Spectral Projection Methods for Balanced Truncation of Descriptor Systems

Peter Benner

Institut für Mathematik
TU Berlin, Germany

DFG research center Berlin
mathematics for key technologies

benner@math.tu-berlin.de

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Outline

- Linear descriptor systems
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- Balanced truncation model reduction applied to descriptor systems
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Linear Descriptor Systems

Linear (time-invariant) descriptor systems:

$$\begin{aligned} E\left(\mathcal{D}x(t)\right) &= Ax(t) + Bu(t), \quad t > 0, \qquad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

Continuous-time systems:

$$t \in \mathbb{R}, \qquad \mathcal{D}x(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t).$$

Discrete-time systems:

$$t \in \mathbb{Z}, \quad \mathcal{D}x(t) = x(t+1).$$

Arise, e.g., in

- control of multibody (mechanical) systems,
- manipulation of fluid flow (e.g., Navier-Stokes equations),
- circuit simulation, VLSI chip design, in particular modeling of interconnet via RLC networks,
- simulation of MEMS and NEMS (micro-/nano-electro-mechanical systems).

Assumptions

- n generalized states, i.e., $x(t) \in \mathbb{R}^n$ (n is the degree of the system);
- $m \ll n$ inputs, i.e., $u(t) \in \mathbb{R}^m$;
- $p \ll n$ outputs, i.e., $y(t) \in \mathbb{R}^p$;
- $A - \lambda E$ regular, i.e., $\exists \lambda \in \mathbb{C}$ with $\det(A - \lambda E) \neq 0$.
- $A - \lambda E$ stable, i.e.,
  - for continuous-time systems: $\Lambda(A, E) \setminus \{\infty\} \subset \mathbb{C}^-$
  - for discrete-time systems: $\Lambda(A, E) \setminus \{\infty\} \subset \{z \in \mathbb{C} \mid |z| < 1\} =: B_1(0)$

Corresponding transfer function:

$$G(s) = C(sE - A)^{-1}B + D.$$

Hence, in frequency domain, $y(s) = G(s)u(s)$ (if $x(0) = 0$).

Model Reduction

Given

$$E\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t),$$

find reduced model

$$\tilde{E}\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \qquad \tilde{y}(t) = \tilde{C}\tilde{x}(t) + \tilde{D}u(t),$$

of degree $\ell \ll n$ with $\tilde{y}(t) \in \mathbb{R}^p$ and output error

$$y - \tilde{y} = Gu - \tilde{G}u = (G - \tilde{G})u$$

such that

$\|y - \tilde{y}\|$ "small" or, respectively, $\|G - \tilde{G}\|$ "small".

Consequence of Paley-Wiener Theorem:

$$\|y - \tilde{y}\|_{\mathcal{L}_2[0,\infty)} = \|y - \tilde{y}\|_{\mathcal{H}_2} \leq \|G - \tilde{G}\|_{\mathcal{H}_\infty} \|u\|_{\mathcal{H}_2}.$$

if $G \in \mathcal{H}_\infty$ (e.g., G stable and $\mathrm{ind}(A, E) \leq 1$) $\Rightarrow$ global error bound!

Model Reduction for Descriptor Systems

Goal: apply balanced truncation technique to descriptor systems, requires

- computing the system Gramians P, Q by solving two Lyapunov equations,
- balancing the Gramians, i.e., find system equivalence transformation

$$(E, A, B, C, D) \rightarrow (TES^{-1}, TAS^{-1}, TB, CS^{-1}, D)$$

  such that new Gramians are diagonal and equal,

- truncate the balanced system.

Solved by Stykel '01.

Model Reduction for Descriptor Systems, continued

Observation: Stykel's procedure is equivalent to

- separating finite and infinite poles such that transfer function decouples into

$$G(s) \;=\; G_0(s) + G_\infty(s),$$

- apply balanced truncation to standard system $(E_0^{-1}A_0, E_0^{-1}B_0, C_0, D_0)$,
- obtain reduced order model as $\tilde{G}(s) = \tilde{G}_0(s) + G_\infty(s)$.

Here: implement the above procedure using spectral projection methods.

Note: reduction of $G_\infty(s)$ to order $\min\{m,p\} \cdot \mathrm{ind}(A,E)$ possible without additional error; see Stykel '03.

System Gramians

Assume E nonsingular; for singular E see [Stykel '01–'03]

Controllability and observability Gramians of G are obtained from linear matrix equations.

- Continuous-time case: generalized Lyapunov equations

$$APE^T + EPA^T + BB^T = 0, \qquad A^T\hat{Q}E + E^T\hat{Q}A + C^TC = 0.$$

- Discrete-time case: generalized Stein equations

$$APA^T + EPE^T + BB^T = 0, \qquad A^T\hat{Q}A + E^T\hat{Q}E + C^TC = 0.$$

and $Q := E^T\hat{Q}E$.

Balanced Truncation

For balanced realization

$$(E,A,B,C,D) = \left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right)$$

of $G(s)$ with $P = Q =: \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$ the reduced-order model

$$\tilde{G}(s) = C_1(sE_{11} - A_{11})^{-1}B_1 + D =: \tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B} + D$$

is balanced, minimal, stable. The Gramians are $\tilde{P} = \tilde{Q} = \tilde{\Sigma} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_\ell \end{bmatrix}$.

$\Longrightarrow$ Computable global error bound, allows adaptive choice of ℓ.

$$\|G - \tilde{G}\|_{\mathcal{H}_\infty} \leq 2 \sum_{k=\ell+1}^{n} \sigma_k.$$

Spectral Projection Methods

Given $A - \lambda E$, compute projector $\mathcal{P}_1$ onto deflating subspace $\mathcal{L}_1$ corresponding to $\Lambda_1 \subset \Lambda(A, E)$.

From $\mathcal{P}_1$ obtain

- orthonormal basis of deflating subspace $\mathcal{L}_1$ from (rank-revealing) QR decomposition of $\mathcal{P}_1$;
- block-triangular decomposition of $A - \lambda E$:

$$Q^T(A - \lambda E)Z = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} - \lambda \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}$$

with Q, Z orthogonal, $\Lambda(A_{11}, E_{11}) = \Lambda_1$.

Determine Q, Z via inverse-free subspace extraction algorithm.

Sign Function Method

[Roberts '71, Gardiner, Laub '86]

For $Z, Y \in \mathbb{R}^{n \times n}$ with $\Lambda(Z, Y) \cap \imath\mathbb{R} = \emptyset$, Z, Y nonsingular, compute spectral projector $\frac{1}{2}(I - Y^{-1}Z_\infty)$ onto stable deflating subspace of $Z - \lambda Y$:

$$Z_0 \leftarrow Z, \quad Z_{j+1} \leftarrow \frac{1}{2}\left(c_j Z_j + \frac{1}{c_j} Y Z_j^{-1} Y\right), \; j = 0, 1, \ldots, \quad \Rightarrow Z_\infty = \lim_{j \to \infty} Z_j.$$

($c_j > 0$ is scaling parameter for convergence acceleration and rounding error minimization.)

Properties

- Sign function undefined if Z has purely imaginary eigenvalues $\Longrightarrow$ problems for eigenvalues close to imaginary axis.
- Usually, computed invariant subspaces are as accurate as their conditioning admits.
  [Byers, He, Mehrmann '97]
- Block-triangular form often better conditioned than computation of Schur form $\Longrightarrow$ often more accurate than computations based on QR/QZ algorithms.

The Disk Function Method

Let $Z - \lambda Y$ be regular with $\Lambda\,(Z, Y) \cap \{|z| = 1\} = \emptyset$ and Weierstraß (Kronecker) canonical form

$$Z - \lambda Y = T \begin{bmatrix} J_0 - \lambda I & 0 \\ 0 & J_\infty - \lambda N \end{bmatrix} S^{-1},$$

where

- J_0 contains Jordan blocks corresponding to $\Lambda\,(Z, Y) \cap B_1(0)$,
- $J_\infty - \lambda N$ contains Jordan and nilpotency structure corresponding to $\Lambda\,(Z, Y) \cap \{|z| > 1\}$, including infinite eigenvalues,

$$\text{disk}\,(Z, Y) \;:=\; S \left(\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 \\ 0 & I_{n-k} \end{bmatrix} \right) S^{-1} \;=:\; D_Z - \lambda D_Y$$

- D_Z is projector onto *right* d-stable deflating subspace of $Z - \lambda Y$.
- D_Y is projector onto *right* d-antistable deflating subspace of $Z - \lambda Y$.

Computation of the Disk Function

Inverse-free iteration [MALYSHEV '93, BAI, DEMMEL, GU '94]

Input: $Z - \lambda Y$, $\Lambda(Z, Y) \cap \{|z| = 1\} = \emptyset$.

Output: $Z_\infty - \lambda Y_\infty$, $\Lambda(Z_\infty, Y_\infty) \subset \{0, \infty\}$

1. Set $Z_0 = Z$, $Y_0 = Y$.
2. FOR $j = 0, 1, 2, \ldots$

   Compute QR decomposition $\begin{bmatrix} Y_j \\ -Z_j \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} R_j \\ 0 \end{bmatrix}$,

   Set $Z_{j+1} = U_{12}^T Z_j, \quad Y_{j+1} = U_{22}^T Y_j$.

   END FOR
3. Set $Z_\infty := \lim_{j \to \infty} Z_j$, $Y_\infty := \lim_{j \to \infty} Y_j$.

$$\Longrightarrow \quad \text{disk}\,(Z, Y) = (Z_\infty + Y_\infty)^{-1}(Y_\infty - \lambda Z_\infty) = D_Z - \lambda D_Y$$

Inverse-Free Subspace Extraction Algorithm for Disk Function

[Quintana-Ortí, Sun '96]

1. Compute rank-revealing LQ factorization $\Pi_1 Y_\infty \tilde{V} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & 0 \end{bmatrix}$,

   Let $r_Y = \operatorname{rank}(Y_\infty)$, then: columns of $V_1 := \tilde{V}(:, r_Y + 1 : n)$ span $\operatorname{Ker}(Y_\infty)$.

2. Compute rank-revealing QR factorization $U^T (ZV_1,\ YV_1)\,\Pi_2 = \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix}$.

3. Set $V := \begin{bmatrix} V_1 & \tilde{V}(:, 1 : r_Y) \end{bmatrix}$.

Then

$$U^T(Z - \lambda Y)V = \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix} - \lambda \begin{bmatrix} Y_{11} & Y_{12} \\ 0 & Y_{22} \end{bmatrix}$$

with $\Lambda(Z_{11}, Y_{11}) \subset B_1(0)$, $\Lambda(Z_{22}, Y_{22}) \subset \overline{\mathbb{C} \setminus B_1(0)}$.

Based on $\operatorname{Ker}(Y_\infty)$ = stable deflating subspace of $Z - \lambda Y$.

Balanced Truncation Using Spectral Projection Method

1. Compute the factors S, R of the controllability and observability Gramians using coupled Newton iteration for the sign function:

$$
\begin{aligned}
& A_0 \leftarrow A, \quad S_0 \leftarrow B, \quad R_0 \leftarrow C \\
& \texttt{for } j = 0, 1, 2, \ldots \\
& \qquad A_{j+1} \leftarrow \frac{1}{\sqrt{2c_j}} \left(A_j + c_j^2 E A_j^{-1} E \right), \\
& \qquad S_{j+1} \leftarrow \text{full-rank factor of } \frac{1}{\sqrt{2c_j}} \left[\begin{array}{cc} R_j & c_j E A_j^{-1} R_j \end{array} \right] \\
& \qquad R_{j+1} \leftarrow \text{full-rank factor of } \frac{1}{\sqrt{2c_j}} \left[\begin{array}{c} R_j \\ c_j R_j A_j^{-1} E \end{array} \right]
\end{aligned}
$$

   Set $S := \frac{1}{\sqrt{2}} \lim_{j\to\infty} S_j E^{-T}$, $R := \frac{1}{\sqrt{2}} \lim_{j\to\infty} R_j$.

2. Compute SVD $\quad SR^T = [U_1 \; U_2] \left[\begin{array}{cc} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{array} \right] \left[\begin{array}{c} V_1^T \\ V_2^T \end{array} \right]$
3. Set $T_l = \Sigma_1^{-1/2} V_1^T R E^{-1}$, $T_r = S^T U_1 \Sigma_1^{-1/2}$.
4. Compute $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (T_l E T_r, T_l A T_r, T_l B, C T_r, D)$.

Discrete-time case: analogous, compute factors of Gramians using coupled squared Smith iteration.

Application to Descriptor Systems

Use additive decomposition of transfer function,

$$G(s) = G_0(s) + G_\infty(s),$$

where $G_0(s), G_\infty(s)$ correspond to finite poles, infinite poles, respectively.

Reduce $G_0(s)$ using BT, keep $G_\infty(s) \Longrightarrow$ obtain reduced-order model as

$$\tilde{G}(s) = \tilde{G}_0(s) + G_\infty(s).$$

$$\Longrightarrow \quad E(s) = G(s) - \tilde{G}(s) = G_0(s) - \tilde{G}_0(s) \text{ is strictly proper}$$

$$\Longrightarrow \quad \|E\|_{\mathcal{H}_\infty} = \|G_0 - \tilde{G}_0\|_{\mathcal{H}_\infty} \leq 2 \sum_{k=\ell+1}^{n_0} \sigma_k$$

$$\Longrightarrow \quad \|y\|_2 \leq 2 \sum_{k=\ell+1}^{n_0} \sigma_k \|u\|_2$$

Additive Decomposition via Block-Diagonalization

Need block-diagonalization of $A - \lambda E$:

$$\hat{A} - \lambda \hat{E} := U(A - \lambda E)V^{-1} = \begin{bmatrix} A_0 & 0 \\ 0 & A_\infty \end{bmatrix} - \lambda \begin{bmatrix} E_0 & 0 \\ 0 & E_\infty \end{bmatrix}$$

$$\Longrightarrow \qquad \hat{B} := UB =: \begin{bmatrix} B_0 \\ B_\infty \end{bmatrix}, \qquad \hat{C} := CV^{-1} =: [\, C_0 \; C_\infty \,],$$

Then

$$\begin{aligned} G(s) &= C(sE - A)^{-1}B + D \\ &= \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} + \hat{D} \\ &= \begin{bmatrix} C_0 & C_\infty \end{bmatrix} \begin{bmatrix} (sE_0 - A_0)^{-1} & \\ & (sE_\infty - A_\infty)^{-1} \end{bmatrix} \begin{bmatrix} B_0 \\ B_\infty \end{bmatrix} + D \\ &= \underbrace{\{C_0(sE_0 - A_0)^{-1}B_0 + D\}}_{=:G_0(s)} + \underbrace{\{C_\infty(sE_\infty - A_\infty)^{-1}B_\infty\}}_{=:G_\infty(s)} \end{aligned}$$

Block-Diagonalization via the Disk Function Method

1. **(Block-triangular form)** Compute disk (A, E) and obtain spectral projector $\mathcal{P}_0 := (A_\infty + E_\infty)^{-1} A_\infty \Rightarrow$ obtain Q, Z orthogonal from inverse-free extraction algorithm such that

$$Q^T(A - \lambda E)Z = \begin{bmatrix} A_0 & W_A \\ 0 & A_\infty \end{bmatrix} - \lambda \begin{bmatrix} E_0 & W_E \\ 0 & E_\infty \end{bmatrix}.$$

2. **(Block-diagonal form)** Solve generalized Sylvester equation

$$A_0 Y + X A_\infty + W_A = 0, \quad E_0 Y + X E_\infty + W_E = 0.$$

Then

$$\begin{aligned} \hat{A} - \lambda \hat{E} \quad &:= \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \left(\begin{bmatrix} A_0 & W_A \\ 0 & A_\infty \end{bmatrix} - \lambda \begin{bmatrix} E_0 & W_E \\ 0 & E_\infty \end{bmatrix} \right) \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} A_0 & 0 \\ 0 & A_\infty \end{bmatrix} - \lambda \begin{bmatrix} E_0 & 0 \\ 0 & E_\infty \end{bmatrix}, \end{aligned}$$

Block-Triangularization via Disk Function

Discrete-time systems

Disk function applied to stable $A - \lambda E$ generates correct spectral projector.

Continuous-time systems

Disk function computes spectral projection w.r.t. unit circle, but stable poles may be anywhere in $\mathbb{C}^-$.

Cure: conformal mapping (Moebius transformation) [Bai, Demmel, Gu '97]

$$(A, E) \to (\alpha A + \beta E, \gamma A + \delta E)$$

- $\alpha = \beta = \gamma = 1$, $\delta = -1$: $B_1(0) \to i\mathbb{R}$ (Cayley transformation)
- $\beta = \gamma = 0$, $\delta = 1$: $B_1(0) \to B_{\frac{1}{\alpha}}((0, 0))$.

In order to split the finite/infinite parts of the spectrum, choose

$$\alpha > \max\{|\lambda| \mid \lambda \in \Lambda(A, E) \setminus \{\infty\}\}.$$

Open problem: estimate α.

Solution of Generalized Sylvester Equation

$\mathrm{ind}(A, E) = 1$ **($\Rightarrow$ no impulsive modes)**

$\Longrightarrow$ $E_\infty = 0$, Sylvester equation decouples:

$E_0 Y = -W_E, \quad X A_\infty = A_0 Y + W_A,$ i.e., solve two linear systems of equations.

$\mathrm{ind}(A, E) > 1$

Several methods available:

- generalized Hessenberg-Schur method,
- generalized Schur method,

and several implementations:

- ACM TOMS, CALGO 705 [Gardiner, Wette, Laub, Amato, Moler '92].
- SLICOT, SB04OD, based on [Kågström, Westin '89].
- Block-recursive method for triangular case [Kågström, Johansson '01].

Unsatisfactory for large-scale problems; requires QZ algorithm (*not efficient, not available in ScaLAPACK*)

SLICOT = Subroutine Library in Control Theory, available from `http://www.win.tue.nl/niconet`

Numerical Example

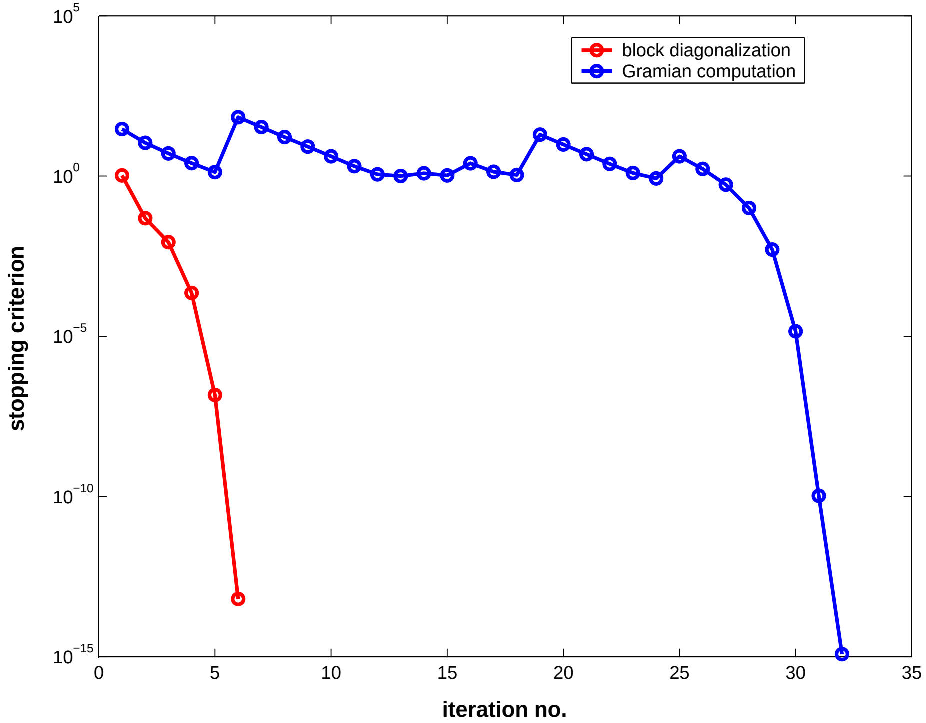
- $n = 500$, $m = 5$, $p = 10$, $\text{ind}(A, E) = 1$;
- $n_0 = 470$ differential, $n_\infty = 30$ algebraic equations;
- "worst" eigenvalues: $\lambda = -3 \cdot 10^{-5} \pm 400$.
- backward error of spectral decomposition:

	disk function	QZ alg.
$\|[A_{21}, E_{21}]\|_F$	$4.2 \cdot 10^{-13}$	$1.0 \cdot 10^{-12}$
$\|E_{22}\|_F$	$1.1 \cdot 10^{-15}$	$1.1 \cdot 10^{-15}$

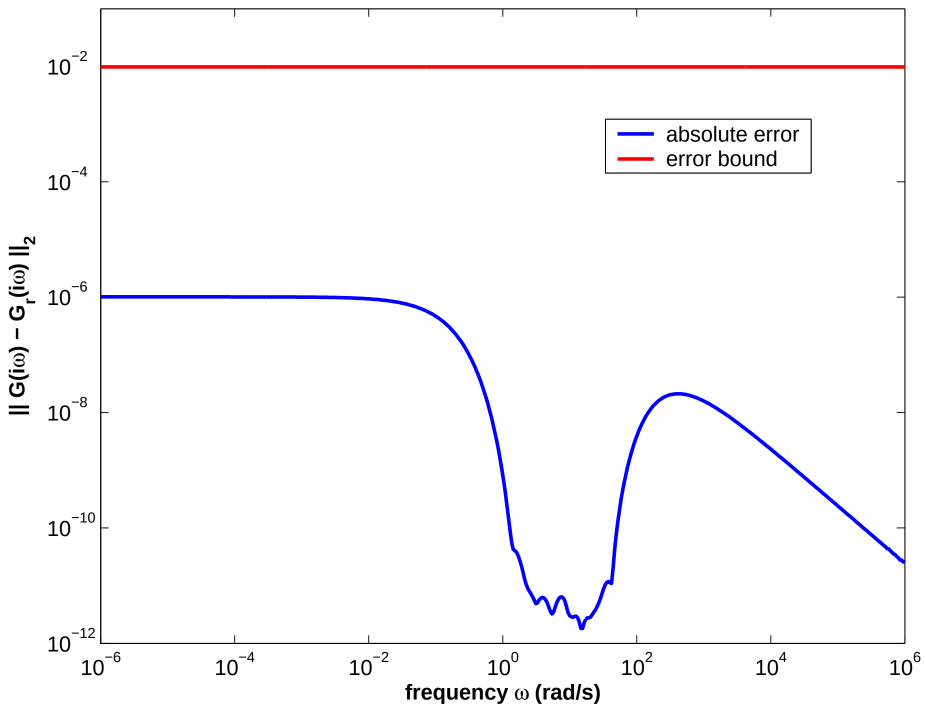
- $\text{rank}\,(R) = 136$, $\text{rank}\,(S) = 89$;
- $\ell = 60$, based on "$\delta \overset{!}{\leq} 0.01$";
- computed error bound: $\delta = 9.796 \cdot 10^{-3}$;

Numerical Example, continued

convergence history



model reduction error



Conclusions

- Parallel implementation for software library PLiCMR under way; integrated into parallel version of SLICOT.
- Model reduction of large-scale systems on Linux cluster in Castellón: E-mail, web server.
- Implementations of methods for sparse systems based on ADI possible [Stykel].
- Circuit simulation:
  - Get global error bounds, error often much smaller than for modal truncation.
  - Computation of passive reduced systems needs balancing of positive real Gramians, i.e., solution of algebraic Riccati equations.
- Exploitation of more structural information desirable.

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