MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Parametric Model Order Reduction for Electro-Thermal Simulation in Nanoelectronics

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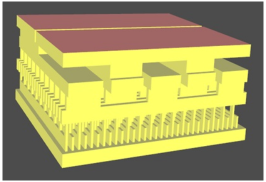
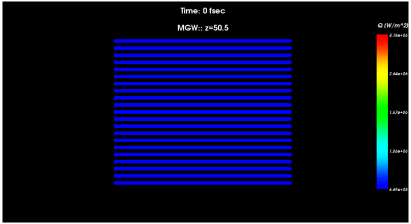
Electro-thermal (ET) Simulation

- (Self-)heating in micro-and nano-electronics is crucial and needs to be limited by design.
- Electro-thermal (ET) simulation is used to study the interaction between the electrical and thermal dynamics of the system.



A Power-MOS device model.

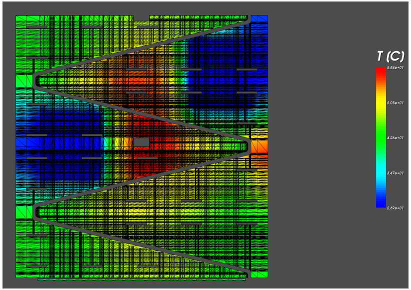


Evolution of the heat flux on the first metal layer.

Electro-thermal (ET) Simulation

$$\text{Electrical:} \quad \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0,$$
$$J = \sigma \vec{E}, \quad \vec{E} = -\nabla \varphi,$$
$$\rho = -\nabla \cdot (\epsilon \nabla \varphi).$$

$$\text{Thermal:} \quad \nabla \cdot \vec{\phi}_q + \frac{\partial w(T)}{\partial t} = \vec{E} \cdot \vec{J},$$
$$\vec{\phi}_q = -\kappa \nabla T,$$
$$w(T) = C_T (T - T_{\text{ref}}).$$



After spatial discretization

$$\begin{aligned} A_E(p) x_E &= -B_E(p) u_E(t), \\ E_T(p) \dot{x}_T &= A_T(p) x_T + B_T(p) u_T(t) + F(p) \times_2 x_E \times_3 x_E, \\ x_T|_{t=0} &= x_T^0, \qquad x_E|_{t=0} = x_E^0, \\ y &= C_E(p) x_E + C_T(p) x_T + D(p) [u_E^T, u_T^T]^T. \end{aligned}$$

Electro-thermal (ET) Simulation

Electrical:
$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0,$$
$$J = \sigma \vec{E}, \quad \vec{E} = -\nabla \varphi,$$
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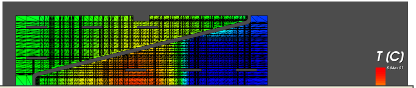


large-scale;
parametrized;
coupled;
weakly nonlinear;
multiple-input and multiple-output.

After spatial discretization

$$\begin{aligned} A_E(p) x_E &= -B_E(p) u_E(t), \\ E_T(p) \dot{x}_T &= A_T(p) x_T + B_T(p) u_T(t) + F(p) \times_2 x_E \times_3 x_E, \\ x_T|_{t=0} &= x_T^0, \qquad x_E|_{t=0} = x_E^0, \\ y &= C_E(p) x_E + C_T(p) x_T + D(p) [u_E^T, u_T^T]^T. \end{aligned}$$

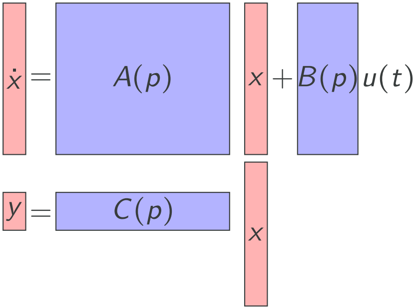
Electro-thermal (ET) Simulation

Basic idea

- Consider a simple parametrized system:

$$\dot{x} = A(p)x + B(p)u(t),$$
$$y = C(p)x.$$

Full-order model (FOM)



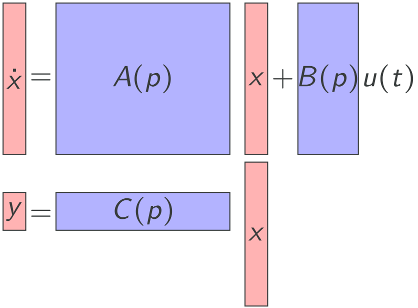
Basic PMOR Concept

Basic idea

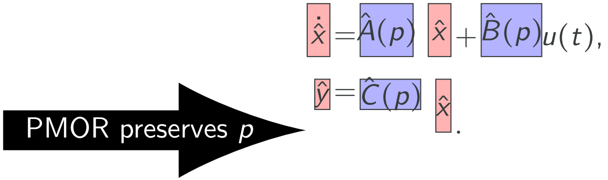
- Consider a simple parametrized system:

$$\dot{x} = A(p)x + B(p)u(t),$$
$$y = C(p)x.$$

Full-order model (FOM)



Reduced-order model (ROM)



- such that $y \approx \hat{y} \;\forall\; p$.
- $x \in \mathbb{R}^n$ is much shorter than $\hat{x} \in \mathbb{R}^r$, i.e. $r \ll n$.

Basic idea

Full-order model (FOM)

$$\dot{x} = A(p)\,x + B(p)\,u(t),$$

$$y = C(p)\,x.$$

PMOR preserves p

Graphical illustration

$p = (d, \theta)$

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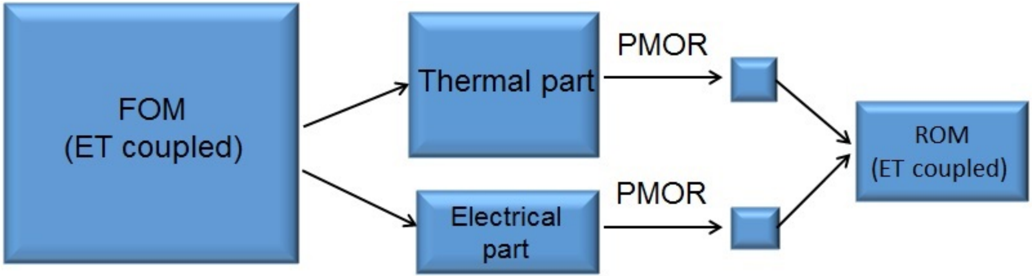
Basic PMOR Concept

Dealing with coupling



PMOR methods overview

See [B./Gugercin/Willcox'15] for a survey

- Interpolatory methods.
- Proper orthogonal decomposition method.
- Reduced basis method.
- Multi-moment-matching method.

Basic PMOR Concept

PMOR methods overview

See [B./Gugercin/Willcox'15] fo

- Interpolatory methods.
- Proper orthogonal decompos
- Reduced basis method.
- Multi-moment-matching method. $\rightsquigarrow$ Our choice.

more flexible for system with varying inputs;
computationally efficient for linear systems;
error bound $\rightsquigarrow$ reliable.

Brief review

For dynamical systems:

$$\begin{aligned} E(p)\dot{x}(t) &= A(p)x(t) + B(p)u(t), \\ y(t) &= C(p)x(t). \end{aligned}$$

Brief review

For dynamical systems:

$$\begin{aligned} E(p)\dot{x}(t) &= A(p)x(t) + B(p)u(t), \\ y(t) &= C(p)x(t). \end{aligned} \quad \xrightarrow{\text{Laplace transform}} \quad \begin{aligned} G(\mu)x(\mu) &= B(\mu)u(\mu), \\ y(\mu) &= C(\mu)x(\mu), \\ &\mu = (p, s). \end{aligned}$$

Transfer function: $H(\mu) = y(\mu)/u(\mu) = C(\mu)x(\mu)/u(\mu) = C(\mu)[G(\mu)]^{-1}B(\mu).$

Multi-Moment-Matching PMOR

Brief review

For dynamical systems:

$$\begin{array}{rcl} E(p)\dot{x}(t) & = & A(p)x(t) + B(p)u(t), \\ y(t) & = & C(p)x(t). \end{array} \quad \xrightarrow{\text{Laplace transform}} \quad \begin{array}{rcl} G(\mu)x(\mu) & = & B(\mu)u(\mu), \\ y(\mu) & = & C(\mu)x(\mu), \\ & & \mu = (p, s). \end{array}$$

Transfer function: $H(\mu) = y(\mu)/u(\mu) = C(\mu)x(\mu)/u(\mu) = C(\mu)[G(\mu)]^{-1}B(\mu).$

For steady systems:

$$\begin{array}{rcl} G(\mu)x(\mu) & = & B(\mu), \\ y(\mu) & = & C(\mu)x(\mu), \quad \mu := p. \end{array}$$

For simplicity, we assume that $G(\mu)$ and $B(\mu)$ have affine structures,

$$G(\mu) = G_0 + \mu_1 G_1 + \ldots + \mu_m G_m, \quad B(\mu) = B_0 + \mu_1 B_1 + \ldots + \mu_k B_k.$$

Brief review

Consider the solution $x(\mu)$ ($u(\mu)$ disappears in the steady case),

$$x(\mu) = [G(\mu)]^{-1}B(\mu)\bar{u}(\mu).$$

$x(\mu)$ can be expanded into power series about an expansion point [Daniel et al.' 04] $\mu^0 = (\mu_1^0, \ldots, \mu_m^0)$,

$$\begin{aligned} x(\mu) &= \sum_{i=0}^{\infty} (\sigma_1 M_1 + \ldots + \sigma_m M_m)^i B_M u(\mu) \\ &\approx \sum_{i=0}^{q} (\sigma_1 M_1 + \ldots + \sigma_m M_m)^i B_M u(\mu), \end{aligned}$$

where $\sigma_i = \mu_i - \mu_i^0, i = 1, 2, \ldots, p,\ M_i = -[G(\mu^0)]^{-1}G_i, i = 1, \ldots, m,$
$B_M = [G(\mu^0)]^{-1}[B_1, \ldots, B_k]$.

Multi-Moment-Matching PMOR

Brief review

Since

$$x(\mu) \approx \sum_{i=0}^{q} (\sigma_1 M_1 + \ldots + \sigma_m M_m)^i B_M u(\mu),$$

$x(\mu) \approx \hat{x}(\mu) \in \mathrm{span}\{B_M, R_1, \ldots, R_q\}$.

Parameter independent terms B_M, R_i, $i = 1, \ldots, q$, satisfy recursion [Feng/B. '07/'14]:

$$\begin{aligned} R_1 &= (M_1, \ldots, M_m) B_M \quad (i = 1), \\ &\vdots \\ R_q &= (M_1, \ldots, M_m) R_{q-1} \quad (i = q). \end{aligned}$$

$$\mathrm{range}(V_{\mu^0}) = \mathrm{span}\{B_M, R_1, \ldots, R_q\}.$$

Multi-Moment-Matching PMOR

Brief review

The ROM can be obtained by Galerkin projection,

dynamical systems

$$\begin{aligned} V_{\mu^0}^T E(p) V_{\mu^0} \frac{dz}{dt} &= V_{\mu^0}^T A(p) V_{\mu^0} z + V_{\mu^0}^T B(p) u(t), \\ \hat{y}(t) &= C(p) V_{\mu^0} z. \end{aligned}$$

steady systems

$$\begin{aligned} V_{\mu^0}^T G(\mu) V_{\mu^0} z &= V_{\mu^0}^T B(\mu), \\ y(\mu) &= C(\mu) V_{\mu^0} z. \end{aligned}$$

- The leading multi-moments $CB_M, CR_i, i = 1, \ldots, q$, (coefficients in the series expansion) of the transfer function $H(\mu)$ are matched by the transfer function $\hat{H}(\mu)$ of the ROM: multi-moment matching.
- For steady systems, $y(\mu)$ plays the role of $H(\mu)$.
- If there are more than three parameters, multiple-point expansion is needed.

Multi-Moment-Matching PMOR

Brief review

Multiple-point expansion: given $\mu^i, i = 1, \dots, l$:

- For each expansion point μ^i, we compute a matrix

$$\mathrm{range}(V_{\mu^i}) = \mathrm{span}\{B_M, R_1, \dots, R_{\tilde{q}}\}, \;\; \tilde{q} \ll q.$$

- The ROM is obtained via $V = \mathrm{orth}\{V_{\mu^1}, \dots, V_{\mu^l}\}$,

$$\begin{aligned} V^T E(p) V \tfrac{dz}{dt} &= V^T A(p) V z + V^T B(p) u(t), \\ \hat{y}(t) &= C(p) V z. \end{aligned} \quad \text{or} \quad \begin{aligned} V^T G(\mu) V z &= V^T B(\mu), \\ y(\mu) &= C(\mu) V z. \end{aligned}$$

How to adaptively choose μ^i?

$\Delta(\mu)$: $|H(\mu) - \hat{H}(\mu)| \leq \Delta(\mu)$ or $|y(\mu) - \hat{y}(\mu)| \leq \Delta(\mu)$ can guide the selection of μ^i. $\rightsquigarrow$

- Reliable ROM.
- Automatic generation of the ROM.

Error bound formulation

Theorem [Feng/Antoulas/B. '15]

Assume that $\sigma_{\min}(G(\mu)) =: \beta(\mu) > 0 \quad \forall\, \mathrm{Re}(s) \geq 0, \forall\, p \in \mathbb{D}$ (recall: $\mu = (p, s)$), then

for dynamical systems:

$$|H(\mu) - \hat{H}(\mu)| \leq \tilde{\Delta}(\mu) + |e(\mu)| =: \Delta(\mu),$$

for steady systems:

$$|y(\mu) - \hat{y}(\mu)| \leq \tilde{\Delta}(\mu).$$

Here, $\Delta(\mu) := \frac{||r^{du}(\mu)||_2 ||r^{pr}(\mu)||_2}{\beta(\mu)}$.

Note: $r^{du}(\mu), r^{pr}(\mu)$, and $e(\mu)$ can be efficiently computed.

Extension to MIMO case possible taking max over all I/O channels.

Automatic generation of the ROM: adaptively select μ^i

Algorithm 1 Automatic generation of the ROM: adaptively select μ^i

Input: $V = []$; $\epsilon > \epsilon_{tol}$; Initial expansion point: $\hat{\mu}$; $i = -1$;
Ξ_{train}: a set of samples of μ covering the parameter domain.

Output: V.

WHILE $\epsilon > \epsilon_{tol}$

$i = i + 1$;

$\mu^i = \hat{\mu}$;

$V_{\mu^i} = span\{R_0, \ldots, R_{\tilde{q}}\}$;

$V = [V, V_{\mu^i}]$;

$\hat{\mu} = \arg\max_{\mu \in \Xi_{train}} \Delta(\mu)$;

$\epsilon = \Delta(\hat{\mu})$;

END WHILE

Automatic PMOR for ET coupled systems

Recall: ET coupled system after spatial discretization

$$\begin{cases} A_E(p)x_E(t) = -B_E(p)u_E(t), & (1a) \\ E_T(p)\dot{x}_T(t) = A_T(p)x_T(t) + B_T(p)u_T(t) + F(p) \times_2 x_E(t) \times_3 x_E(t), & (1b) \\ y(t) = C_E(p)x_E(t) + C_T(p)x_T(t) + D(p)[u_E(t)^T, u_T(t)^T]^T. & (1c) \end{cases}$$

- Coupling term, $F(p) \times_2 x_E \times_3 x_E$: quadratic.
- Apply Algorithm 1 to (1a) to generate V_E.
- For (1b), apply Algorithm 1 only to the **linear part** to generate V_T:

$$E_T(p)\dot{x}_T = A_T(p)x_T + B_T(p)u_T(t).$$

Automatic PMOR for ET coupled systems

Recall: ET coupled system after spatial discretization

$$\begin{cases} A_E(p)x_E(t) = -B_E(p)u_E(t), & (1a) \\ E_T(p)\dot{x}_T(t) = A_T(p)x_T(t) + B_T(p)u_T(t) + F(p) \times_2 x_E(t) \times_3 x_E(t), & (1b) \\ y(t) = C_E(p)x_E(t) + C_T(p)x_T(t) + D(p)[u_E(t)^T, u_T(t)^T]^T. & (1c) \end{cases}$$

ROM: coupled again

$$\begin{cases} V_E^T A_E(p) V_E z_E = -V_E^T B_E(p) u_E, & (2a) \\ V_T^T E_T(p) V_T \dot{z}_T = V_T^T A_T(p) V_T z_T + V_T^T B_T(p) u_T + V_T^T F(p) \times_2 V_E z_E \times_3 V_E z_E, & (2b) \\ y = C_E(p) V_E z_E + C_T(p) V_T z_T + D(p)[u_E^T, u_T^T]^T. & (2c) \end{cases}$$

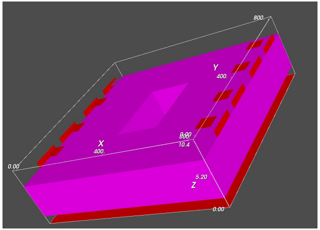
Automatic PMOR for ET coupled systems: a package model

The parameter is chosen to be the top layer thickness $h(\mu m)$ of the package.

- Finite-integration technique (FIT) leads to thermal fluxes that are proportional to the dual areas of the mesh cells and inversely proportional to the lengths of the edges in the mesh cells.
- Considering meshes that are topologically equivalent for different package thicknesses, the system matrices take the parametric form

$$M(h) = M_0 + hM_1 + \frac{1}{h}M_2,$$

$$(M = A_E, B_E, E_T, A_T, B_T, F, C_E, C_T, D \text{ from } (1)).$$



A package model.

Automatic PMOR for ET coupled systems: a package model

- Electrical subsystem: $x_E \in \mathbb{R}^{n_E}$, $n_E = 1,122$.
- Thermal subsystem: $x_T \in \mathbb{R}^{n_T}$, $n_T = 8,071$.
- A MIMO ET-coupled system: number of inputs: 34, number of outputs: 68.
- Feasible parameter domain: $h \in (0,100]\mu m$, frequency domain $f \in [0,10^2]Hz$.

Using the pMOR method proposed, $n_E = 1,122 \rightsquigarrow r_E = 68$, $n_T = 8,071 \rightsquigarrow r_T = 606$.

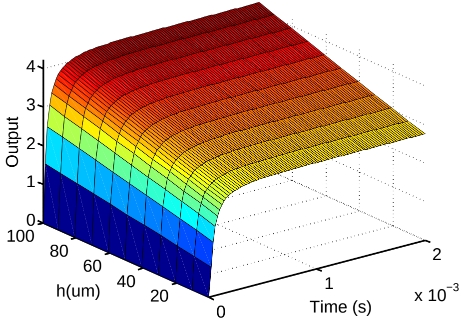
Automatic PMOR for ET coupled systems: a package model

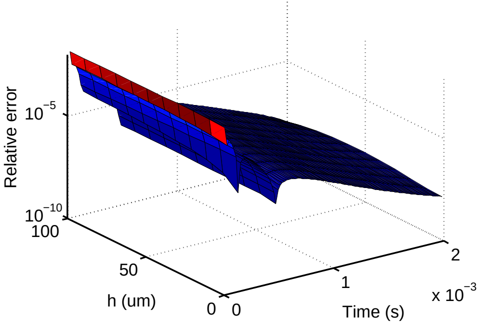
Convergence behavior of Algorithm 1 for the package model ($\epsilon_{tol} = 10^{-3}$).

	Electrical sub-system		Thermal sub-system	
Iteration	Selected sample h	Error bound	Selected sample (h, s)	Error bound
1	1.0×10^0	2.1×10^3	$(7.591, 8.1339)$	7.3×10^6
2	1.0×10^2	3.7×10^0	$(2.9653 \times 10^1, 4.1065 \times 10^1)$	2.3×10^1
3	9.0×10^1	6.6×10^{-2}	$(1.5121 \times 10^1, 1.7494 \times 10^1)$	1.3×10^{-1}
4	8.0×10^1	6.4×10^{-3}	$(4.6942, 1.6455 \times 10^1)$	7.8×10^{-5}
5	7.0×10^1	5.3×10^{-3}	—	—
6	6.0×10^1	4.2×10^{-3}	—	—
7	5.0×10^1	3.1×10^{-3}	—	—
8	4.0×10^1	1.8×10^{-3}	—	—
9	3.0×10^1	8.9×10^{-4}	—	—

Automatic PMOR for ET coupled systems: a package model

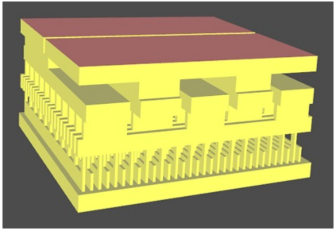
- Output response in time domain: thermal flux at port 36.
- Maximal relative error is below 1×10^{-2}.





Automatic PMOR for ET coupled systems: a power-MOS model

- Commonly used in energy harvesting, where energy from external sources is collected in order to power small devices, e.g., implanted biosensors [Spirito, et al. '02].
- The conductivity (S/m) of the third metal layer σ is chosen to be the parameter.
- FIT assembles fluxes that are proportional to the conductivity of each mesh cell material, so that



A power-MOS model.

$$M(\sigma) = M_0 + \sigma M_1, \quad (M = A_E, B_E, E_T, A_T, B_T, F, C_E, C_T, D).$$

Automatic PMOR for ET coupled systems: a power-MOS model

- Electrical subsystem: $x_E \in \mathbb{R}^{n_E}$, $n_E = 1,160$.
- Thermal subsystem: $x_T \in \mathbb{R}^{n_T}$, $n_T = 11,556$.
- A MIMO ET-coupled system: number of inputs: 6, number of outputs: 12.
- Feasible parameter domain: $\sigma \in [10^7, 5 \times 10^7] S/m$, frequency domain $f \in [0, 10^6] Hz$.

Using the pMOR method proposed, $n_E = 1,160 \rightsquigarrow r_E = 2$, $n_T = 11,556 \rightsquigarrow r_T = 35$.

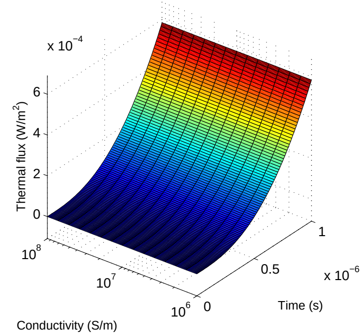
Automatic PMOR for ET coupled systems: a power-MOS model

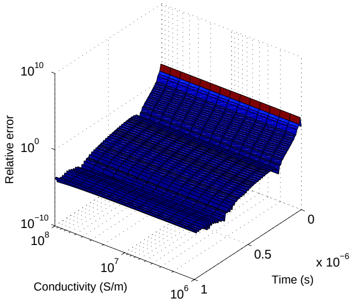
Convergence behavior of Algorithm 1 for the power-MOS model ($\epsilon_{tol} = 10^{-12}$).

	Electrical sub-system		Thermal sub-system	
Iteration	Selected sample σ	Error bound	Selected sample (σ, s)	Error bound
1	10^7	7.165399×10^{-24}	$(2.736 \times 10^7, 0)$	43.73
2	—	—	$(2.537 \times 10^7, 10^6)$	4.225×10^{-4}
3	—	—	$(1.694 \times 10^7, 2.632 \times 10^5)$	4.345×10^{-8}
4	—	—	$(2.687 \times 10^7, 5.790 \times 10^5)$	9.774×10^{-11}
5	—	—	$(2.836 \times 10^7, 5.263 \times 10^4)$	4.041×10^{-13}

Automatic PMOR for ET coupled systems: a power-MOS model

- Output response in time domain: output at port 7, thermal flux at the drain.
- The relative error is large in the beginning because the thermal flux is still very close to zero (the circuit is hardly heated up).The ROM approximates the thermal flux accurately after the thermal flux dominates the numerical error ($t > 2 \times 10^{-7}$).





UQ results for the outputs at $t = 10^{-6}$ s.

LHS: latin hypercube sampling, SC: stochastic collocation.

	LHS (FOM)	LHS (ROM)	SC (FOM)	SC (ROM)
$E(I_{drain})$	7.4621e-04	7.4621e-04	7.4602e-04	7.4602e-04
$\sigma(I_{drain})$	2.4794e-04	2.4794e-04	2.4867e-04	2.4867e-04
$E(I_{source})$	-7.4621e-04	-7.4621e-04	-7.4602e-04	-7.4602e-04
$\sigma(I_{source})$	2.4794e-04	2.4794e-04	2.4867e-04	2.4867e-04
$E(I_{back})$	0	0	0	0
$\sigma(I_{back})$	0	0	0	0
$E(\phi_{drain})$	5.8479e-04	5.8478e-04	5.8479e-04	5.8479e-04
$\sigma(\phi_{drain})$	1.5838e-10	1.5677e-10	1.5985e-10	1.5719e-10
$E(\phi_{source})$	4.1977e-04	4.1975e-04	4.1977e-04	4.1977e-04
$\sigma(\phi_{source})$	1.8528e-10	9.1986e-11	4.6370e-11	9.2124e-11
$E(\phi_{back})$	6.6781e-07	6.6773e-07	6.6781e-07	6.6781e-07
$\sigma(\phi_{back})$	1.5682e-14	1.7778e-14	1.1199e-14	1.6189e-14
CPU time	6001.14 s	94.19 s	733.64 s	30.51 s

(P)MOR for electrical subsystem

FOM

$$\begin{aligned} A_E(p)x_E &= -B_E(p)u_E(t), \\ y_E &= C_E x_E. \end{aligned}$$

(P)MOR for electrical subsystem

FOM

$$\begin{aligned} A_E(p)x_E &= -B_E(p)u_E(t), \\ y_E &= C_E x_E. \end{aligned}$$

standard PMOR

((multi-)moment-matching)

$\longrightarrow$

ROM: dense

$A_{E_r}(p)$

Sparse (P)MOR for electrical subsystem based on superposition principle

E-subsystem

$$\begin{aligned} A_E(p)x_E &= -B_E(p)u_E(t), \qquad u_E(t) \in \mathbb{R}^{m_E}, \quad m_E \gg 10. \\ y_E &= C_E x_E. \end{aligned}$$

Sparse (P)MOR for electrical subsystem based on superposition principle

E-subsystem

$$\begin{aligned} A_E(p)x_E &= -B_E(p)u_E(t), \qquad u_E(t) \in \mathbb{R}^{m_E}, \quad m_E \gg 10. \\ y_E &= C_E x_E. \end{aligned}$$

⇕ **superposition principle**

Equivalent E-subsystem in block-diagonal structure

$$\begin{pmatrix} A_E & 0 & \cdots & 0 \\ 0 & A_E & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_E \end{pmatrix} \begin{pmatrix} x_{E_1} \\ x_{E_2} \\ \vdots \\ x_{E_{m_E}} \end{pmatrix} = -\begin{pmatrix} b_{E_1} & 0 & \cdots & 0 \\ 0 & b_{E_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & b_{E_{m_E}} \end{pmatrix} u_E,$$

$$y_E = C_E x_{E_1} + \ldots + C_E x_{E_{m_E}} \qquad (\text{where } B_E = (b_{E_1}, \ldots, b_{E_m})\ ).$$

Sparse (P)MOR based on superposition principle

Equivalent FOM

$$\begin{pmatrix} A_E & 0 & \cdots & 0 \\ 0 & A_E & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_E \end{pmatrix} \begin{pmatrix} x_{E_1} \\ x_{E_2} \\ \vdots \\ x_{E_{m_E}} \end{pmatrix} = - \begin{pmatrix} b_{E_1} & 0 & \cdots & 0 \\ 0 & b_{E_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & b_{E_{m_E}} \end{pmatrix} u_E(t),$$

$$y_E = C_E x_{E_1} + \ldots + C_E x_{E_{m_E}}.$$

Sparse (P)MOR based on superposition principle

Equivalent FOM

$$\begin{pmatrix} A_E & 0 & \cdots & 0 \\ 0 & A_E & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_E \end{pmatrix} \begin{pmatrix} x_{E_1} \\ x_{E_2} \\ \vdots \\ x_{E_{m_E}} \end{pmatrix} = -\begin{pmatrix} b_{E_1} & 0 & \cdots & 0 \\ 0 & b_{E_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & b_{E_{m_E}} \end{pmatrix} u_E(t),$$

$$y_E = C_E x_{E_1} + \ldots + C_E x_{E_{m_E}}.$$

Blcok-wise sparse PMOR

$$\begin{pmatrix} V_{E_1}^T A_E V_{E_1} & 0 & \cdots & 0 \\ 0 & V_{E_2}^T A_E V_{E_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & V_{E_m}^T A_E V_{E_m} \end{pmatrix} \begin{pmatrix} x_{E_{1r}} \\ x_{E_{2r}} \\ \vdots \\ x_{E_{m_{Er}}} \end{pmatrix} = -\begin{pmatrix} V_{E_1}^T b_{E_1} & 0 & \cdots & 0 \\ 0 & V_{E_2}^T b_{E_2 r} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & V_{E_m}^T b_{E_{m_E}} \end{pmatrix} u_E,$$

$$y_{Er} = C_E V_{E_1} x_{E_{1r}} + \ldots + C_E V_{E_m} x_{E_{m_{Er}}}.$$

Sparse (P)MOR based on superposition principle

Equivalent FOM

$$\begin{pmatrix} A_E & 0 & \cdots & 0 \\ 0 & A_E & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_E \end{pmatrix} \begin{pmatrix} x_{E_1} \\ x_{E_2} \\ \vdots \\ x_{E_{m_E}} \end{pmatrix} = - \begin{pmatrix} b_{E_1} & 0 & \cdots & 0 \\ 0 & b_{E_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & b_{E_{m_E}} \end{pmatrix} u_E(t),$$
$$y_E = C_E x_{E_1} + \ldots + C_E x_{E_{m_E}}.$$

Blcok-wise sparse PMOR

$$\begin{pmatrix} V_{E_1}^T A_E V_{E_1} & 0 & \cdots & 0 \\ 0 & V_{E_2}^T A_E V_{E_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & V_{E_m}^T A_E V_{E_m} \end{pmatrix} \begin{pmatrix} x_{E_{1r}} \\ x_{E_{2r}} \\ \vdots \\ x_{E_{m_{Er}}} \end{pmatrix} = - \begin{pmatrix} V_{E_1}^T b_{E_1} & 0 & \cdots & 0 \\ 0 & V_{E_2}^T b_{E_2 r} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & V_{E_m}^T b_{E_{m_E}} \end{pmatrix} u_E,$$
$$y_{Er} = C_E V_{E_1} x_{E_{1r}} + \ldots + C_E V_{E_m} x_{E_{m_{Er}}}.$$

Sparse (P)MOR based on superposition principle

Block-wise sparse PMOR

$$\begin{pmatrix} V_{E_1}^T A_E V_{E_1} & 0 & \cdots & 0 \\ 0 & V_{E_2}^T A_E V_{E_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & V_{E_m}^T A_E V_{E_m} \end{pmatrix} \begin{pmatrix} x_{E_{1r}} \\ x_{E_{2r}} \\ \vdots \\ x_{E_{m_{Er}}} \end{pmatrix} = - \begin{pmatrix} V_{E_1}^T b_{E_1} & 0 & \cdots & 0 \\ 0 & V_{E_2}^T b_{E_2r} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & V_{E_m}^T b_{E_{m_E}} \end{pmatrix} u_E,$$

$$y_{Er} = C_E x_{E_{1r}} + \ldots + C_E x_{E_{m_{Er}}}.$$

V_{E_i} is constructed from the ith **SI**MO system, using, e.g., (multi-)moment-matching

$$\begin{aligned} A_E(p) x_{E_i} &= -b_{E_i} u_{E_i}(t), \\ y_{E_i} &= C_E x_{E_i}, \qquad i = 1, \ldots, m_E, \\ u_E(t) &= (u_{E_1}(t), \ldots, u_{E_m}(t))^T. \end{aligned}$$

(P)MOR for thermal subsystem

Thermal subsystem

$$\begin{aligned} E_T(p)\dot{x}_T &= A_T(p)x_T + B_T(p)u_T(t) + F(p) \times_2 x_E \times_3 x_E, \\ y &= C_T(p)x_T, \qquad u_T(t) \in \mathbb{R}^{m_T}, \quad m_T \gg 10. \end{aligned}$$

$$\Downarrow$$

T-subsystem after E-subsystem is reduced

$$\begin{aligned} E_T(p)\dot{x}_T &\approx A_T(p)x_T + B_T(p)u_T(t) + \mathcal{F}_r(p) \times_2 \xi_{Er} \times_3 \xi_{Er}, \\ y &= C_T(p)x_T. \end{aligned}$$

(P)MOR for thermal subsystem

T-subsystem after E-subsystem is reduced

$$\begin{aligned} E_T(p)\dot{x}_T &= A_T(p)x_T + \underbrace{B_T(p)u_T(t) + \mathcal{F}_r(p) \times_2 \xi_{Er} \times_3 \xi_{Er}}_{\text{new input}}, \\ y &= C_T(p)x_T. \end{aligned}$$

⇓ **Superposition principle**

Equivalent block-diagonal T-subsystem

$$\begin{pmatrix} E_T & 0 \\ 0 & \mathcal{E}_{T_I} \end{pmatrix} \begin{pmatrix} \dot{x}_{T_1} \\ \dot{x}_{T_I} \end{pmatrix} = \begin{pmatrix} A_T & 0 \\ 0 & \mathcal{A}_{T_I} \end{pmatrix} \begin{pmatrix} x_{T_1} \\ x_{T_I} \end{pmatrix} + \begin{pmatrix} \text{tensor part} + b_{T_1} u_{T_1} & \\ & \mathcal{B}_{T_I} u_{T_I} \end{pmatrix},$$

$$y_T = (C_T, \mathcal{C}_{T_I}) \begin{pmatrix} x_{T_1} \\ x_{T_I} \end{pmatrix}, \qquad \begin{aligned} B_T &= (b_{T_1}, \ldots, b_{T_{m_T}}), \\ \mathcal{B}_{T_I} &= \text{blkdiag}(b_{T_2}, \ldots, b_{T_{m_T}}). \end{aligned}$$

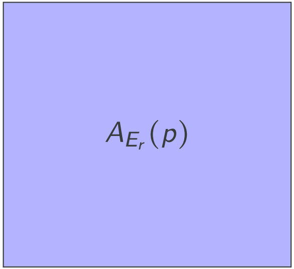
Results for a power-cell model

- Electrical subsystem: $x_E \in \mathbb{R}^{n_E}$, $n_E = 392,773$.
- Thermal subsystem: $x_T \in \mathbb{R}^{n_T}$, $n_T = 532,513$.
- Non-parametric, coupled term (the tensor part) is not considered.
- A linear MIMO system: number of inputs: 408, number of outputs: 816.

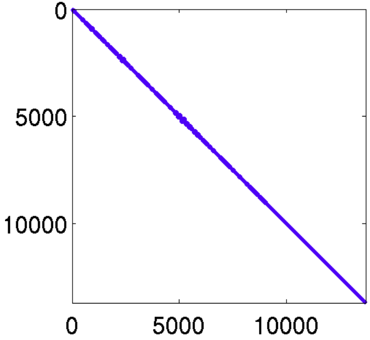
MOR results

- $n_E = 392,773 \rightsquigarrow r_E = 9,396$, $n_T = 532,513 \rightsquigarrow r_T = 4,305$.
- Standard MOR (e.g., moment-matching) fails due to excessive memory demands.
- Proposed sparse MOR achieves 98.5% reduction in size and a speedup factor of 972.7, with output error 7×10^{-7}.

Results for a power-cell model: sparsity comparison on the algebraic subsystem



Dense reduced matrix A_{E_r}
by standard moment-matching.



Block-diagonal structure of A_{E_r} by sparse MOR.

Motivation

- Many strongly nonlinear systems can be written in the form of quadratic-bilinear systems [Gu '12], e.g., nonlinear transmission line models.
- Spatial discretization of many well-known problems results in quadratic-bilinear systems, e.g., Burgers' equation, Navier-Stokes equations, FitzHugh-Nagumo system (a neuron model).
- Idea: Apply the proposed error bound to PMOR for quadratic-bilinear systems in order to realize adaptivity.

MOR for quadratic bilinear systems

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + H(x(t) \otimes x(t)) + Nx(t)u(t) + Bu(t), \\
y(t) &= Cx(t), \quad x(0) = x_0.
\end{aligned}
$$

$$
E, A, N \in \mathbb{R}^{n \times n}, H \in \mathbb{R}^{n \times n^2}, B, C^T \in \mathbb{R}^n.
$$

MOR for quadratic bilinear systems

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + H(x(t) \otimes x(t)) + Nx(t)u(t) + Bu(t),\\
y(t) &= Cx(t), \quad x(0) = x_0.
\end{aligned}
$$

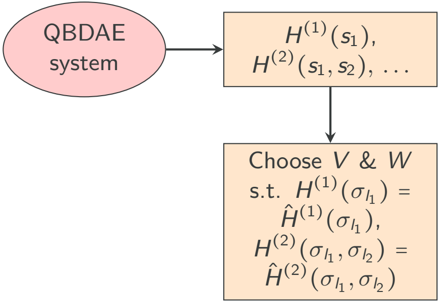
$$E, A, N \in \mathbb{R}^{n\times n}, H \in \mathbb{R}^{n\times n^2}, B, C^T \in \mathbb{R}^n.$$

MOR

$$
\begin{aligned}
E_r\dot{x}_r(t) &= A_r x_r(t) + H_r(x_r(t) \otimes x_r(t)) + N_r x_r(t)u(t) + B_r u(t),\\
y_r(t) &= C_r x_r(t), \quad x_r(0) = x_{r0}.
\end{aligned}
$$

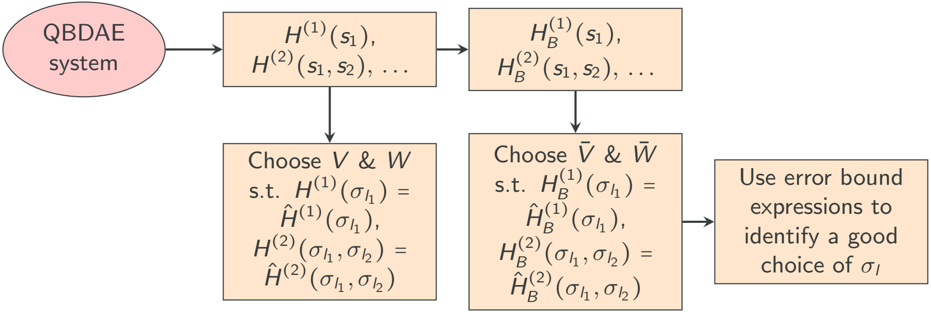
$$
\begin{aligned}
&E_r = W^T EV,\ A_r = W^T AV,\ N_r = W^T NV \in \mathbb{R}^{r\times r}, \quad H_r = W^T H(V \otimes V) \in \mathbb{R}^{r\times r^2},\\
&B_r = W^T B, C_r^T = V^T C^T \in \mathbb{R}^r.
\end{aligned}
$$

Problem statement



Question: How to choose the interpolation points?

Problem statement



Question: How to choose the interpolation points? Use error bound.

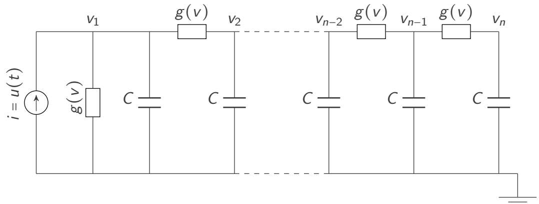
Our technique

- Compute $\bar{V}, \bar{W}$ from the bilinear part of the system:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + pNx(t) + Bu(t), \quad p(t) \equiv u(t), \\ y(t) &= Cx(t). \end{aligned}$$

- Treat the bilinear system as a parametric system, so that the error bound can be used to realize automatic PMOR $\rightsquigarrow$ automatic selection of the interpolation points.
- Use $\bar{V}, \bar{W}$ to get the ROM of the quadratic-bilinear system: $V = \bar{V}, W = \bar{W}$.

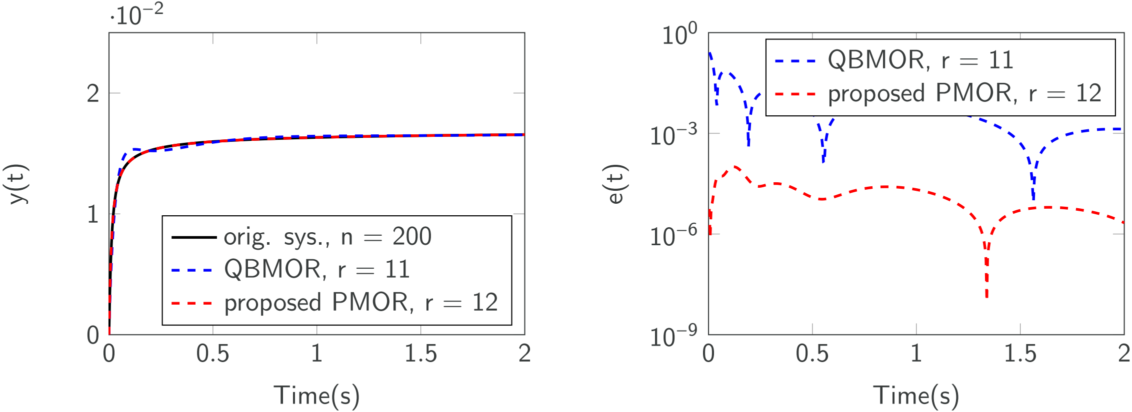
A nonlinear RC circuit



$$\dot{v}(t) = f(v(t), g(v(t))) + Bu(t),$$
$$y(t) = v_1(t).$$

- Strongly nonlinear.
- Transformation to quadratic-bilinear form exists, by doubling the state dimension.

A nonlinear RC circuit



Left: output. Right: relative output error. $u(t) \equiv 1$ $(t > 0)$, QBMOR [B./Breiten '14].

A PCB example

- System in time domain:

$$\begin{aligned} E\frac{dx(t)}{dt} &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned}$$

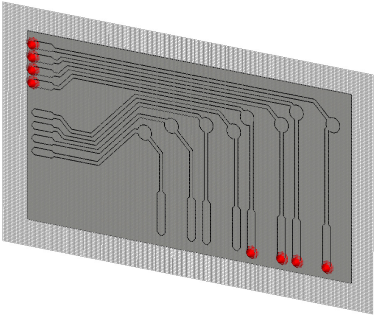
System in frequency domain:

$$\begin{aligned} sEx(s) &= Ax(s) + Bu(s), \\ y(s) &= Cx(s). \end{aligned}$$

- Reduced basis method considers s as a parameter, and use the system in frequency domain to compute

$$\mathrm{range}(V) = span\{x(s_1), \ldots, x(s_m)\}.$$

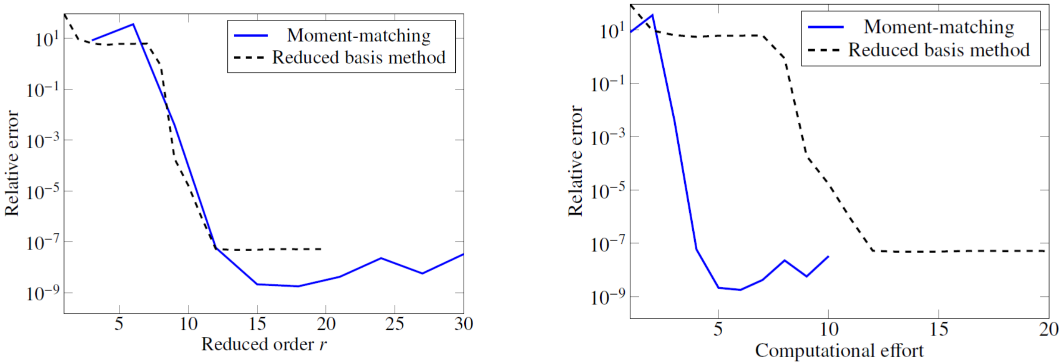
The ROM is obtained by Galerkin projection with V.



Printed circuit board model, $n = 233,060$.

Courtesy of TEMF, TU Darmstadt.

A PCB example



Moment-matching vs. reduced basis method.

Conclusions

- We have developed an adaptive PMOR method for ET coupled problems based on an *a posteriori* error bound.
- Results for a power-MOS model, a package model, and a power-cell model are promising.
- Advanced sparse (P)MOR techniques for systems with numerous inputs and outputs.
- Adaptive PMOR for a quadratic-bilinear system based on an *a posteriori* error bound.
- Comparison between reduced basis method and moment-matching shows advantage in efficiency for the latter.
- We have developed an output error bound/estimation for general nonlinear dynamical systems in time domain. $\rightsquigarrow$ Reliable ROM obtained by PMOR methods based on snapshots.

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