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Interpolatory Model Reduction for Second Order Descriptor Systems

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Outline

Linear Descriptor Systems

- Linear time invariant descriptor/DAE system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \qquad E \in \mathbb{R}^{n\times n} \text{ is singular}$$

- The transfer matrix is

$$G(s) = C(sE - A)^{-1}B + D.$$

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Linear Descriptor Systems

Strictly Proper and Polynomial Parts

- The Weierstraß cannonical form is

$$P^{-1}(sE - A)Q = \begin{bmatrix} sI_f - J & 0 \\ 0 & sN - I_\infty \end{bmatrix},$$

P and Q are nonsingular,

J - Jordan block ($\lambda_j(J)$ are finite eigenvalues of $\lambda E - A$),

N - nilpotent ($N^{v-1} \neq 0,\ N^v = 0 \rightarrow v$ is index of $\lambda E - A$).

- Spectral projectors onto the deflating subspaces of $\lambda E - A$

$$P_l = P \begin{bmatrix} I_f & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \qquad Q_l = I - P_l = P \begin{bmatrix} 0 & 0 \\ 0 & I_\infty \end{bmatrix} P^{-1}$$

$$P_r = Q \begin{bmatrix} I_f & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \qquad Q_r = I - P_r = Q \begin{bmatrix} 0 & 0 \\ 0 & I_\infty \end{bmatrix} Q^{-1}$$

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Linear Descriptor Systems

Strictly Proper and Polynomial Parts

- The spectral projectors decompose $G(s)$ as,

$$G(s) = \underbrace{CP_r(sE-A)^{-1}P_lB}_{G_{sp}(s)} + \underbrace{CQ_r(sE-A)^{-1}Q_lB + D}_{P(s)}$$

$$G(s) = CQ\begin{bmatrix} (sI_f - J)^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}B + CQ\begin{bmatrix} 0 & 0 \\ 0 & (sN - I_\infty)^{-1} \end{bmatrix} P^{-1}B + D$$

- This partitioning is useful for model reduction but is computationaly expensive.

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Linear Descriptor Systems

Index Concept and Examples

- Index of a DAE system is the number of differentiations needed to transform the DAE into an ODE.
- Any solution of the DAE is also a solution of the underlying ODE.
- For linear DAEs this is equal to nilpotency index v.

Examples

- Index 1 DAE (semi-explicit systems)

$$E = \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $E_{11} - E_{12}A_{22}^{-1}A_{21}$ and A_{22} are both nonsingular

$$N^0 = I, \qquad N^1 = 0.$$

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Linear Descriptor Systems

Index Concept and Examples

- Index 2 DAE (Stokes-like systems)

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix},$$

where E_{11} is nonsingular and A_{12}, A_{21}^T have full column rank.

$$N^1 \neq 0, \quad N^2 = 0.$$

- Index 3 DAE (Mechanical systems)

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Linear Descriptor Systems

Second order DAE's

- Second order system can also be written as,

$$\begin{aligned} M\ddot{q}(t) &= D\dot{q}(t)+Kq(t)+Bu(t) \\ y(t) &= Cq(t) \end{aligned} \quad \rightarrow \quad \begin{aligned} \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \dot{x}(t) &= \begin{bmatrix} 0 & I \\ D & K \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ B \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} C & 0 \end{bmatrix} x(t) \end{aligned}$$

where $x(t) = \begin{bmatrix} q(t)^T & \dot{q}(t)^T \end{bmatrix}^T$

- Special second order structure,

$M = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} K_1 & G_1 \\ G_2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$, $C^T = \begin{bmatrix} C_1^T \\ 0 \end{bmatrix}$

in which M_1 is invertable and G_1, G_2^T have full rank then,

$$\begin{bmatrix} I & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & I & 0 \\ K_1 & D_1 & G_1 \\ G_2 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ B_1 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} C_1 & 0 & 0 \end{bmatrix} x(t),$$

where $x(t) = [q_1(t)^T \ \dot{q}_1(t)^T \ q_2(t)^T]^T$

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MOR for DAE's

Model Reduction Via Projection

- Given a descriptor system,

$$\Sigma : \begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \qquad \dim(\Sigma) = n$$

$$G(s) = C(sE - A)^{-1}B + D$$

find a reduced system,

$$\tilde{\Sigma} : \begin{cases} \tilde{E}\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \\ \tilde{y}(t) = \tilde{C}\tilde{x}(t) + \tilde{D}u(t) \end{cases} \qquad \dim(\tilde{\Sigma}) = r < n$$

$$\tilde{G}(s) = \tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B} + \tilde{D}$$

MOR for DAE's

Model Reduction Via Projection

- $\tilde{G}(s)$ tangentially interpolates $G(s)$ at $s = \sigma \in \mathbb{C}$ along right and left directions $b, c \in \mathbb{C}^n$ if

$$\tilde{G}(\sigma)b = G(\sigma)b, \quad c^T\tilde{G}(\sigma) = c^T G(\sigma)$$

Standard Projection

- Compute basis matrices $V,\ W \in \mathbb{R}^{n\times r}$
- Approximate $x(t)$ by $V\tilde{x}(t)$
- Ensure Petrov-Galerkin condition:

$$W^T(EV\dot{\tilde{x}}(t) - AV\tilde{x}(t) - Bu(t)) = 0,$$
$$y(t) = CV\tilde{x}(t) + Du(t)$$

- Reduced system matrices

$$\tilde{E}=W^TEV,\ \tilde{A}=W^TAV,\ \tilde{B}=W^TB,\ \tilde{C}=CV,\ \tilde{D}=D$$

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MOR for DAE's

Standard Subspaces

- Interpolatory subspaces

$$\begin{aligned}\mathrm{Ran}(V) &= \mathrm{span}\{(\sigma_1 E - A)^{-1} B b_1, \ldots, (\sigma_r E - A)^{-1} B b_r\} \\ \mathrm{Ran}(W) &= \mathrm{span}\{(\sigma_1 E - A^T)^{-1} C^T c_1, \ldots, (\sigma_r E - A^T)^{-1} C^T c_r\}\end{aligned}$$

$\sigma_k \in \mathbb{C},\ b_k \in \mathbb{C}^p,\ c_k \in \mathbb{C}^q$ for $k = 1, \ldots, r$

- Interpolating approximation

$$\tilde{G}(s) = CV(sW^T EV - W^T AV)^{-1} W^T B + D$$

$$\begin{aligned}\tilde{G}(\sigma_k) b_k &= G(\sigma_k) b_k, \quad c_k^T \tilde{G}(\sigma_k) = c_k^T G(\sigma_k), \\ c_k^T \tilde{G}'(\sigma_k) b_k &= c_k^T G'(\sigma_k) b_k.\end{aligned}$$

$G'(\sigma)$ is derivative of $G(s)$ w.r.t. s, evaluated at $s = \sigma$

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$G'(\sigma)$ is derivative of $G(s)$ w.r.t. s, evaluated at $s = \sigma$

MOR for DAE's

Modified Subspaces

- The interpolation conditions hold as long as the inverses $(\sigma_k E - A)^{-1},\ k = 1, \ldots, r$ exist
- The conditions are independent of the singularity of E
- In E singular case, $G(s)$ might be improper while $\tilde{E} = W^T EV$ is, in general, nonsingular and $\tilde{G}(s)$ proper.
- This may produce an unbounded error
- To ensure bounded error, Weierstraß canonical form is used to decompose $G(s) = G_{sp}(s) + P(s)$ and the subspaces V and W are modified such that

$$\tilde{G}(s) = \tilde{G}_{sp}(s) + \tilde{P}(s)$$

in which $\tilde{G}_{sp}(s)$ interpolates $G_{sp}(s)$ and $\tilde{P}(s) = P(s)$.

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MOR for DAE's

Modified Subspaces

Theorem (Gugercin et al 2013)

Let $V = [V_f \quad V_\infty]$ *and* $W = [W_f \quad W_\infty]$*. Also*

- $Ran(V_f) = span\{(\sigma_1 E - A)^{-1} P_l B b_1, \ldots, (\sigma_r E - A)^{-1} P_l B b_r)\}$
- $Ran(W_f) = span\{(\sigma_1 E - A^T)^{-1} P_r^T C^T c_1, \ldots, (\sigma_r E - A^T)^{-1} P_r^T C^T c_r)\}$
- $Ran(V_\infty) = Ran(Q_r)$
- $Ran(W_\infty) = Ran(Q_l^T)$

Then,

$$\tilde{P}(s) = CV_\infty (W_\infty^T (sE - A) V_\infty)^{-1} W_\infty^T B + D = P(s),$$

$$\tilde{G}(\sigma_k) b_k = G(\sigma_k) b_k, \quad c_k^T \tilde{G}(\sigma_k) = c_k G(\sigma_k)$$

MOR for DAE's

Transformation based MOR

- Index-1 DAE transformation:

$$\begin{aligned} E_{11}\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \\ 0 &= A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) \\ y(t) &= C_1x_1(t) + C_2x_2(t) + Du(t) \end{aligned} \quad \rightarrow \quad \begin{aligned} E_{11}\dot{x}_1(t) &= \mathcal{A}_1x_1(t) + \mathcal{B}_1u(t) \\ y(t) &= \mathcal{C}_1x_1(t) + \mathcal{D}_1u(t) \end{aligned}$$

$\mathcal{A}_1 = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $\mathcal{B}_1 = B_1 - A_{12}A_{22}^{-1}B_2$, $\mathcal{C}_1 = C_1 - C_2A_{22}^{-1}A_{21}$ and $\mathcal{D}_1 = D - C_2A_{22}^{-1}B_2$

- Index-2 DAE transformation:[Heinkenschloss et al 2008]

$$\begin{aligned} E_{11}\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \\ 0 &= A_{21}x_1(t) \\ y(t) &= C_1x_1(t) + C_2x_2(t) + Du(t) \end{aligned} \quad \rightarrow \quad \begin{aligned} \Delta_l E_{11}\Delta_r\dot{x}_1(t) &= \Delta_l A_{11}\Delta_r x_1(t) + \Delta_l B_1u(t) \\ y(t) &= \mathcal{C}_1\Delta_r x_1(t) + \mathcal{D}_1u(t) \end{aligned}$$

$\Delta_l = I - A_{12}(A_{21}E_{11}^{-1}A_{12})^{-1}A_{21}E_{11}^{-1}$, $\mathcal{C}_1 = C - A_{22}^{-1}A_{21}$ and $\mathcal{D} = D - C_2A_{22}^{-1}A_{21}B_1$. Also $\Delta_r = I - E_{11}^{-1}A_{12}(A_{21}E_{11}^{-1}A_{12})^{-1}A_{21}$ and $\Delta_r x_1(t) = x_1(t)$.

MOR for DAE's

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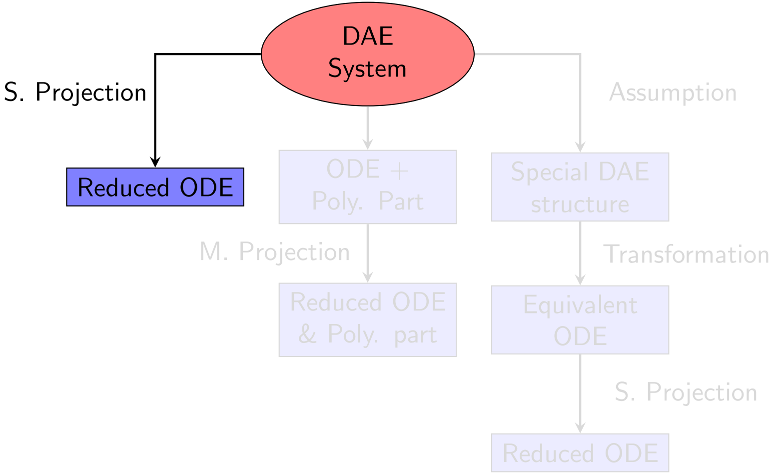
$\mathcal{A}_1 = A_{11} - A_{12}A_{22}^{-1}A_{21},\ \mathcal{B}_1 = B_1 - A_{12}A_{22}^{-1}B_2,\ \mathcal{C}_1 = C_1 - C_2A_{22}^{-1}A_{21}$ and $\mathcal{D}_1 = D - C_2A_{22}^{-1}B_2$

- Index-2 DAE transformation:[Heinkenschloss et al 2008]

$$\begin{aligned} E_{11}\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \\ 0 &= A_{21}x_1(t) \\ y(t) &= C_1x_1(t) + C_2x_2(t) + Du(t) \end{aligned} \quad \rightarrow \quad \begin{aligned} \Delta_l E_{11}\Delta_r\dot{x}_1(t) &= \Delta_l A_{11}\Delta_r x_1(t) + \Delta_l B_1u(t) \\ y(t) &= \mathcal{C}_1\Delta_r x_1(t) + \mathcal{D}_1u(t) \end{aligned}$$

$\Delta_l = I - A_{12}(A_{21}E_{11}^{-1}A_{12})^{-1}A_{21}E_{11}^{-1},\ \mathcal{C}_1 = C - A_{22}^{-1}A_{21}$ and $\mathcal{D} = D - C_2A_{22}^{-1}A_{21}B_1$. Also $\Delta_r = I - E_{11}^{-1}A_{12}(A_{21}E_{11}^{-1}A_{12})^{-1}A_{21}$ and $\Delta_r x_1(t) = x_1(t)$.

MOR for DAE's


DAE System
S. Projection
Reduced ODE
ODE + Poly. Part
M. Projection
Reduced ODE & Poly. part
Assumption
Special DAE structure
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Equivalent ODE
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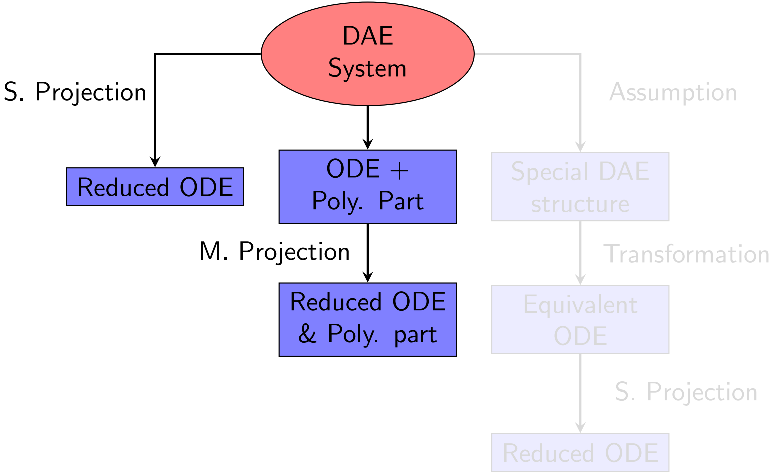
MOR for DAE's


DAE
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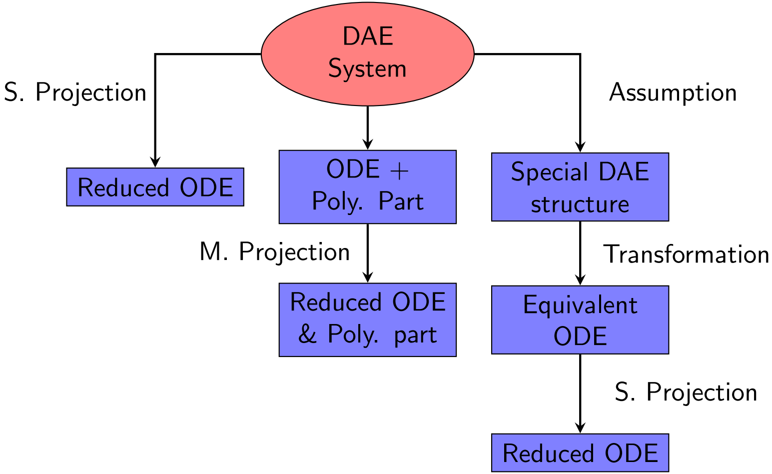
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MOR of Index-3 DAE

Equivalent ODE System

- Second order descriptor system is equivalent to,

$$\begin{bmatrix} I & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & I & 0 \\ K_1 & D_1 & G_1 \\ G_2 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ B_1 \\ 0 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} C_1 & 0 & 0 \end{bmatrix} x(t),$$

- Defining $\Pi_l = I - G_1 G M_1^{-1}$ where $G = (G_2 M_1^{-1} G_1)^{-1} G_2$ and replacing x_3,

$$\begin{bmatrix} I & 0 \\ 0 & M_1 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Pi_l K_1 & \Pi_l D_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \Pi_l B_1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

- The structure implies $G_2 x_1(t) = 0$ and $G_2 x_2(t) = 0$, since $x_2(t) = \dot{x}_1(t)$. Then

$$G_2 v(t) = 0 \quad \text{iff} \quad \Pi_r v(t) = v(t)$$

where $\Pi_r = I - M_1^{-1} G_1 G$. [Heinkenschloss et al 2008]

MOR of Index-3 DAE

Equivalent ODE System

- These results give,

$$\begin{bmatrix} \Pi_r & 0 \\ 0 & M_1\Pi_r \end{bmatrix}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \Pi_r \\ \Pi_l K_1 \Pi_r & \Pi_l D_1 \Pi_r \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \Pi_l B_1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} C_1\Pi_r & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

where, $\Pi_L = \begin{bmatrix} \Pi_l & 0 \\ 0 & \Pi_l \end{bmatrix}$, $\Pi_R = \begin{bmatrix} \Pi_r & 0 \\ 0 & \Pi_r \end{bmatrix}$

- Decomposing Π_L and Π_R into full rank matrices such that

$$\Pi_L = V_L W_L^T, \quad \Pi_R = V_R W_R^T \text{ and } W_L^T V_L = W_R^T V_R = I$$

$$\boxed{\begin{aligned} W_L^T \mathcal{E} V_R \dot{\tilde{x}}(t) &= W_L^T \mathcal{A} V_R \tilde{x}(t) + W_L^T \mathcal{B} u(t), \\ y(t) &= \mathcal{C} V_R \tilde{x}(t) \end{aligned}}$$

where, $\tilde{x}(t) = W_R^T \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T$

MOR of Index-3 DAE

Equivalent ODE System

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$$\begin{bmatrix} \Pi_l\Pi_r & 0 \\ 0 & \Pi_l M_1 \Pi_r \end{bmatrix}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \Pi_l\Pi_r \\ \Pi_l K_1 \Pi_r & \Pi_l D_1 \Pi_r \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \Pi_l B_1 \end{bmatrix} u(t),$$

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where, $\tilde{x}(t) = W_R^T \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T$

MOR of Index-3 DAE

Equivalent ODE System

- These results give,

$$\Pi_L \underbrace{\begin{bmatrix} I & 0 \\ 0 & M_1 \end{bmatrix}}_{\mathcal{E}} \Pi_R \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \Pi_L \underbrace{\begin{bmatrix} 0 & I \\ K_1 & D_1 \end{bmatrix}}_{\mathcal{A}} \Pi_R \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \Pi_L \underbrace{\begin{bmatrix} 0 \\ B_1 \end{bmatrix}}_{\mathcal{B}} u(t),$$

$$y(t) = \underbrace{\begin{bmatrix} C_1 & 0 \end{bmatrix}}_{\mathcal{C}} \Pi_R \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where, $\Pi_L = \begin{bmatrix} \Pi_l & 0 \\ 0 & \Pi_l \end{bmatrix}$, $\Pi_R = \begin{bmatrix} \Pi_r & 0 \\ 0 & \Pi_r \end{bmatrix}$

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where, $\tilde{x}(t) = W_R^T \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T$

MOR of Index-3 DAE

Equivalent ODE System

- These results give,

$$\Pi_L \underbrace{\begin{bmatrix} I & 0 \\ 0 & M_1 \end{bmatrix}}_{\mathcal{E}} \Pi_R \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \Pi_L \underbrace{\begin{bmatrix} 0 & I \\ K_1 & D_1 \end{bmatrix}}_{\mathcal{A}} \Pi_R \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \Pi_L \underbrace{\begin{bmatrix} 0 \\ B_1 \end{bmatrix}}_{\mathcal{B}} u(t),$$

$$y(t) = \underbrace{\begin{bmatrix} C_1 & 0 \end{bmatrix}}_{\mathcal{C}} \Pi_R \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where, $\Pi_L = \begin{bmatrix} \Pi_l & 0 \\ 0 & \Pi_l \end{bmatrix}$, $\Pi_R = \begin{bmatrix} \Pi_r & 0 \\ 0 & \Pi_r \end{bmatrix}$

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$$\boxed{\begin{array}{rcl} W_L^T \mathcal{E} V_R \dot{\tilde{x}}(t) & = & W_L^T \mathcal{A} V_R \tilde{x}(t) + W_L^T \mathcal{B} u(t), \\ y(t) & = & \mathcal{C} V_R \tilde{x}(t) \end{array}}$$

where, $\tilde{x}(t) = W_R^T \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T$

MOR of Index-3 DAE

Efficient MOR of Equivalent System

Lemma

v satisfies $v = \Pi_R v$ and $v = V_R(\sigma W_L^T \mathcal{E} V_R - W_L^T \mathcal{A} V_R)^{-1} W_L^T \mathcal{B} b$ iff

$$\begin{bmatrix} \sigma I & -I & G_1 & 0 \\ -K_1 & \sigma M_1 - D_1 & 0 & G_1 \\ G_2 & 0 & 0 & 0 \\ 0 & G_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ B_1 b \\ 0 \\ 0 \end{bmatrix}$$

w satisfies $w = \Pi_L^T w$ and $w = W_L(\sigma W_L^T \mathcal{E} V_R - W_L^T \mathcal{A} V_R)^{-T} V_R^T \mathcal{C}^T c$ iff

$$\begin{bmatrix} \sigma I & -I & G_2^T & 0 \\ -K_1^T & \sigma M_1^T - D_1^T & 0 & G_2^T \\ G_1^T & 0 & 0 & 0 \\ 0 & G_1^T & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} C_1^T c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

MOR of Index-3 DAE

Efficient MOR of Equivalent System

Lemma

v satisfies $v = \Pi_R v$ *and* $v = V_R(\sigma W_L^T \mathcal{E} V_R - W_L^T \mathcal{A} V_R)^{-1} W_L^T \mathcal{B} b$ *iff*

$$\begin{bmatrix} \sigma I & -I & G_1 & 0 \\ -K_1 & \sigma M_1 - D_1 & 0 & G_1 \\ G_2 & 0 & 0 & 0 \\ 0 & G_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ B_1 b \\ 0 \\ 0 \end{bmatrix}$$

w satisfies $w = \Pi_L^T w$ *and* $w = W_L(\sigma W_L^T \mathcal{E} V_R - W_L^T \mathcal{A} V_R)^{-T} V_R^T \mathcal{C}^T c$ *iff*

$$\begin{bmatrix} \sigma I & -I & G_2^T & 0 \\ -K_1^T & \sigma M_1^T - D_1^T & 0 & G_2^T \\ G_1^T & 0 & 0 & 0 \\ 0 & G_1^T & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} C_1^T c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

MOR of Index-3 DAE

Efficient MOR of Equivalent System

- let $v = V_R(\sigma W_L^T \mathcal{E} V_R - W_L^T \mathcal{A} V_R)^{-1} W_L^T \mathcal{B} b$, then using $W_R^T V_R = I$, and $V_R W_R^T = \Pi_R$,

$$W_L^T(\sigma\mathcal{E} - \mathcal{A})\Pi_R v = W_L^T \mathcal{B} b$$

- Also note that $v = \Pi_R v$,

$$\Pi_L\left((\sigma\mathcal{E} - \mathcal{A})v - \mathcal{B}b\right) = 0.$$

- Since, $\text{null}(\Pi_L) = \text{range}\begin{pmatrix} G_1 & 0 \\ 0 & G_1 \end{pmatrix}$,

$$\begin{pmatrix} \sigma I & -I \\ -K_1 & \sigma M_1 - D_1 \end{pmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} 0 \\ B_1 b \end{bmatrix} = -\begin{bmatrix} G_1 y_1 \\ G_1 y_2 \end{bmatrix}.$$

MOR of Index-3 DAE

IRKA for index-3 DAE

1. Make an initial selection of shifts $S_m = [\sigma_1, \ldots, \sigma_r]$ and tangent directions b_i, c_i, $i = 1, \ldots, r$
2. while (not converged)
   - Solve the linear systems for $x_{\sigma_i} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\tilde{x}_{\sigma_i} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ associated with each interpolation and corresponding tangents.
   - $V = [x_{\sigma_1} \cdots x_{\sigma_m}]$ and $W = [\tilde{x}_{\sigma_i} \cdots \tilde{x}_{\sigma_i}]$
   - Update the interpolation points and tangent directions
3. Return $\tilde{E} = W^T EV$, $\tilde{A} = W^T AV$, $\tilde{B} = W^T B$ and $\tilde{C} = CV$.

MOR of Index-3 DAE

Structure Preserving MOR

- $G(s)$ has a second order structure while $\tilde{G}(s)$ loses this structure.
- Let V and W be partitioned as,

$$V = \begin{bmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{bmatrix}, \quad W = \begin{bmatrix} \mathcal{W}_1 \\ \mathcal{W}_2 \end{bmatrix},$$

where $\mathcal{V}_i, \ \mathcal{W}_i \in \mathbb{R}^{n_1 \times r}, \ i = 1, 2.$ Defining $\mathcal{V}, \mathcal{W} \in \mathbb{R}^{2n_1 \times 2r}$ as,

$$\mathcal{V} = \begin{bmatrix} \mathcal{V}_1 & 0 \\ 0 & \mathcal{V}_2 \end{bmatrix}, \quad \mathcal{W} = \begin{bmatrix} \mathcal{W}_1 & 0 \\ 0 & \mathcal{W}_2 \end{bmatrix},$$

and $\qquad \tilde{H}(s) = \mathcal{C}\mathcal{V}(\mathcal{W}^T(s\mathcal{E} - \mathcal{A})\mathcal{V})^{-1}\mathcal{W}^T\mathcal{B}$

MOR of Index-3 DAE

Structure Preserving MOR

- $\tilde{H}(s)$ has a second order structure like,

$$G(s) = \mathcal{C} V_r (W_l^T (s^2 M_1 - sD_1 - K_1) V_r)^{-1} W_l^T B,$$

provided that $\mathcal{W}_1^T \mathcal{V}_1$ and $\mathcal{W}_1^T \mathcal{V}_2$ are invertible.
[Vandendorpe/Van Dooren 2004]

- $\tilde{H}(s)$ also tangentially interpolates $G(s)$ similar to $\tilde{G}(s)$

$$Im(V) \subset Im(\mathcal{V}), \quad Im(W) \subset Im(\mathcal{W})$$

- $\tilde{H}(s)$ however has degree $2r$ instead of r

Numerical Results

Example: Constrained damped mass-spring system: [Mehrmann/Stykel, 2005]

$$n = 10001, \; p = 1, \; q = 3, \; r = 20 \text{ and } \tilde{r} = 40$$

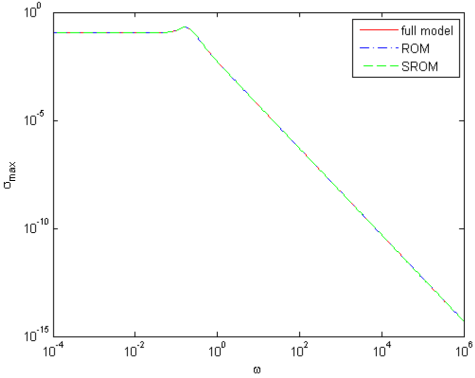

Figure : $\mathcal{H}_\infty$ norm of $G(s)$, $\tilde{G}(s)$ and $\tilde{H}(s)$

Numerical Results

Example: Constrained damped mass-spring system: [Mehrmann/Stykel, 2005]

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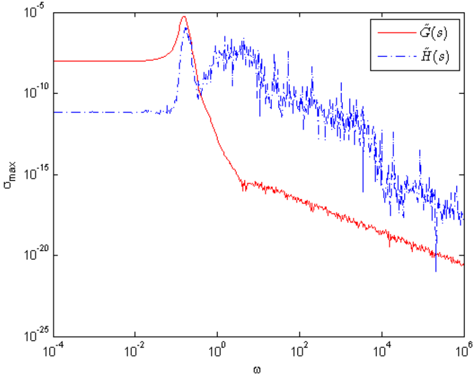

Figure : Absolute error in $\mathcal{H}_\infty$ norm for $\tilde{G}(s)$ and $\tilde{H}(s)$

Summary

- Special second order DAE's can be transformed to equivalent ODE systems
- Efficient reduction of the equivalent ODE sysem is possible with out computing or decomposing the oblique projectors.
- IRKA iterations can be used to select the optimal choice of interpolation points.
- Structure preserving approximation of the second order system can be computed.

Thanks for your attention