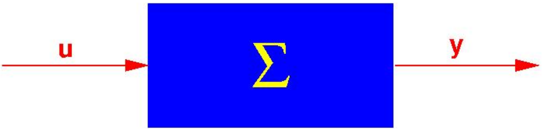
Dynamical Systems

$$\Sigma : \begin{cases} E\dot{x}(t) &= f(t, x(t), u(t)), \quad x(t_0) = x_0, & \text{(a)} \\ y(t) &= g(t, x(t), u(t)) & \text{(b)} \end{cases}$$

with

- (generalized) states $x(t) \in \mathbb{R}^n$ ($E \in \mathbb{R}^{n \times n}$),
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$, (b) is called output equation.

E singular $\Rightarrow$ (a) is system of differential-algebraic equations (DAEs)
otherwise $\Rightarrow$ (a) is system of ordinary differential equations (ODEs)



Model Reduction for Dynamical Systems

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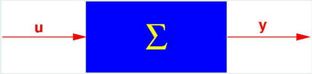

Reduced-Order System

$$\widehat{\Sigma} : \begin{cases} \hat{E}\dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
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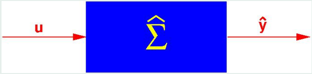

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$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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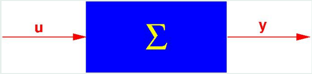

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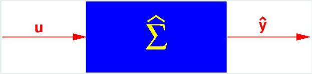

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Linear Systems

Linear, Time-Invariant (LTI) / Descriptor Systems

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \qquad A, E \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y(t) &= Cx(t) + Du(t), \qquad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{aligned}$$

Laplace Transformation / Frequency Domain

Application of Laplace transformation ($x(t) \mapsto x(s)$, $\dot{x}(t) \mapsto sx(s)$) to linear system with $x(0) = 0$:

$$sEx(s) = Ax(s) + Bu(s), \quad y(s) = Bx(s) + Du(s),$$

yields I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sE - A)^{-1}B + D}_{=:G(s)}\Big)u(s)$$

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Model Reduction for Linear Systems

Problem

Approximate the dynamical system

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by reduced-order system

$$\begin{aligned} \hat{E}\dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \qquad \hat{A}, \hat{E} \in \mathbb{R}^{r\times r}, \quad \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, \qquad \hat{C} \in \mathbb{R}^{p\times r}, \quad \hat{D} \in \mathbb{R}^{p\times m}, \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\|\|u\| < \text{tolerance} \cdot \|u\|.$$

$\Longrightarrow$ Approximation problem: $\min_{\text{order}\,(\hat{G})\leq r} \|G - \hat{G}\|$.

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Application Areas

General assumptions

Here:

- linear systems,
- $n \gg m, p$,
- n so large, that $A(, E)$ cannot be stored in main memory (RAM) as $n \times n$ array: $n > 5000$, say, e.g., from
  - semi-discretization of PDEs,
  - finite element modeling of MEMS,
  - VLSI design/circuit simulation, ...
- $A(, E)$ sparse or data-sparse, i.e., $A(, E)$ can be stored in $\mathcal{O}(n)$ or $\mathcal{O}(n \log n)$ memory locations, but matrix manipulations like similarity transformations are too expensive (possible exception: permutations, sparse factorizations).

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Time-domain simulation

Evaluation of variation-of-constants formula

$$y(t) = C\exp(At)\left(x^0 + \int_0^t \exp(-A\tau)Bu(\tau)d\tau\right),$$

usually too expensive $\rightsquigarrow$ numerical simulation, e.g., using backwards Euler

$$y_h(t_{k+1}) = C(E - h_k A)^{-1}\left(Ex_h(t_k) + h_k Bu(t_{k+1})\right) + Du(t_{k+1}),$$

Bottleneck: solution of $(E - h_k A)z = b$, computation time can be significantly reduced by using reduced-order model:

$$\hat{y}_h(t_{k+1}) = \hat{C}(\hat{E} - h_k \hat{A})^{-1}\left(\hat{E}x_h(t_k) + h_k \hat{B}u(t_{k+1})\right) + \hat{D}u(t_{k+1}).$$

Frequency-domain simulation

Frequency response analysis, e.g., for Bode, Nyquist or Nichols plots, requires evaluation of transfer function

$$G(\imath\omega_k) = C(\imath\omega_k E - A)^{-1}B + D, \quad \omega_k \geq 0,\ k = 1, \ldots, N_f.$$

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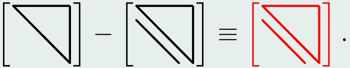
But: the cost for solving the linear systems in time/frequency domain simulation may not benefit from smaller order, if efficient sparse direct solver for full-size sparse system matrices is available.

Application Areas
Simulation

An easy improvement

Significant reduction can be achieved by transforming $(\hat{A}, \hat{E})$ to Hessenberg-triangular form using QZ algorithm, i.e., compute orthogonal Q, Z such that

$$Q(\lambda \hat{E} - \hat{A})Z = \lambda$$


New reduced-order system: $(Q\hat{E}Z, Q\hat{A}Z, Q\hat{B}, \hat{C}Z)$, linear systems of equations

$$\begin{aligned} (\jmath\omega \hat{E} - \hat{A})x &= b, \\ (\hat{E} - h_k \hat{A})x_{k+1} &= \hat{E}x_k + \ldots, \quad \text{etc.} \end{aligned}$$

have Hessenberg form and can thus be solved using $r-1$ Givens rotations only! (Needs Hessenberg solver inside simulator.)

For symmetric systems, further reduction can be achieved.

Application Areas

(Optimal) Control

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input = output of plant,
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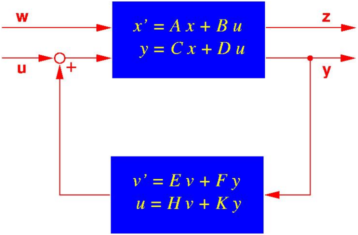
Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



Practical controllers require small N ($N \sim 10$, say) due to

- real-time constraints,
- increasing fragility for larger N.

$\Longrightarrow$ reduce order of plant (n) and/or controller (N).

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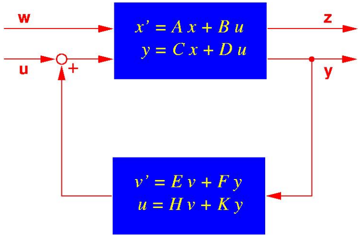

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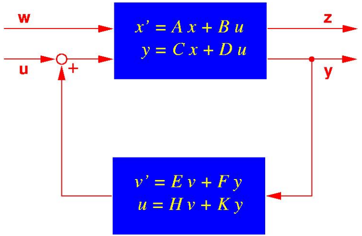

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System inversion

Assume $m = p$, $D \in \mathbb{R}^{m \times m}$ invertible (generalizations possible!), then

$$G^{-1}(s) = -D^{-1}C(sE - (A - BD^{-1}C))^{-1}BD^{-1} + D^{-1}.$$

Some applications like

- inverse-based control,
- identification of source terms,

reconstruct input function from reference trajectory/measured outputs: given $Y(s)$, the Laplace transform of $y(t)$, compute $U(s) = G^{-1}(s)Y(s)$.

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Goal: reduced-order transfer function $\hat{G}(s)$ such that

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has small error

$$\|U - \hat{U}\| = \|G^{-1}Y - \hat{G}^{-1}Y\| \leq \|G^{-1} - \hat{G}^{-1}\|\|Y\| \leq \text{tolerance} \cdot \|Y\|.$$

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- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),
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  - passivity ("system does not generate energy"),

All this can be achieved by system-theoretic methods based on balancing!

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Balancing Basics
($E = I_n$ for ease of notation)

Linear, Time-Invariant (LTI) Systems

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Model Reduction Based on Balancing

Given $P, Q \in \mathbb{R}^{n\times n}$ symmetric positive definite (spd), and a contragredient transformation $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$TPT^T = T^{-T}QT^{-1} = \mathrm{diag}(\sigma_1, \ldots, \sigma_n), \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0.$$

Balancing Σ w.r.t. P, Q:

$$\Sigma \equiv (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D) \equiv \Sigma.$$

Generalization to $P, Q \geq 0$ possible: if $\hat{n}$ is McMillan degree of Σ, then

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Basic Model Reduction Procedure

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$$\begin{aligned}(A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right)\end{aligned}$$

2. Truncation $\rightsquigarrow$ reduced-order model:

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Implementation: SR Method

1. Compute Cholesky (square) or full-rank (maybe rectangular, "thin") factors of P, Q

$$P = S^T S, \quad Q = R^T R.$$

2. Compute SVD

$$SR^T = [\, U_1, \; U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

3. Set

$$W = R^T V_1 \Sigma_1^{-1/2}, \qquad V = S^T U_1 \Sigma_1^{-1/2}.$$

4. Reduced-order model is

$$(\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (W^T A V, W^T B, CV, D) \quad (\equiv (A_{11}, B_1, C_1, D).)$$

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$$SR^T = [\, U_1,\ U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

3. Set

$$W = R^T V_1 \Sigma_1^{-1/2}, \qquad V = S^T U_1 \Sigma_1^{-1/2}.$$

4. Reduced-order model is

$$(\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (W^T A V, W^T B, CV, D) \quad (\equiv (A_{11}, B_1, C_1, D).)$$

Balancing for Simulation, Control

Truncate realization, balanced w.r.t. $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma$,
$\sigma_1 \geq \ldots \geq \sigma_r > \sigma_{r+1} \geq \ldots \sigma_n \geq 0$ at size r.

Classical Balanced Truncation (BT) — MULLIS/ROBERTS '76, MOORE '81

- $P/Q =$ controllability/observability Gramian of $\Sigma \equiv (A, B, C, D)$.
- For asymptotically stable systems, P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0.$$

- $\{\sigma_1^{\mathrm{BT}}, \ldots, \sigma_n^{\mathrm{BT}}\}$ are the Hankel singular values (HSVs) of Σ.
- Preserves stability, extends to unstable systems w/o purely imaginary poles using frequency domain definition of the Gramians [ZHOU/SALOMON/WU '99].
- Preserves passivity for certain symmetric systems.
- Computable error bound comes for free:

$$\|G - \hat{G}^{\mathrm{BT}}\|_{H_\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_j^{\mathrm{BT}},$$

allows adaptive choice of r!

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Linear-Quadratic Gaussian Balanced Truncation (LQGBT)

JONCKHEERE/SILVERMAN '83

- P/Q = controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual algebraic Riccati equations (AREs)

$$
\begin{aligned}
0 &= AP + PA^T - PC^TCP + B^TB, \\
0 &= A^TQ + QA - QBB^TQ + C^TC.
\end{aligned}
$$

- Applies to unstable systems!
  (Only stabilizability & detectability are required.)
- Computable error bound comes for free: if $G = M^{-1}N$, $\hat{G} = \hat{M}^{-1}\hat{N}$ are left coprime factorizations with stable factors, then

$$
\| \begin{bmatrix} N & M \end{bmatrix} - \begin{bmatrix} \hat{N} & \hat{M} \end{bmatrix} \|_{H_\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_j^{\mathrm{LQG}} \left(1 + (\sigma_j^{\mathrm{LQG}})^2\right)^{\frac{1}{2}},
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Positive-Real Balanced Truncation (PRBT) — Green '88

- Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.
- For $m = p$, P, Q solve dual AREs

$$
\begin{aligned}
0 &= \bar{A}P + P\bar{A}^T + PC^T\bar{R}^{-1}CP + B\bar{R}^{-1}B^T, \\
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where $\bar{R} = D + D^T$, $\bar{A} = A - B\bar{R}^{-1}C$.

- Preserves stability, strict passivity; needs stability of $\bar{A}$.
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$$
\|G - \hat{G}^{\mathrm{PR}}\|_{H_\infty} \leq 2\|R\|^2\|\hat{G}_D\|_\infty\|G_D\|_\infty \sum_{k=r+1}^{n} \sigma_k^{\mathrm{PR}}.
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($G_D(s) := G(s) + D^T$, $\hat{G}_D(s) := \hat{G}(s) + D^T$.)

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Balanced Stochastic Truncation (BST) — Desai/Pal '84, Green '88

- $P =$ controllability Gramian of $\Sigma \equiv (A, B, C, D)$, i.e., solution of Lyapunov equation $AP + PA^T + BB^T = 0$.
- $Q =$ observability Gramian of right spectral factor of power spectrum of Σ, i.e., solution of ARE

$$A_W^T Q + Q A_W + Q B_W (DD^T)^{-1} B_W^T Q + C^T (DD^T)^{-1} C = 0,$$

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$\rightsquigarrow$ uniform approximation quality over full frequency range.

Note: $|\sigma_j^{\mathrm{BST}}| \leq 1$.

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- Zeros of $G(s)$ are preserved in $\hat{G}(s)$. $\Longrightarrow$ *$G(s)$ minimum-phase $\Longrightarrow$ $\hat{G}(s)$ minimum-phase.*
- Error bound for inverse system [B. '03]

If $G(s)$ is square, minimal, stable, minimum-phase, nonsingular on $\jmath\mathbb{R}$, then

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Balanced Truncation and Relatives

Basic Principle of Balanced Truncation

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [Opdenacker/Jonckheere '88];
- H_∞ balanced truncation (HinfBT) – closed-loop balancing based on H_∞ compensator [Mustafa/Glover '91].

Both approaches require solution of dual AREs.

- Frequency-weighted versions of the above approaches.

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All balancing-related methods have nice theoretical properties that make them appealing for applications in simulation, control, optimization, inverse problems.

But: computationally demanding w.r.t. to memory and CPU time; need efficient solvers for linear (Lyapunov) and nonlinear (Riccati) matrix equations!

General form for $A, G = G^T, W = W^T \in \mathbb{R}^{n\times n}$ given and $P \in \mathbb{R}^{n\times n}$ unknown:

$$\begin{aligned} 0 &= \mathcal{L}(Q) := A^T Q + QA + W, \\ 0 &= \mathcal{R}(Q) := A^T Q + QA - QGQ + W. \end{aligned}$$

In large scale applications from semi-discretized control problems for PDEs,

- $n = 10^3 - 10^6$ ($\Longrightarrow 10^6 - 10^{12}$ unknowns!),
- A has sparse representation ($A = -M^{-1}K$ for FEM),
- G, W low-rank with $G, W \in \{BB^T, C^T C\}$, where $B \in \mathbb{R}^{n\times m}$, $m \ll n$, $C \in \mathbb{R}^{p\times n}$, $p \ll n$.
- Standard (eigenproblem-based) $\mathcal{O}(n^3)$ methods are not applicable!

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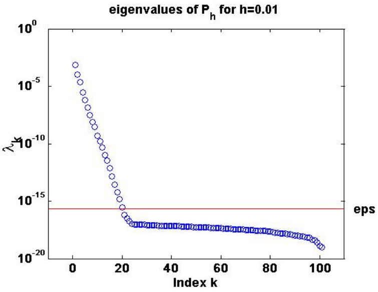
Consider spectrum of ARE solution (analogous for Lyapunov equations).

Example:

- Linear 1D heat equation with point control,
- $\Omega = [0, 1]$,
- FEM discretization using linear B-splines,
- $h = 1/100 \Longrightarrow n = 101$.



Idea: $Q = Q^T \geq 0 \Longrightarrow$

$$Q = ZZ^T = \sum_{k=1}^{n} \lambda_k z_k z_k^T \approx Z^{(r)}(Z^{(r)})^T = \sum_{k=1}^{r} \lambda_k z_k z_k^T.$$

Low-Rank Approximation

ARE $0 = A^T Q + QA - QBB^T Q + CC^T$

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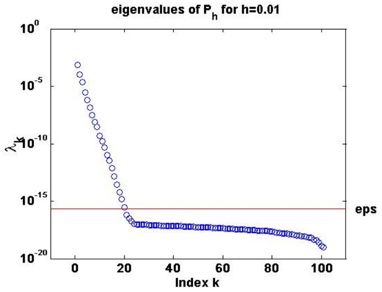

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$$AX + XA^T = -BB^T.$$

- ADI Iteration: [Wachspress 1988]

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with parameters $p_k \in \mathbb{C}^-$ and $p_{k+1} = \overline{p_k}$ if $p_k \notin \mathbb{R}$.

- For $X_0 = 0$ and proper choice of p_k: $\lim_{k\to\infty} X_k = X$ superlinear.

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Lyapunov equation $0 = AX + XA^T + BB^T$.

Setting $X_k = Y_k Y_k^T$, some algebraic manipulations $\Longrightarrow$

Algorithm [Penzl '97/'00, Li/White '99/'02, B. 04, B./Li/Penzl '99/'08]

$$V_1 \leftarrow \sqrt{-2\mathrm{Re}\,(p_1)}(A + p_1 I)^{-1}B, \qquad Y_1 \leftarrow V_1$$

FOR $j = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{Re}\,(p_k)}{\mathrm{Re}\,(p_{k-1})}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1}V_{k-1}\right)$$

$$Y_k \leftarrow \left[\begin{array}{cc} Y_{k-1} & V_k \end{array}\right]$$

$$Y_k \leftarrow \mathrm{rrlq}(Y_k, \tau) \qquad \text{% column compression}$$

At convergence, $Y_{k_{\max}} Y_{k_{\max}}^T \approx X$, where

$$Y_{k_{\max}} = \left[\begin{array}{ccc} V_1 & \ldots & V_{k_{\max}} \end{array}\right], \quad V_k = \left[\,\right] \in \mathbb{C}^{n \times m}.$$

Note: Implementation in real arithmetic possible by combining two steps.

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FOR $j = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{Re}\,(p_k)}{\mathrm{Re}\,(p_{k-1})}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1} V_{k-1}\right)$$

$$Y_k \leftarrow \begin{bmatrix} Y_{k-1} & V_k \end{bmatrix}$$

$$Y_k \leftarrow \mathrm{rrlq}(Y_k, \tau) \qquad \text{\% column compression}$$

At convergence, $Y_{k_{\max}} Y_{k_{\max}}^T \approx X$, where

$$Y_{k_{\max}} = \begin{bmatrix} V_1 & \ldots & V_{k_{\max}} \end{bmatrix}, \qquad V_k = \Big[\,\Big] \in \mathbb{C}^{n \times m}.$$

Note: Implementation in real arithmetic possible by combining two steps.

Factored Galerkin-ADI Iteration

Lyapunov equation $0 = AX + XA^T + BB^T$

Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis $\mathrm{range}(Z)$, $Z \in \mathbb{R}^{n\times r}$, for subspace $\mathcal{Z} \subset \mathbb{R}^n$, $\dim \mathcal{Z} = r$.
2. Set $\hat{A} := Z^T A Z$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A}\hat{X} + \hat{X}\hat{A}^T + \hat{B}\hat{B}^T = 0$.
4. Use $X \approx Z\hat{X}Z^T$.

Examples:

- Krylov subspace methods, i.e., for $m = 1$:

$$\mathcal{Z} = \mathcal{K}(A, B, r) = \mathrm{span}\{B, AB, A^2B, \ldots, A^{r-1}B\}$$

[Jaimoukha/Kasenally '94, Jbilou '02–'08].

- K-PIK [Simoncini '07],

$$\mathcal{Z} = \mathcal{K}(A, B, r) \cup \mathcal{K}(A^{-1}, B, r).$$

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Examples:

- ADI subspace [B./R.-C. Li/Truhar '08]:

$$\mathcal{Z} = \mathrm{colspan}\left[\ V_1, \quad \ldots, \quad V_r\ \right].$$

Note: ADI subspace is rational Krylov subspace [J.-R. Li/White '02].

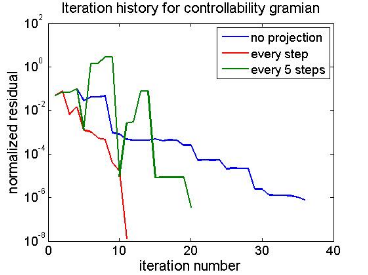
Factored Galerkin-ADI Iteration

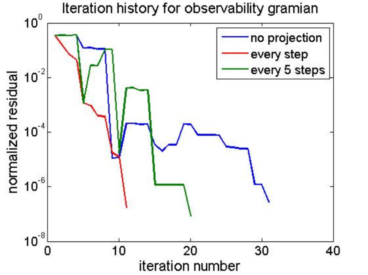
Numerical example

FEM semi-discretized control problem for parabolic PDE:

- optimal cooling of rail profiles ($\rightsquigarrow$ later),
- $n = 20,209,\ m = 7,\ p = 6.$

Good ADI shifts





CPU times: 80s (projection every 5th ADI step) vs. 94s (no projection).

Computations by Jens Saak.

Factored Galerkin-ADI Iteration
Numerical example

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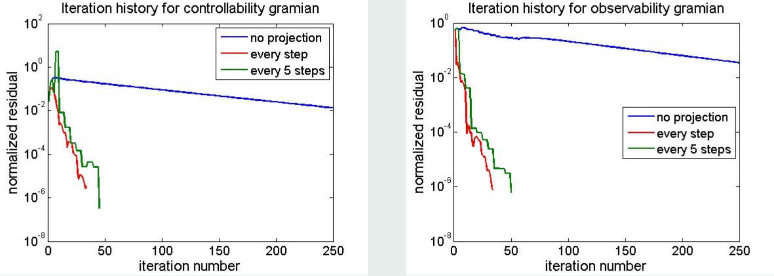

CPU times: 368s (projection every 5th ADI step) vs. 1207s (no projection).

Computations by Jens Saak.

Newton's Method for AREs
[Kleinman '68, Mehrmann '91, Lancaster/Rodman '95, B./Byers '94/'98, B. '97, Guo/Laub '99]

- Consider $0 = \mathcal{R}(Q) = C^T C + A^T Q + QA - QBB^T Q.$
- Frechét derivative of $\mathcal{R}(Q)$ at Q:

$$\mathcal{R}'_Q : Z \to (A - BB^T Q)^T Z + Z(A - BB^T Q).$$

- Newton-Kantorovich method:

$$Q_{j+1} = Q_j - \left(\mathcal{R}'_{Q_j}\right)^{-1} \mathcal{R}(Q_j), \quad j = 0, 1, 2, \ldots$$

Newton's method (with line search) for AREs

FOR $j = 0, 1, \ldots$

1. $A_j \leftarrow A - BB^T Q_j =: A - BK_j$.
2. Solve the Lyapunov equation $A_j^T N_j + N_j A_j = -\mathcal{R}(Q_j)$.
3. $Q_{j+1} \leftarrow Q_j + t_j N_j$.

END FOR j

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Low-Rank Newton-ADI for AREs

Re-write Newton's method for AREs

$$A_j^T N_j + N_j A_j = -\mathcal{R}(Q_j)$$

$$\Longleftrightarrow$$

$$A_j^T \underbrace{(Q_j + N_j)}_{=Q_{j+1}} + \underbrace{(Q_j + N_j)}_{=Q_{j+1}} A_j = \underbrace{-C^T C - Q_j B B^T Q_j}_{=:-W_j W_j^T}$$

Set $Q_j = Z_j Z_j^T$ for $\operatorname{rank}(Z_j) \ll n \Longrightarrow$

$$A_j^T \left(Z_{j+1} Z_{j+1}^T\right) + \left(Z_{j+1} Z_{j+1}^T\right) A_j = -W_j W_j^T$$

Factored Newton Iteration [B./Li/Penzl '99/'08]

Solve Lyapunov equations for Z_{j+1} directly by factored ADI iteration and use 'sparse + low-rank' structure of A_j.

Low-Rank Newton-ADI for AREs

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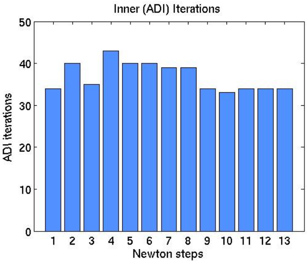
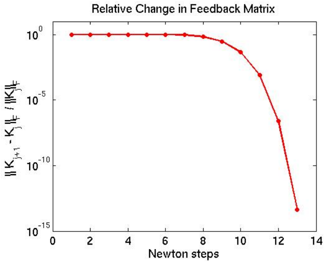
Factored Newton Iteration [B./Li/Penzl '99/'08]

Solve Lyapunov equations for Z_{j+1} directly by factored ADI iteration and use 'sparse + low-rank' structure of A_j.

- Linear 2D heat equation with homogeneous Dirichlet boundary and point control/observation.
- FD discretization on uniform 150×150 grid.
- $n = 22.500$, $m = p = 1$, 10 shifts for ADI iterations.
- Convergence of large-scale matrix equation solvers:





Performance of Newton's method for accuracy $\sim 1/n$

grid	unknowns	$\frac{\|\mathcal{R}(P)\|_F}{\|P\|_F}$	it. (ADI it.)	CPU (sec.)
8×8	2,080	4.7e-7	2 (8)	0.47
16×16	32,896	1.6e-6	2 (10)	0.49
32×32	524,800	1.8e-5	2 (11)	0.91
64×64	8,390,656	1.8e-5	3 (14)	7.98
128×128	134,225,920	3.7e-6	3 (19)	79.46

Here,

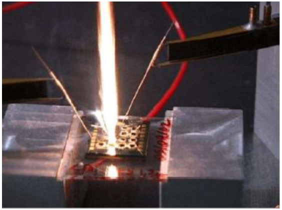
- Convection-diffusion equation,
- $m = 1$ input and $p = 2$ outputs,
- $P = P^T \in \mathbb{R}^{n \times n} \Rightarrow \frac{n(n+1)}{2}$ unknowns.

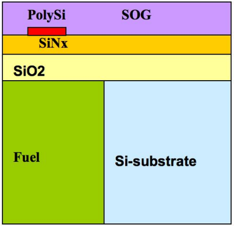
Confirms mesh independence principle for Newton-Kleinman [Burns/Sachs/Zietsmann 2006].

Numerical Examples: Simulation

Microthruster (MEMS)

- Co-integration of solid fuel with silicon micro-machined system.
- Goal: Ignition of solid fuel cells by electric impulse.
- Application: nano satellites.
- Thermo-dynamical model, ignition via heating an electric resistance by applying voltage source.
- Design problem: reach ignition temperature of fuel cell w/o firing neighboring cells.
- Spatial FEM discretization of thermo-dynamical model $\rightsquigarrow$ linear system, $m = 1$, $p = 7$.





Source: The Oberwolfach Benchmark Collection http://www.imtek.de/simulation/benchmark

Courtesy of C. Rossi, LAAS-CNRS/EU project "Micropyros".

Numerical Examples: Simulation

Microthruster (MEMS)

- axial-symmetric 2D model
- FEM discretization using linear (quadratic) elements $\rightsquigarrow$ $n = 4,257$ $(11,445)$ $m = 1$, $p = 7$.
- Reduced model computed using SpaRed, modal truncation using ARPACK, and Z. Bai's PVL implementation.

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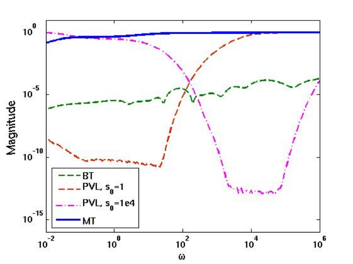
Relative error $n = 4,257$



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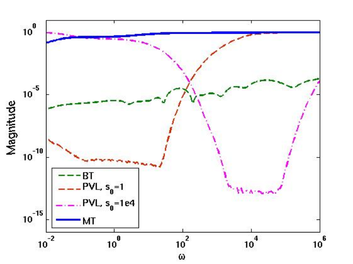
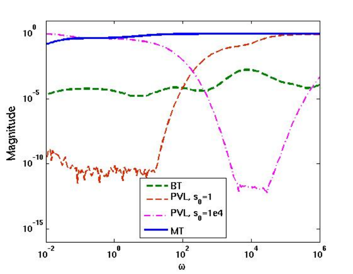


Relative error $n = 11,445$



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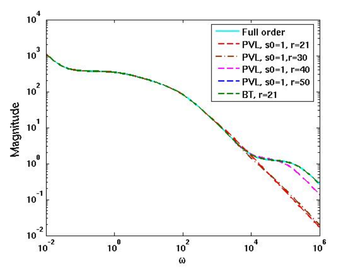
Frequency Response BT/PVL



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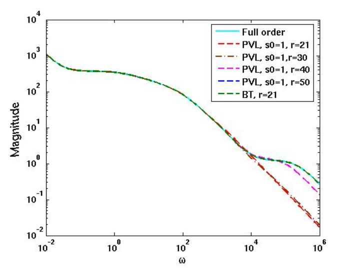
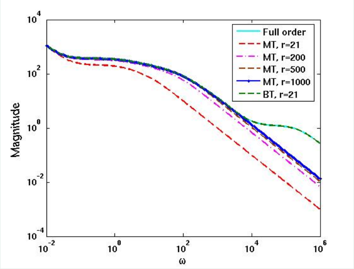


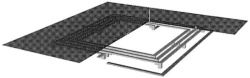
Frequency Response BT/MT



Numerical Examples: Simulation

Spiral Inductor (Micro Electronics)

- Passive device used for RF filters etc.
- $n = 1,434,\ m = 1,\ p = 1.$

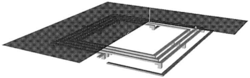


Source: The Oberwolfach Benchmark Collection http://www.imtek.de/simulation/benchmark

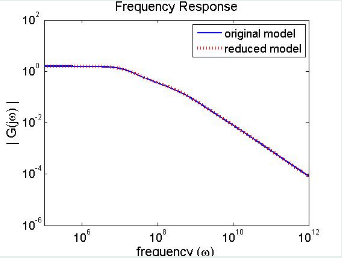
Numerical Examples: Simulation

Spiral Inductor (Micro Electronics)

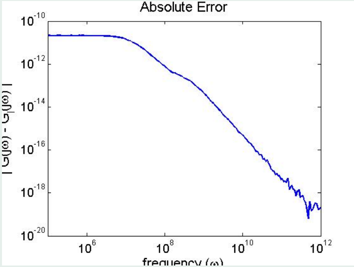
- Passive device used for RF filters etc.
- $n = 1,434$, $m = 1$, $p = 1$.
- Numerical rank of Gramians is 34/41.
- $r = 20$ passive model computed by PRBT (MorLab).



Frequency Response Analysis



Absolute Error



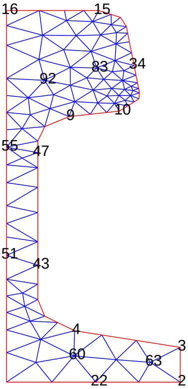
Source: The Oberwolfach Benchmark Collection http://www.imtek.de/simulation/benchmark

- Mathematical model: boundary control for linearized 2D heat equation.

$$
\begin{aligned}
c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, & \xi \in \Omega \\
\lambda \frac{\partial}{\partial n} x &= \kappa (u_k - x), & \xi \in \Gamma_k,\ 1 \leq k \leq 7, \\
\frac{\partial}{\partial n} x &= 0, & \xi \in \Gamma_7.
\end{aligned}
$$

$\Longrightarrow m = 7, p = 6.$

- FEM Discretization, different models for initial mesh ($n = 371$), 1, 2, 3, 4 steps of mesh refinement $\Rightarrow$ $n = 1357, 5177, 20209, 79841$.



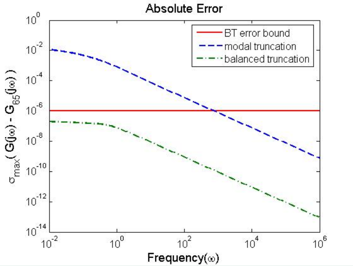
Source: Physical model: courtesy of Mannesmann/Demag.

Math. model: Tröltzsch/Unger '99/'01, Penzl '99, Saak '03.

Numerical Examples: Control

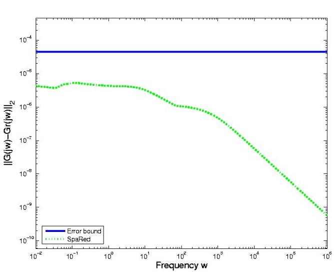
Optimal Cooling of Steel Profiles

$n = 1357$, Absolute Error



- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.

$n = 79841$, Absolute error



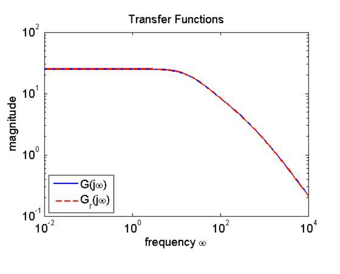
- BT model computed using SpaRed,
- computation time: 8 min.

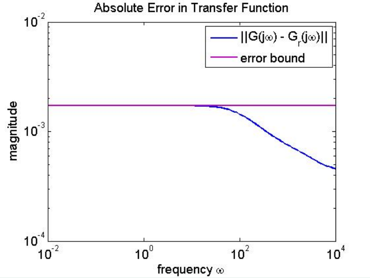
Numerical Examples: Control

2D Heat Control

- FD discretized linear 2D heat equation with homogeneous Dirichlet boundary and point control/observation.
- $n = 22.500$, $m = p = 1$.
- Computed reduced-order model (BT): $r = 6$, BT error bound $\delta = 1.7 \cdot 10^{-3}$.
- Solve LQR problem: quadratic cost functional, solution is linear state feedback.

Transfer function approximation





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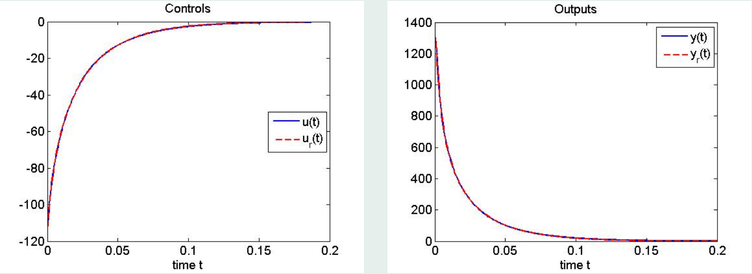
Computed controls and outputs (implicit Euler)



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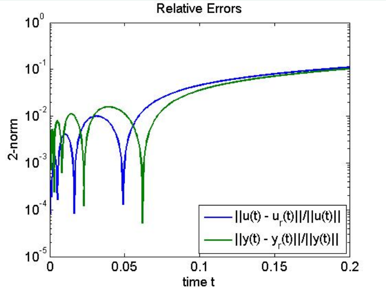
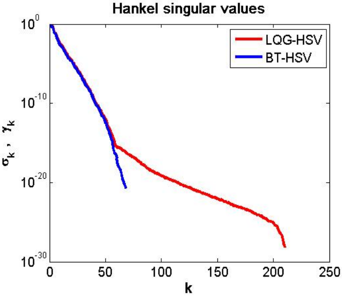
Errors in controls and outputs



Numerical Examples: Control

BT vs. LQG BT

- Boundary control problem for 2D heat flow in copper on rectangular domain; control acts on two sides via Robins BC.
- FDM $\rightsquigarrow$ $n = 4496$, $m = 2$; 4 sensor locations $\rightsquigarrow$ $p = 4$.
- Numerical ranks of BT Gramians are 68 and 124, respectively, for LQG BT both have rank 210.
- Computed reduced-order model: $r = 10$.



Source: *COMPl$_e$ib* v1.1, `www.compleib.de`.

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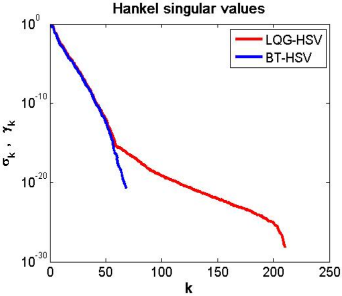

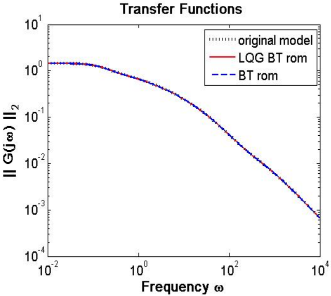

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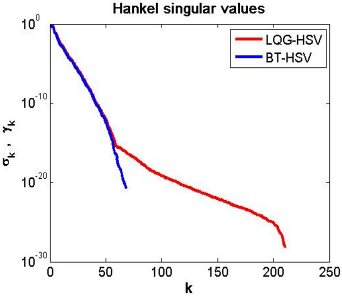

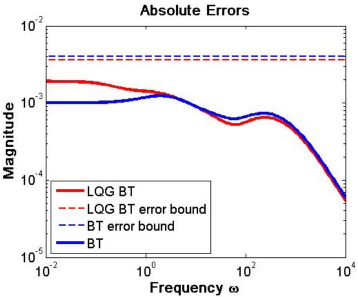

Source: *COMPl_eib* v1.1, `www.compleib.de`.

Conclusions and Outlook

- Main message:

  Balanced truncation and family are applicable to large-scale systems.

  (If efficient numerical algorithms are employed.)
- Applications: nanoelectronics, microsystems technology, optimal control, machine tool design, systems biology, ...
- Efficiency of numerical algorithms can be further enhanced, several details require deeper investigation.
- Algorithms for data-sparse systems using formatted arithmetic for $\mathcal{H}$-matrices [Baur/B. '06/'08].
- Application to 2nd order systems $\rightsquigarrow$ talk of Jens Saak.
- Extension to descriptor systems possible.

  [Stykel since '02, B. 03/'08, Freitas/Martins/Rommes '08, Heinkenschloss/Sorensen/Sun '06/'08].
- Combination of BT with sparse grid interpolation for parametric model reduction [Baur/B. '08/'09].

Conclusions and Outlook

- Main message:

  Balanced truncation and family are applicable to large-scale systems.

  (If efficient numerical algorithms are employed.)
- Applications: nanoelectronics, microsystems technology, optimal control, machine tool design, systems biology, ...
- Efficiency of numerical algorithms can be further enhanced, several details require deeper investigation.
- Extension to nonlinear systems employing Carleman bilinearization and tensor product structure of Krylov subspaces in combination with balanced truncation for bilinear systems [*B./Damm '09*] quite promising, in particular for polynomial nonlinearities as often encountered in biological systems.
- Theory and numerical algorithm for application to stochastic systems: [*B./Damm '09*]; need algorithmic enhancements for really large-scale problems.

Support

BMBF research network **SyreNe**

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System Reduction for Nanoscale IC Design

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Support

DFG Projects

- Automatic, Parameter-Preserving Model Reduction for Applications in Microsystems Technology.
  Jointly with *Jan Gerrit Korvink* (IMTEK/U Freiburg and FRIAS).
- Integrated Simulation of the System "Machine Tool – Drive System – Stock Removal Process" Using Reduced-Order Structural FE Models.
  Jointly with *Michael Zäh* (iwb/TU München) and *Heike Faßbender* (ICM/TU Braunschweig).