

Combined Reduction for EEG Model Inversion

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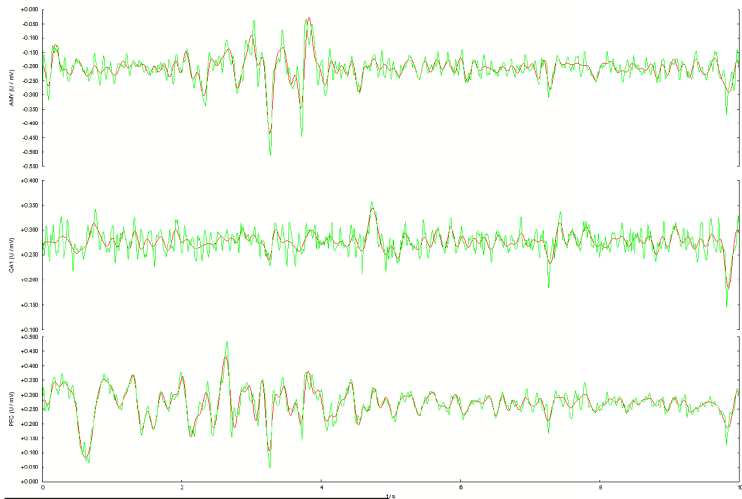
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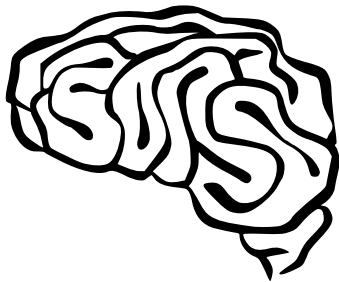
Motivation

Intracranial EEG and tuned model output¹:



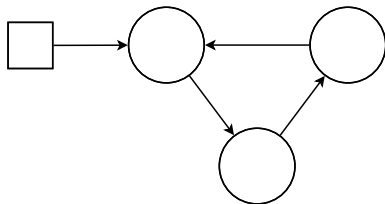
Dynamic Causal Modelling

- How are brain regions connected?
- How does sensory input disperse?
- How does the brain learn and unlearn?



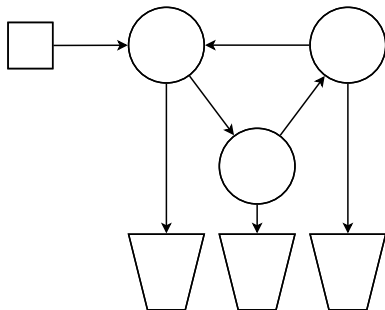
Dynamic Causal Modelling

- How are brain regions connected?
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 - How does the brain learn and unlearn?
- Parametrized Dynamical System



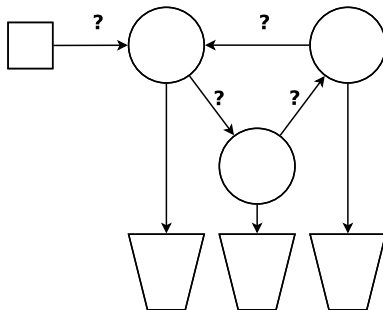
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- Output System



Dynamic Causal Modelling

- How are brain regions connected?
- How does sensory input disperse?
- How does the brain learn and unlearn?
- Parametrized Dynamical System
- Output System
- Bayesian Inference



Jansen Neural Mass Model

Synaptic Input: $u(t)$

Impulse Response: $h(t) = H\kappa t \exp(-t\kappa)$

Postsynaptic Membrane Potential:

$$v(t) = h(t) * u(t)$$

$$\Rightarrow \ddot{v}(t) = H\kappa u(t) - 2\kappa \dot{v}(t) - \kappa^2 v(t)$$

$$\Rightarrow \begin{cases} \dot{v}(t) = x(t) \\ \dot{x}(t) = H\kappa u(t) - 2\kappa x(t) - \kappa^2 v(t) \end{cases}$$

Jansen Neural Mass Model

Synaptic Input: $u(t) = S(v(t)) + u_x(t)$

Impulse Response: $h(t) = H\kappa t \exp(-t\kappa)$

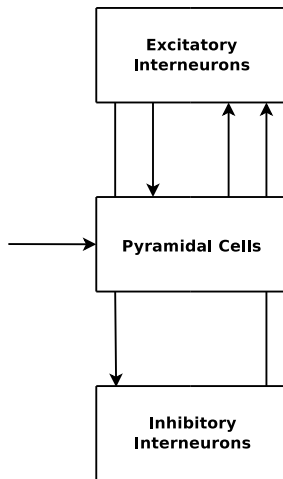
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$$\Rightarrow \begin{cases} \dot{v}(t) = x(t) \\ \dot{x}(t) = H\kappa u(t) - 2\kappa x(t) - \kappa^2 v(t) \end{cases}$$

3-Layer Model²



$$v_0(t) = h_e(t) * u(v_2)$$

$$v_1(t) = h_i(t) * u(v_3, v_4)$$

$$v_2(t) = h_e(t) * u(v_0, v_1, u_x)$$

$$v_3(t) = h_e(t) * u(v_0, v_1)$$

$$v_4(t) = h_i(t) * u(v_3, v_4)$$

² see [Moran'07]

Single Region³ (SISO)

$$\dot{x}_0 = x_5$$

$$\dot{x}_1 = x_6$$

$$\dot{x}_2 = x_7$$

$$\dot{x}_3 = x_8$$

$$\dot{x}_4 = x_9$$

$$\dot{x}_5 = \kappa_e H_e \gamma_2 S(x_2) - 2\kappa_e x_5 - \kappa_e^2 x_0$$

$$\dot{x}_6 = \kappa_i H_i \gamma_4 S(x_3 - x_4) - 2\kappa_i x_6 - \kappa_i^2 x_1$$

$$\dot{x}_7 = \kappa_e H_e \gamma_1 S(x_0 - x_1) - 2\kappa_e x_7 - \kappa_e^2 x_2 + \kappa_e H_e \gamma_1 u$$

$$\dot{x}_8 = \kappa_e H_e \gamma_3 S(x_0 - x_1) - 2\kappa_e x_8 - \kappa_e^2 x_3$$

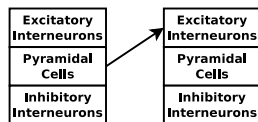
$$\dot{x}_9 = \kappa_i H_i \gamma_5 S(x_3 - x_4) - 2\kappa_i x_9 - \kappa_i^2 x_4$$

$$y = x_0 - x_1$$

³ see [Moran'07]

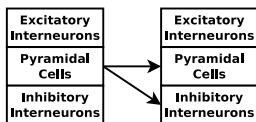
Connectivity⁴

Forward Connection:



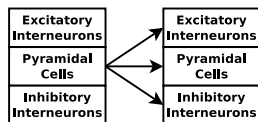
A_F

Backward Connection:



A_B

Lateral Connection:



A_L

Multiple Regions⁵ (MIMO)

$$\dot{X}_0 = X_5$$

$$\dot{X}_1 = X_6$$

$$\dot{X}_2 = X_7$$

$$\dot{X}_3 = X_8$$

$$\dot{X}_4 = X_9$$

$$\dot{X}_5 = \kappa_e H_e (A_B + A_L + \gamma_2 \mathbb{1}) S(X_2) - 2\kappa_e X_5 - \kappa_e^2 X_0$$

$$\dot{X}_6 = \kappa_i H_i \gamma_4 \mathbb{1} S(X_3 - X_4) - 2\kappa_i X_6 - \kappa_i^2 X_1$$

$$\dot{X}_7 = \kappa_e H_e (A_F + A_L + \gamma_1 \mathbb{1}) S(X_0 - X_1) - 2\kappa_e X_7 - \kappa_e^2 X_2 + \kappa_e H_e \gamma_1 U$$

$$\dot{X}_8 = \kappa_e H_e (A_B + A_L + \gamma_3 \mathbb{1}) S(X_0 - X_1) - 2\kappa_e X_8 - \kappa_e^2 X_3$$

$$\dot{X}_9 = \kappa_i H_i \gamma_5 \mathbb{1} S(X_3 - X_4) - 2\kappa_i X_9 - \kappa_i^2 X_4$$

$$Y = C(X_0 - X_1)$$

⁵ see [David'06]

Linear Control System:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

General Control System:

$$\dot{x} = f(x, u, \theta)$$

$$y = g(x, u, \theta)$$

DCM-EEG Neural Mass Model:

$$\dot{x} = \begin{pmatrix} 0 & \mathbb{1} \\ -\kappa^2 \mathbb{1} & -2\kappa \mathbb{1} \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ \kappa H \Sigma(\theta) & 0 \end{pmatrix} S(x) + Bu$$

$$y = Cx$$

→ Nonlinear Second-Order Control System!

(Nonlinear) Model Reduction

State-Space Reduction:

- (Empirical) Gramian-Based
 - 1 Balanced Truncation
 - 2 Approximate Balancing (Cross Gramian)

(Nonlinear) Model Reduction

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Parameter-Space Reduction:

- (Empirical) Gramian-Based
 - 1 Controllability-Based (parameters as additional inputs)
 - 2 Observability-Based (parameters as additional states)
 - 3 Cross-Gramian-Based (parameters as additional states)

(Nonlinear) Model Reduction

State-Space Reduction:

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 - 1 Controllability-Based (parameters as additional inputs)
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Combined (State and Parameter) Reduction:

- (Empirical) Gramian-Based
 - 1 Controllability-Based
 - 2 Observability-Based
 - 3 Cross-Gramian-Based

Second-Order Balanced Truncation⁶

Second-Order System Gramians:

Controllability Gramian:

Observability Gramian:

Cross Gramian:

$$W_C = \begin{pmatrix} W_{C,P} & W_{C,PV} \\ W_{C,VP} & W_{C,V} \end{pmatrix}$$

$$W_O = \begin{pmatrix} W_{O,P} & W_{O,PV} \\ W_{O,VP} & W_{O,V} \end{pmatrix}$$

$$W_X = \begin{pmatrix} W_{X,P} & W_{X,PV} \\ W_{X,VP} & W_{X,V} \end{pmatrix}$$

$$\left. \begin{array}{l} \text{Position Gramian: } W_{?,P} \\ \text{Velocity Gramian: } W_{?,V} \end{array} \right\} \rightarrow \{U_P, U_V, V_P, V_P\}$$

Second-Order Projections:

$$\left. \begin{array}{l} \dot{x} = \left(f \left(\begin{pmatrix} x_P \\ x_V \end{pmatrix}, u, \theta \right) \right) \\ y = g \left(\begin{pmatrix} x_P \\ x_V \end{pmatrix}, u, \theta \right) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \dot{\tilde{x}} = \left(U_V f \left(\begin{pmatrix} \tilde{x}_V \\ U_P \tilde{x}_P \\ U_V \tilde{x}_V \end{pmatrix}, u, \theta \right) \right) \\ \tilde{y} = g \left(\begin{pmatrix} U_P \tilde{x}_P \\ U_V \tilde{x}_V \end{pmatrix}, u, \theta \right) \end{array} \right.$$

⁶ see [Reis'07]

A zoo of methods⁷

Balanced Decomposition:

$$BAL(W_{C,P}, W_{O,P}) \rightarrow \{U_P, V_P\} \equiv \{U_V, V_V\}$$

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Singular Value Decomposition:

$$SVD(W_{C,P}) \rightarrow \{U_P, V_P\} \equiv \{U_V, V_V\}$$

$$SVD(W_{C,V}) \rightarrow \{U_V, V_V\} \equiv \{U_P, V_P\}$$

$$SVD(W_{O,P}) \rightarrow \{U_P, V_P\} \equiv \{U_V, V_V\}$$

$$SVD(W_{O,V}) \rightarrow \{U_V, V_V\} \equiv \{U_P, V_P\}$$

$$\left. \begin{array}{l} SVD(W_{C,P}) \\ SVD(W_{C,V}) \end{array} \right\} \rightarrow \{U_P, V_P, U_V, V_V\}$$

$$\left. \begin{array}{l} SVD(W_{O,P}) \\ SVD(W_{O,V}) \end{array} \right\} \rightarrow \{U_P, V_P, U_V, V_V\}$$

$$SVD(W_{X,P}) \rightarrow \{U_P, V_P\} \equiv \{U_V, V_V\}$$

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$$\left. \begin{array}{l} SVD(W_{X,P}) \\ SVD(W_{X,V}) \end{array} \right\} \rightarrow \{U_P, V_P, U_V, V_V\}$$

Empirical Gramians⁸ I

POD-based method:

- Empirical Controllability Gramian: $W_C = \langle \int_0^\infty x_U(t)x_U^*(t)dt \rangle_U$

- Empirical Observability Gramian: $W_O = \langle \int_0^\infty \rho(y_X^*(t)y_X(t))dt \rangle_X$

- Empirical Cross Gramian: $W_X = \langle \int_0^\infty \varphi(x_U(t), y_X(t))dt \rangle_{U \times X}$

with perturbation spaces:

- 1 U perturbing the input u ,
- 2 X perturbing the initial state x_0

assembled from:

- rotations, sets of orthogonal matrices Q_U, Q_X
- scales, sets of real numbers R_U, R_X
- for each input / state, the unit normal vectors E_U, E_X

determined by the operating range of the underlying control system.

⁸ see [Lall'99]

Empirical Gramians⁹ II

Empirical Controllability Gramian:

$$W_C = \frac{1}{|Q_u||R_u|} \sum_{h=1}^{|Q_u|} \sum_{i=1}^{|R_u|} \sum_{j=1}^m \frac{1}{c_h^2} \int_0^\infty \Psi^{hij}(t) dt$$

$$\Psi^{hij}(t) = (x^{hij}(t) - \bar{x})(x^{hij}(t) - \bar{x})^* \in \mathbb{R}^{n \times n}, u^{hij}(t) = c_h S_i e_j u(t) + \bar{u}$$

Empirical Observability Gramian:

$$W_O = \frac{1}{|Q_x||R_x|} \sum_{k=1}^{|Q_x|} \sum_{l=1}^{|R_x|} \frac{1}{d_k^2} T_l \int_0^\infty \Psi^{kl}(t) dt T_l^*$$

$$\Psi_{ab}^{kl}(t) = (y^{kla}(t) - \bar{y})^* (y^{klb}(t) - \bar{y}) \in \mathbb{R}, x_0^{kla} = d_k T_l f_a + \bar{x}$$

Empirical Cross Gramian:

$$W_X = \frac{1}{|Q_u||R_u| m |Q_x||R_x|} \sum_{h=1}^{|Q_u|} \sum_{i=1}^{|R_u|} \sum_{j=1}^m \sum_{k=1}^{|Q_x|} \sum_{l=1}^{|R_x|} \frac{1}{c_h d_k} \int_0^\infty T_l \Psi^{hijkl}(t) T_l^* dt$$

$$\Psi_{ab}^{hijkl}(t) = f_b^* T_k^* \Delta x^{hij}(t) e_i^* S_h^* \Delta y^{kla}(t) \in \mathbb{R}$$

$$\Delta x^{hij}(t) = (x^{hij}(t) - \bar{x}), u^{hij}(t) = c_h S_i e_j u(t) + \bar{u}$$

$$\Delta y^{kla}(t) = (y^{kla}(t) - \bar{y}), x_0^{kla} = d_k T_l f_a + \bar{x}$$

⁹[Hahn'02], [Streif'06], [Himpe'13a]

Empirical Gramians¹⁰ III

Empirical Sensitivity Gramian (Controllability-Based) W_S :

$$\hat{u} = \begin{pmatrix} u \\ \theta \end{pmatrix} \rightarrow \dot{x} = f(x, \hat{u}) = f(x, u) + \sum_{k=1}^P f(x, \theta_k) \Rightarrow W_C = W_{C,0} + \sum_{k=1}^P W_{C,k}$$

$$\rightarrow W_S = \delta_{i,j} \text{trace}(W_{C,i})$$

Empirical Identifiability Gramian (Observability-Based) W_I :

$$\hat{x} = \begin{pmatrix} x \\ \theta \end{pmatrix} \rightarrow \dot{\hat{x}} = f(\hat{x}, u) = \begin{pmatrix} f(x, u, \theta) \\ 0 \end{pmatrix}, \hat{x}(0) = \begin{pmatrix} x_0 \\ \theta \end{pmatrix} \Rightarrow \hat{W}_O = \left(\begin{array}{c|c} W_O & W_M \\ \hline W_M^* & W_P \end{array} \right)$$

$$\rightarrow W_I = W_P - W_M^* W_O^{-1} W_M \approx W_P$$

Empirical Joint Gramian (Cross-Gramian-Based) W_J :

$$\hat{x} = \begin{pmatrix} x \\ \theta \end{pmatrix} \Rightarrow W_J := \hat{W}_X = \left(\begin{array}{c|c} W_X & W_M \\ \hline 0 & 0 \end{array} \right)$$

$$\rightarrow W_J = -W_M^* (W_X + W_X^T)^{-1} W_M \approx -W_M^* \text{diag}(W_X + W_X^T)^{-1} W_M$$

¹⁰[Sun'06], [Geffen'08], [Himpe'13a]

Combined Reduction

Controllability-Based

- 1 Compute W_S
 $\rightarrow W_C$
- 2 Decompose W_S
- 3 Truncate θ
- 4 Compute W_O
- 5 Balance W_C, W_O
- 6 Decompose W_{CO}
- 7 Truncate x

Observability-Based

- 1 Compute W_I
 $\rightarrow W_O$
- 2 Decompose W_I
- 3 Truncate θ
- 4 Compute W_C
- 5 Balance W_C, W_O
- 6 Decompose W_{CO}
- 7 Truncate x

Cross-Gramian-Based

- 1 Compute W_j
 $\rightarrow W_X$
- 2 Decompose W_j
- 3 Truncate θ
- 4 Decompose W_X
- 5 Truncate x

Gramians:

- Empirical Controllability Gramian
- Empirical Observability Gramian
- Empirical Cross Gramian
- Empirical Sensitivity Gramian
- Empirical Identifiability Gramian
- Empirical Joint Gramian

Features:

- Uniform Interface
- Compatible with MATLAB & OCTAVE
- Vectorized & Parallelizable
- Open-Source licensed

More info at: <http://gramian.de>

¹¹ see [Himpe'13]

Numerical Experiments

Setup:

- Nonlinear DCM-EEG model
- Inverse problem on connectivity parameters: A_F, A_B, A_L
- Synthetic data for uniformly random connectivity
- with additive Gaussian noise: $y_d = y + N(0, \nu)$
- Combined reduction with: $\{W_S, W_O\}, \{W_C, W_I\}, W_J$
- Leapfrog integration

Dimensions:

- Regions: $n = \{4, \dots, 10\}$
- States: $\dim(x) = N = 10n$
- Parameters: $\dim(\theta) = 3n^2$
- Reduced States: $\dim(\tilde{x}) = 4n$
- Reduced Parameters: $\dim(\tilde{\theta}) = 6n$

Combined Reduction for Inverse Problems

Inverse Problem:

- Bayesian inference: $P(\theta|y_d) \propto P(y_d|\theta)P(\theta)$
- Prior distribution: $A_{F,prior} = A_{B,priors} = A_{L,prior} = N(-\mathbb{1}, \mathbb{1})$
- Fixed biological parameters: $\kappa_e, H_e, \kappa_i, H_i$

Combined Reduction:

- 1 Offline Phase:
Assembly of reduced order model using combined reduction.
- 2 Online Phase:
Optimization reduced model using least-squares.

Applicability

Balanced Decomposition:

$BAL(W_{C,P}, W_{O,P})$	✓
$BAL(W_{C,V}, W_{O,V})$	✓
$BAL(W_{C,P}, W_{O,V})$	✗
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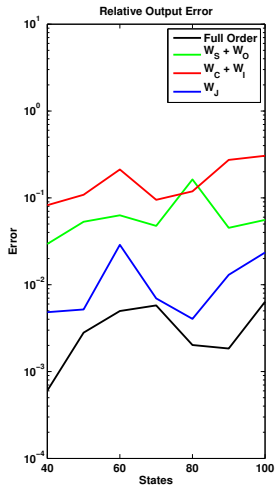
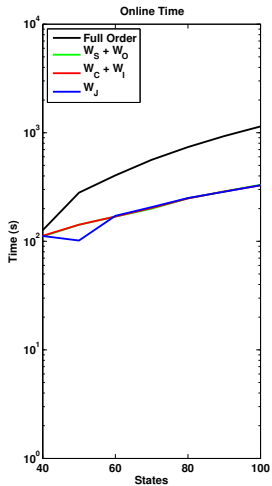
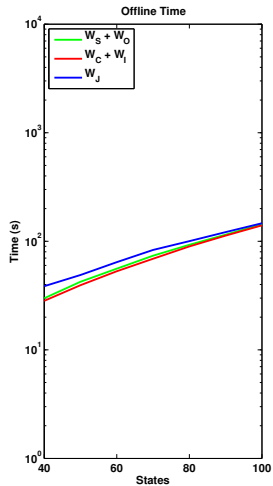
Singular Value Decomposition:

$SVD(W_{C,P})$	✗
$SVD(W_{C,V})$	✗
$SVD(W_{O,P})$	✗
$SVD(W_{O,V})$	✗
$SVD(W_{C,P})$	✗
$SVD(W_{C,V})$	
$SVD(W_{O,P})$	✗
$SVD(W_{O,V})$	
$SVD(W_{X,P})$	✓
$SVD(W_{X,V})$	✓
$SVD(W_{X,P})$	✗
$SVD(W_{X,V})$	

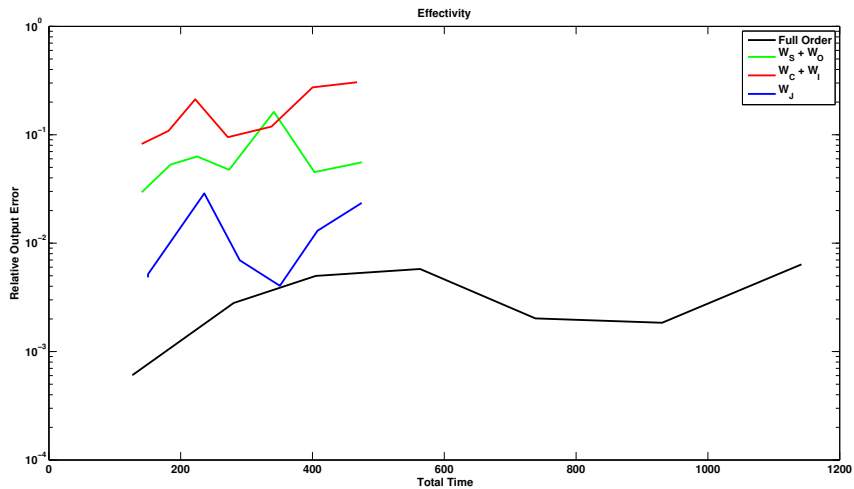
For experiments:

- $BAL(W_{C,P}, W_{O,P})$ for $\{W_S, W_O\}$
- $BAL(W_{C,P}, W_{O,P})$ for $\{W_C, W_I\}$
- $SVD(W_{X,P})$ for W_J

Numerical Results



Reduction Effectivity



Summary:

- Model Reduction prior to inversion,
 - of Dynamic-Causal-EEG-Model.
 - Nonlinear Model Reduction: Empirical Gramians.
 - Combined Reduction: Reduction of States and Parameters.
 - Cross-Gramian-Based Joint Gramian: very efficient.
-
- Get the Source Code: <http://j.mp/modred13> .

Outlook:

- Add delay elements
- Towards Neural-Field-Model

Thanks!