Parameterised Model Order Reduction for Uncertainty Quantification

Roland Pulch

Department of Mathematics and Computer Science

ERNST MORITZ ARNDT
UNIVERSITÄT GREIFSWALD

Wissen lockt. Seit 1456

Contents

- Linear dynamical systems with uncertain parameters
- Stochastic modelling of uncertainties
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System of ODEs :

$$\begin{aligned} \dot{\mathbf{x}}(t,\mathbf{p}) &= A(\mathbf{p})\mathbf{x}(t,\mathbf{p}) + B\mathbf{u}(t) \\ \mathbf{y}(t,\mathbf{p}) &= C\mathbf{x}(t,\mathbf{p}) \end{aligned} \qquad \Sigma = \left(\begin{array}{c|c} A(\mathbf{p}) & B \\ \hline C & 0 \end{array}\right)$$

Parameters : $\mathbf{p} = (p_1, \ldots, p_Q)$, $\mathbf{p} \in \Pi \subseteq \mathbb{R}^Q$

State variables : $\mathbf{x} : [t_0, t_1] \times \Pi \to \mathbb{R}^N$

Inputs : $\mathbf{u} : [t_0, t_1] \to \mathbb{R}^{N_{\text{in}}}$

Outputs : $\mathbf{y} : [t_0, t_1] \times \Pi \to \mathbb{R}^{N_{\text{out}}}$

Stability : $\lambda(\mathbf{p})$ eigenvalue of $A(\mathbf{p}) \longrightarrow \text{Re}(\lambda(\mathbf{p})) < 0$ f.a. $\mathbf{p} \in \Pi$

Dimension N is small or large!

Assumption : $\mathbf{x}(0,\mathbf{p}) = \mathbf{0}$

Laplace transforms :

$$\mathbf{X}(s,\mathbf{p}) := \mathcal{L}(\mathbf{x}(\cdot,\mathbf{p}))(s) := \int_0^\infty \mathrm{e}^{-st}\,\mathbf{x}(t,\mathbf{p})\,\mathrm{d}t \qquad \text{for } s \in \mathbb{C}$$

Likewise $\mathbf{U}(s)$, $\mathbf{Y}(s,\mathbf{p})$ for input and output.

Input-output relation : $\mathbf{Y}(s,\mathbf{p}) = H(s,\mathbf{p})\mathbf{U}(s)$

Transfer function : $H(s,\mathbf{p}) \in \mathbb{C}^{N_{\text{out}} \times N_{\text{in}}}$

$$H(s,\mathbf{p}) = C\left(sI_N - A(\mathbf{p})\right)^{-1} B$$

Domain of interest : $s = \mathrm{i}\,\omega \quad \text{for } \omega \in \mathbb{R}$

Parameters : $\mathbf{p} = (p_1, \ldots, p_Q), \quad \mathbf{p} \in \Pi \subseteq \mathbb{R}^Q$

Probability space : $(\Omega, \mathcal{A}, \mu)$

Random variables : $\mathbf{p}(\omega) = (p_1(\omega), \ldots, p_Q(\omega)), \quad \mathbf{p} : \Omega \to \Pi$

independent, Gaussian, uniform, beta, etc.

Expected value : $\langle f(\mathbf{p}) \rangle = \int_\Omega f(\mathbf{p}(\omega)) \, \mathrm{d}\mu(\omega) = \int_\Pi f(\mathbf{p})\rho(\mathbf{p}) \, \mathrm{d}\mathbf{p}$

Inner product : $\langle f(\mathbf{p}) g(\mathbf{p}) \rangle = \int_\Pi f(\mathbf{p}) g(\mathbf{p}) \rho(\mathbf{p}) \, \mathrm{d}\mathbf{p}$

Hilbert space : $L^2(\Pi, \rho) := \{ f : \Pi \to \mathbb{R} : \langle f(\mathbf{p})^2 \rangle < \infty \}$

Polynomial Chaos [Wiener 1938], [Xiu/Karniadakis 2002]

Random variables : $\mathbf{p}(\omega) = (p_1(\omega), \ldots, p_Q(\omega))$

Assumption : $\langle x_j(t, \mathbf{p})^2 \rangle < \infty$ for $j = 1, \ldots, N$

Polynomial Chaos expansion :

$$\mathbf{x}(t, \mathbf{p}(\omega)) = \sum_{i=0}^{\infty} \mathbf{v}_i(t) \Phi_i(\mathbf{p}(\omega))$$

Basis polynomials : $\Phi_i : \mathbb{R}^Q \to \mathbb{R}$ with $\langle \Phi_i \Phi_j \rangle = \delta_{ij}$, $\Phi_0 \equiv 1$

Gaussian distribution: Hermite polynomials
Uniform distribution: Legendre polynomials
Beta distribution: Jacobi polynomials

Coefficient functions : $\mathbf{v}_i : [t_0, t_1] \to \mathbb{R}^N$

Projection : $\mathbf{v}_i(t) = \langle \mathbf{x}(t, \mathbf{p}) \Phi_i(\mathbf{p}) \rangle$

Dynamical system :
$$\dot{\mathbf{x}}(t,\mathbf{p}) = A(\mathbf{p})\mathbf{x}(t,\mathbf{p}) + B\mathbf{u}(t)$$
$$\mathbf{y}(t,\mathbf{p}) = C\mathbf{x}(t,\mathbf{p})$$

Truncated P.C. :

$$\tilde{\mathbf{x}}(t,\mathbf{p}) = \sum_{i=0}^{M-1} \mathbf{v}_i(t)\Phi_i(\mathbf{p}), \qquad \tilde{\mathbf{y}}(t,\mathbf{p}) = \sum_{i=0}^{M-1} \mathbf{w}_i(t)\Phi_i(\mathbf{p})$$

Residual : $\mathbf{r}(t,\mathbf{p}) := \dot{\tilde{\mathbf{x}}}(t,\mathbf{p}) - A(\mathbf{p})\tilde{\mathbf{x}}(t,\mathbf{p}) - B\mathbf{u}(t)$

Galerkin method : $\langle \mathbf{r}(t,\mathbf{p})\Phi_l(\mathbf{p})\rangle = \mathbf{0}$ for $l = 0, 1, \ldots, M-1$

Larger system :
$$\begin{aligned} \dot{\mathbf{v}}(t) &= \hat{A}\mathbf{v}(t) + \hat{B}\mathbf{u}(t) \\ \mathbf{w}(t) &= \hat{C}\mathbf{v}(t) \end{aligned}$$

dimension $M \cdot N$, $\mathbf{v} := (\mathbf{v}_0, \ldots, \mathbf{v}_{M-1})$, $\mathbf{w} := (\mathbf{w}_0, \ldots, \mathbf{w}_{M-1})$

$\hat{A} := (\langle A(\mathbf{p})\Phi_i(\mathbf{p})\Phi_j(\mathbf{p})\rangle)_{i,j}$,

$\hat{B} := (1, 0, \ldots, 0)^\top \otimes B$, $\hat{C} := I_M \otimes C$

S: union of spectra of $A(\mathbf{p})$ for all $\mathbf{p} \in \Pi$

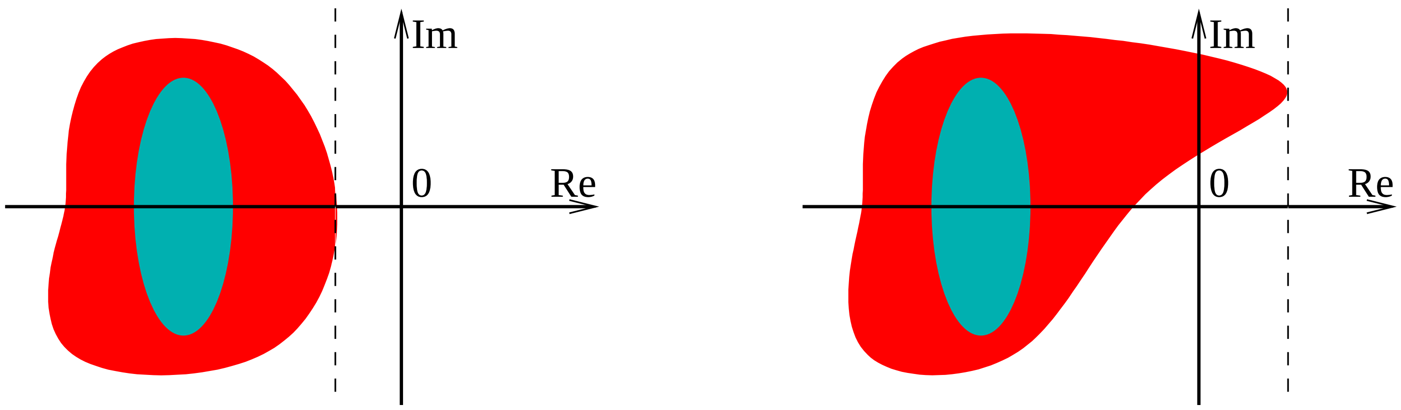
R: union of numerical ranges of $A(\mathbf{p})$ for all $\mathbf{p} \in \Pi$

$S \subseteq R$ in general, $R \subseteq$ convex hull of S for normal matrices

spectrum of $\widehat{A} \subset$ convex hull of R [Sonday et al. 2011]

blue : S (spectrum), red : R (numerical range)

stability guaranteed stability not guaranteed



Dynamical system : $\dot{\mathbf{v}}(t) = \hat{A}\mathbf{v}(t) + \hat{B}\mathbf{u}(t), \quad \mathbf{w}(t) = \hat{C}\mathbf{v}(t)$

Transfer function : $\mathbf{W}(s) = \hat{H}(s)\mathbf{U}(s)$

Approximations : $\hat{H}_i(s) \approx H_i(s) := \langle H(s, \mathbf{p})\Phi_i(\mathbf{p})\rangle$

High potential for reduction :

- huge dimension MN with $M = \frac{(Q+D)!}{Q!D!}$ for degree D.
- typically many P.C. coefficients in $\mathbf{v}$ are relatively small:
  - degree D may be chosen unnecessarily large,
  - for low degree D and large random dimension Q.

    [Doostan et al. 2009], [Conrad/Marzouk 2012]

MOR by moment matching :

[Mi et al. 2007], [Zou et al. 2007] Hermite P.C.

[P., ter Maten, Augustin 2013], [P., ter Maten 2013] generalised P.C.

Idea : analyse behaviour of all state variables: $C := I_N$, $\hat{C} := I_{MN}$

Controllability and Observability Gramian :

$$\widehat{A}W_{\mathsf{C}} + W_{\mathsf{C}}\widehat{A}^{\top} = -\widehat{B}\widehat{B}^{\top}, \quad \widehat{A}^{\top}W_{\mathsf{O}} + W_{\mathsf{O}}\widehat{A} = -\widehat{C}^{\top}\widehat{C}$$

Cholesky decompositions : $W_{\mathsf{C}} = L_{\mathsf{C}}L_{\mathsf{C}}^{\top}, \quad W_{\mathsf{O}} = L_{\mathsf{O}}L_{\mathsf{O}}^{\top}$

Singular value decomposition : $USV^{\top} = L_{\mathsf{C}}^{\top}L_{\mathsf{O}}$

Hankel singular values : $S = \mathrm{diag}(\sigma_1, \ldots, \sigma_{MN}), \quad \sigma_i \geq \sigma_{i+1}$

Truncation : $S_{\mathsf{red}} := \mathrm{diag}(\sigma_1, \ldots, \sigma_{N_{\mathsf{red}}})$

Transformation matrices : $P := S_{\mathsf{red}}^{-\frac{1}{2}}V_{\mathsf{red}}^{\top}L_{\mathsf{O}}^{\top}, \; Q := L_{\mathsf{C}}U_{\mathsf{red}}S_{\mathsf{red}}^{-\frac{1}{2}}$

Reduced matrices : $\widehat{A}_{\mathsf{red}} := P\widehat{A}Q, \quad \widehat{B}_{\mathsf{red}} := P\widehat{B}, \quad \widehat{C}_{\mathsf{red}} := \widehat{C}Q$

From Galerkin method : $$\tilde{\mathbf{y}}(t,\mathbf{p}) = \sum_{i=0}^{M-1} \mathbf{w}_i(t)\Phi_i(\mathbf{p})$$

From MOR : $$\tilde{\mathbf{w}}(t) = \widehat{C}_{\text{red}}\mathbf{v}^{\text{red}}(t)$$

$$\left\| \mathbf{y}(t,\mathbf{p}) - \sum_{i=0}^{M-1} \tilde{\mathbf{w}}_i(t)\Phi_i(\mathbf{p}) \right\|_{L^2(\Pi,\rho)}$$

$$\leq \|\mathbf{y}(t,\mathbf{p}) - \tilde{\mathbf{y}}(t,\mathbf{p})\|_{L^2(\Pi,\rho)} + \left\| \sum_{i=0}^{M-1} \left[\mathbf{w}_i(t) - \tilde{\mathbf{w}}_i(t)\right] \Phi_i(\mathbf{p}) \right\|_{L^2(\Pi,\rho)}$$

$$= \underbrace{\|\mathbf{y}(t,\mathbf{p}) - \tilde{\mathbf{y}}(t,\mathbf{p})\|_{L^2(\Pi,\rho)}}_{E_1(t)} + \underbrace{\|\mathbf{w}(t) - \tilde{\mathbf{w}}(t)\|_2}_{E_2(t)}$$

Balanced Truncation : $$\|\widehat{H}(s) - \widehat{H}^{\text{red}}(s)\|_{\mathcal{H}_\infty} \leq 2 \sum_{k=N_{\text{red}}+1}^{N} \sigma_k$$

$$\|E_2(t)\|_{L^2[0,\infty)} \leq \|\widehat{H}(s) - \widehat{H}^{\text{red}}(s)\|_{\mathcal{H}_\infty} \, \|\|\mathbf{u}(t)\|_2\|_{L^2[0,\infty)}$$

$$\ddot{\mathbf{z}} + \beta\dot{\mathbf{z}} + \omega_0^2\mathbf{z} + T(\mathbf{p})\mathbf{z} = \tilde{B}u(t)$$

$$T(\mathbf{p}) := \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_Q \end{pmatrix} \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \qquad \tilde{B} := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

space-discretisation of the linearised Sine-Gordon PDE with damping

$$A = \left(\begin{array}{c|c} 0 & I_Q \\ \hline -T - \omega_0^2 I_Q & -\beta I_Q \end{array}\right) \qquad B = \left(\begin{array}{c} 0 \\ \hline \tilde{B} \end{array}\right)$$

$$C := \left(\begin{array}{c|c} 0 \cdots 0\ 1 & 0 \end{array}\right) \qquad \text{or} \qquad C := I_{2Q}$$

$$\beta = 1 \qquad \omega_0 = 0.1$$

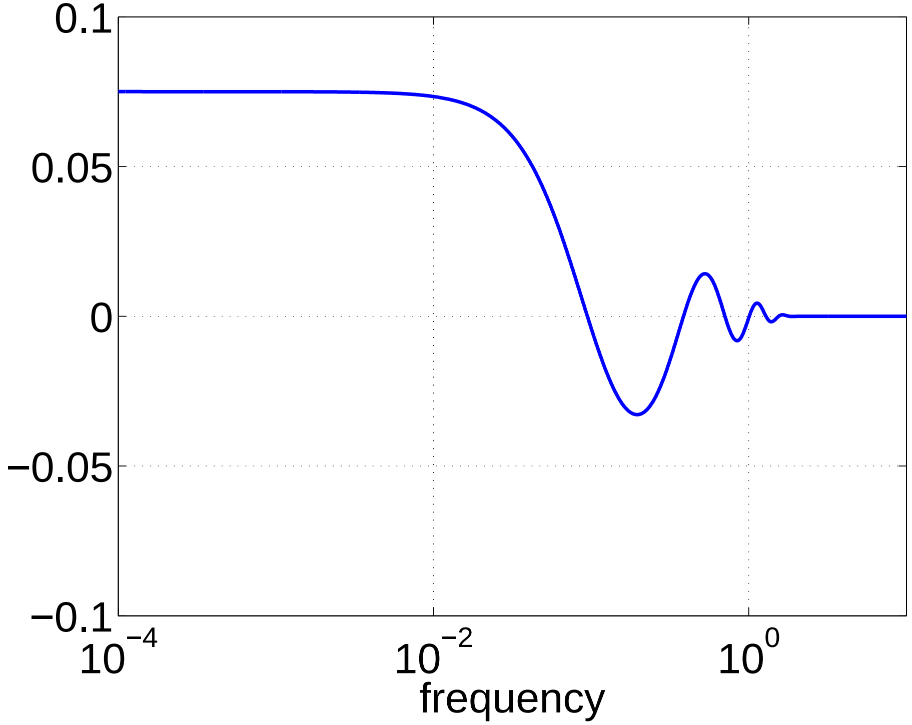
Transfer Function of Final Oscillator

$$Q = 10 \text{ oscillators}$$

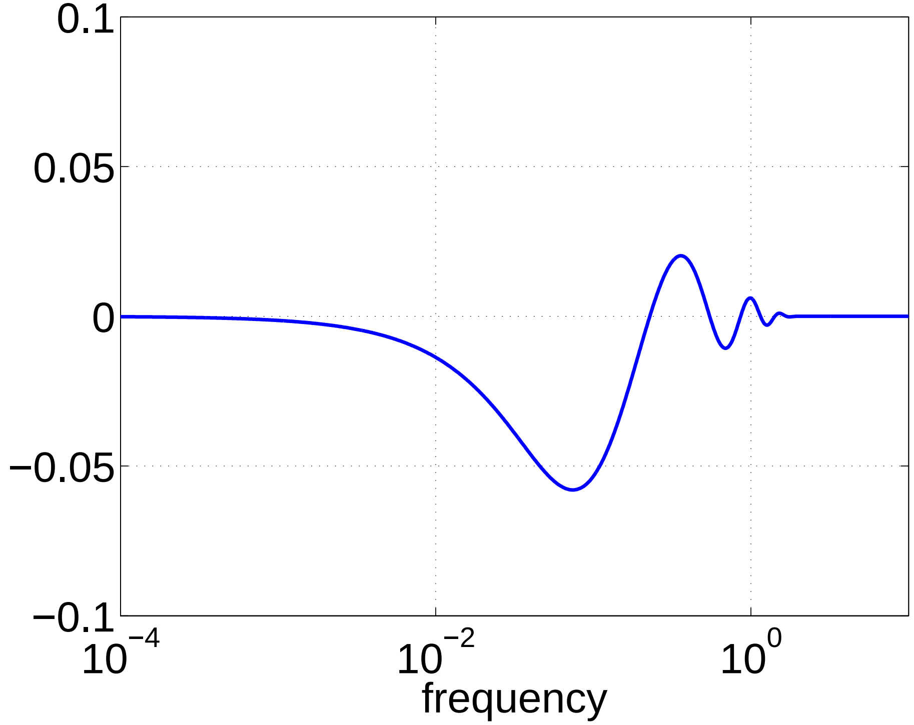
$$p_1 = p_2 = \cdots = p_{10} = 1$$

transfer function $H(\mathrm{i}\omega)$ for $\omega \in \mathbb{R}$

real part



imaginary part



Test Example: Stochastic Modelling

Dimension : $Q = 10 \quad N = 2Q = 20$

Random parameters :

$p_q \in [0.8, 1.2]$ for $q = 1, \ldots, 10$

independent uniform distributions

(variations of 20% around $\langle p_q \rangle = 1$)

P.C. expansion :

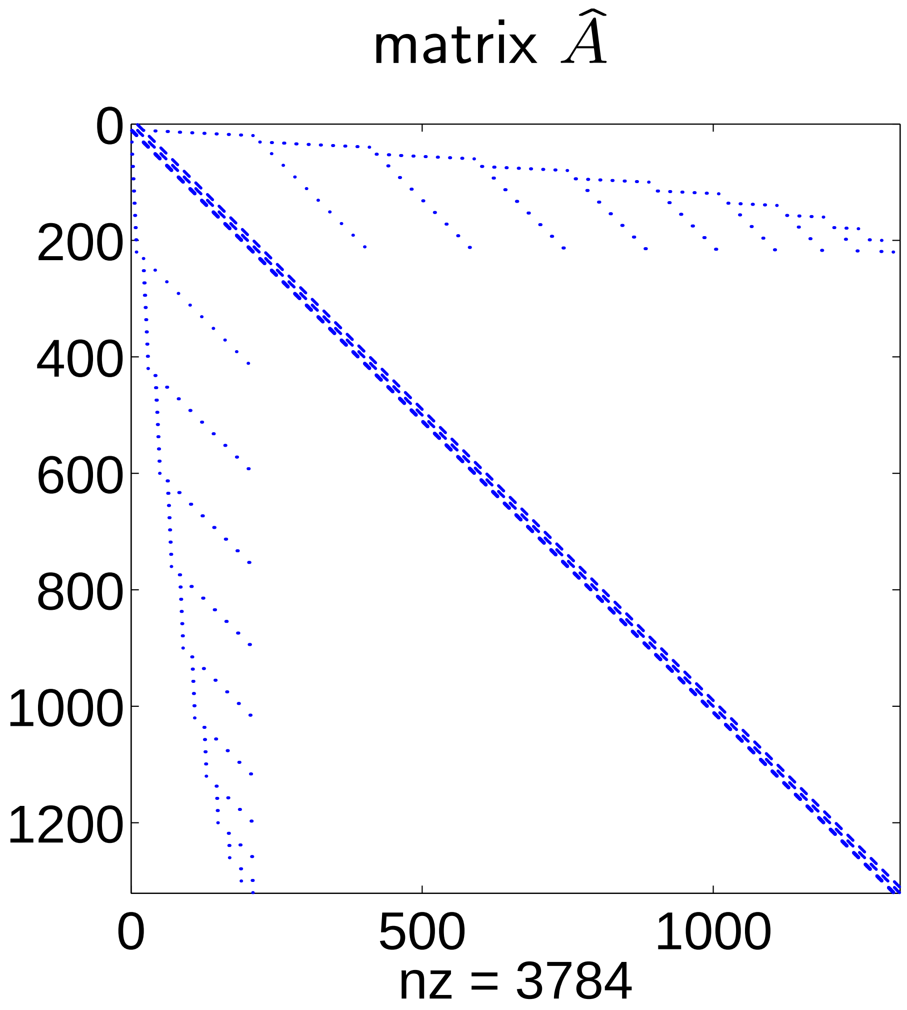
Legendre polynomials up to degree 2

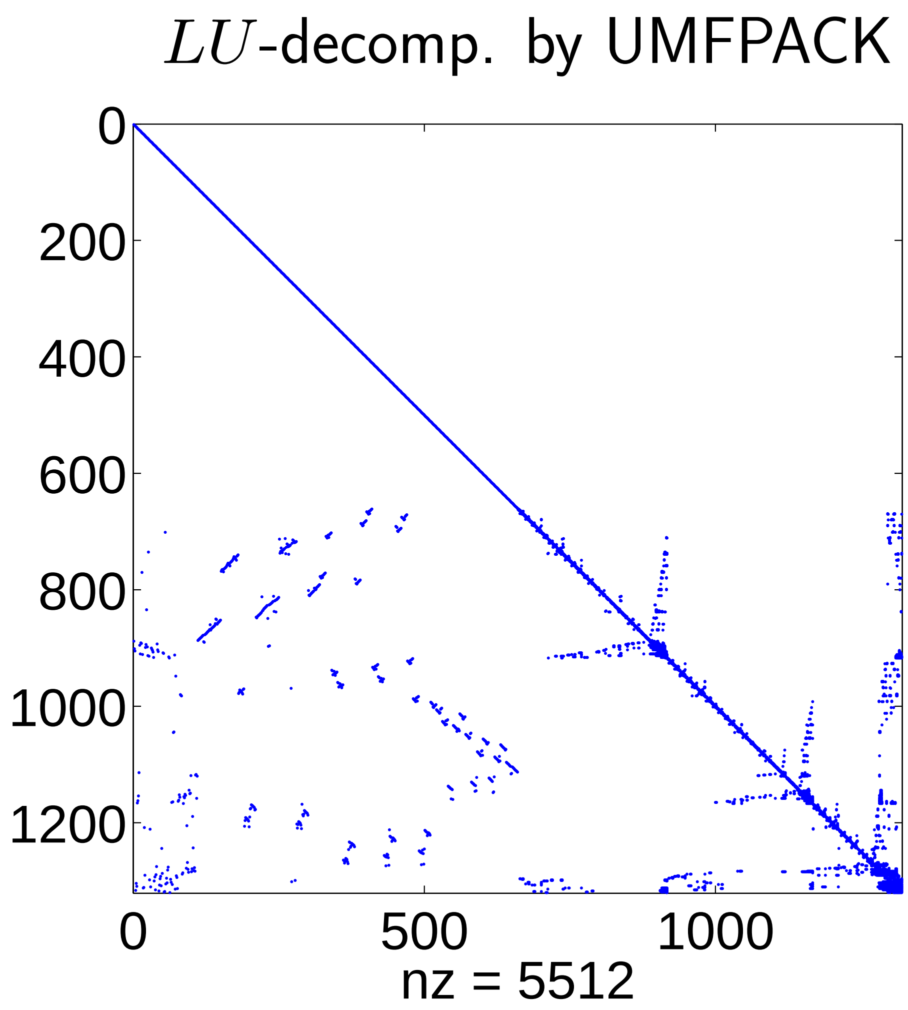
$M = 66$ basis functions

Galerkin method :

system of dimension $MN = 1320$

computation of matrix $\hat{A}$ by Stroud-5-quadrature

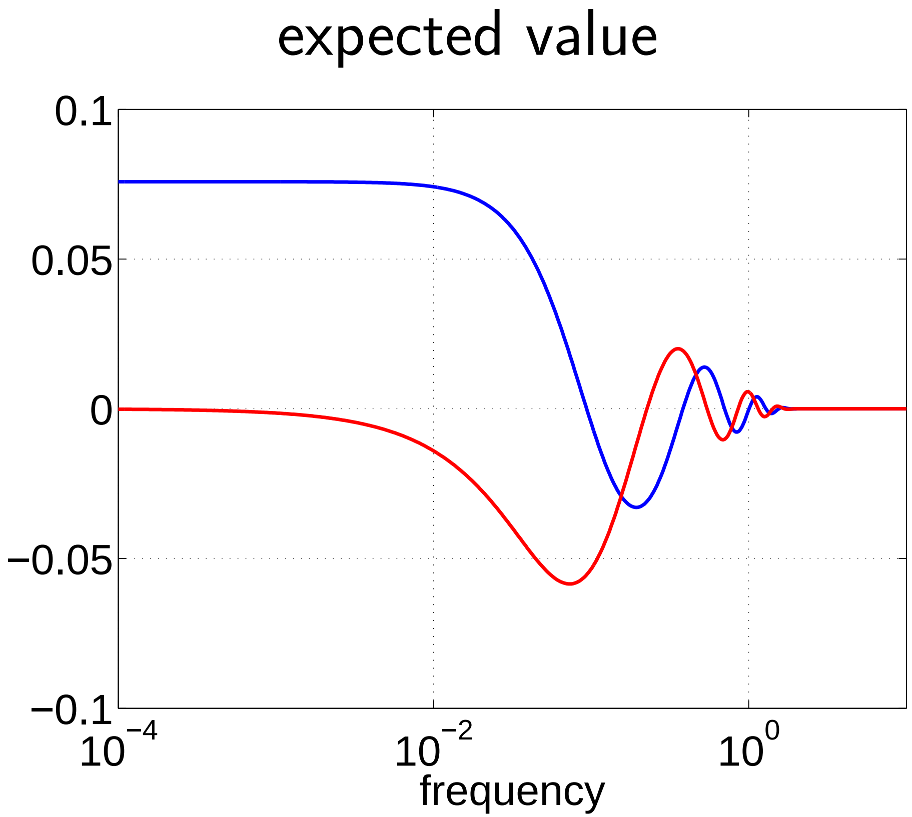

matrix $\widehat{A}$
0
200
400
600
800
1000
1200
0
500
1000
nz = 3784

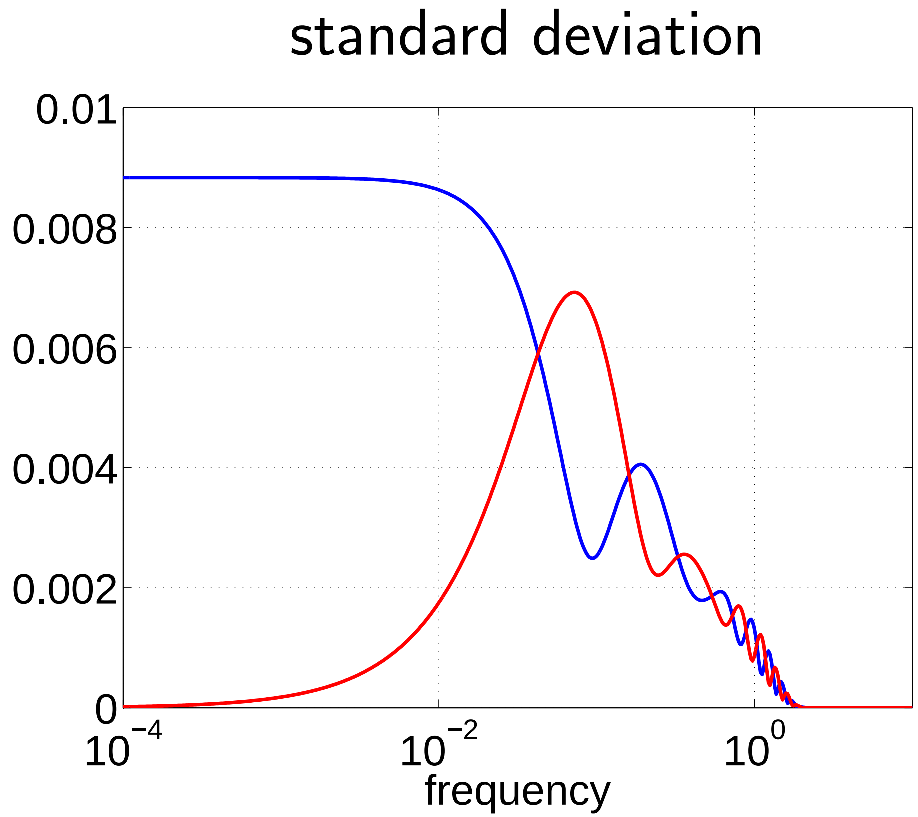

LU-decomp. by UMFPACK
0
200
400
600
800
1000
1200
0
500
1000
nz = 5512

$H(\mathrm{i}\omega, \mathbf{p})$ for single-output

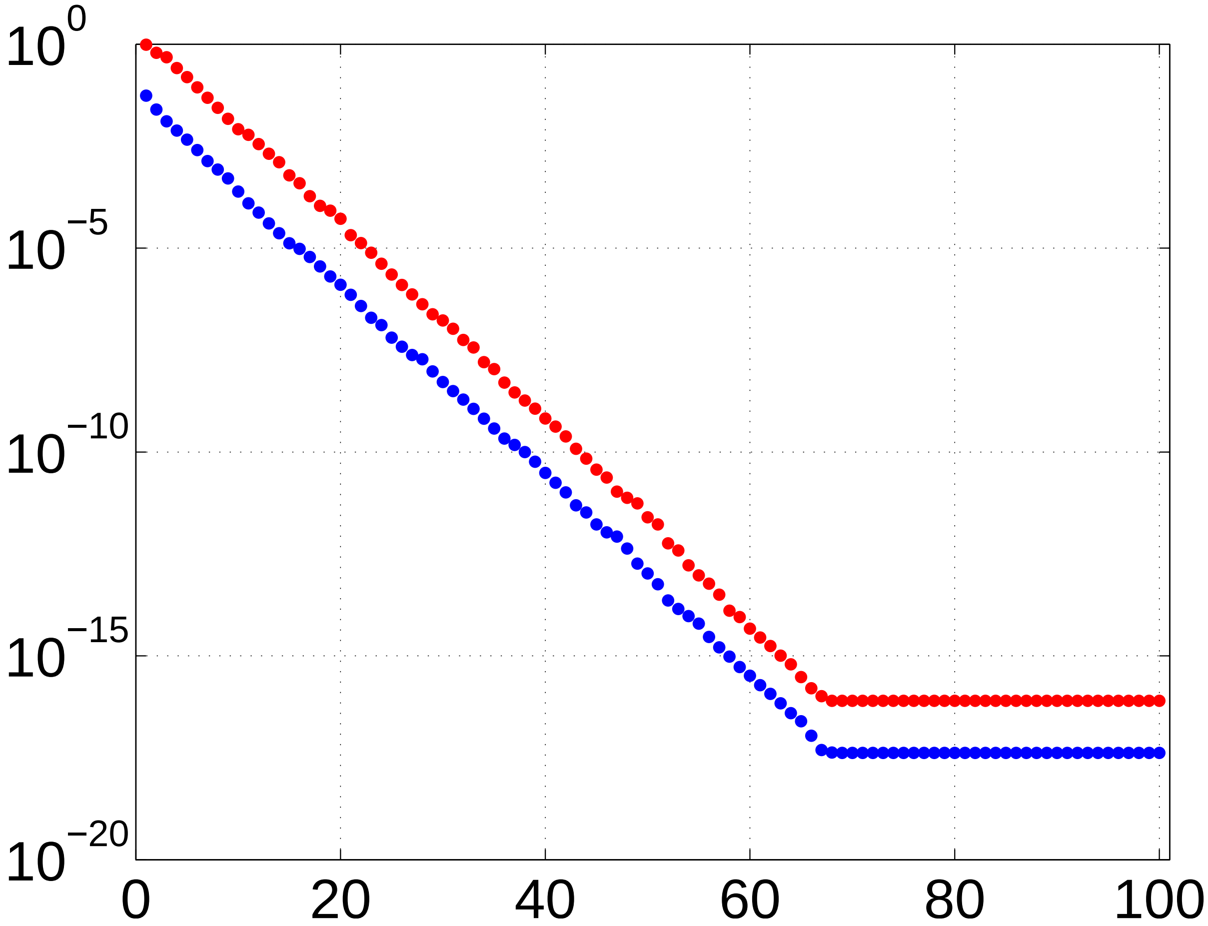
reconstructed by $\widehat{H}(\mathrm{i}\omega)$ from Galerkin system

real part imaginary part





$MN = 66 \cdot 20 = 1320$ singular values

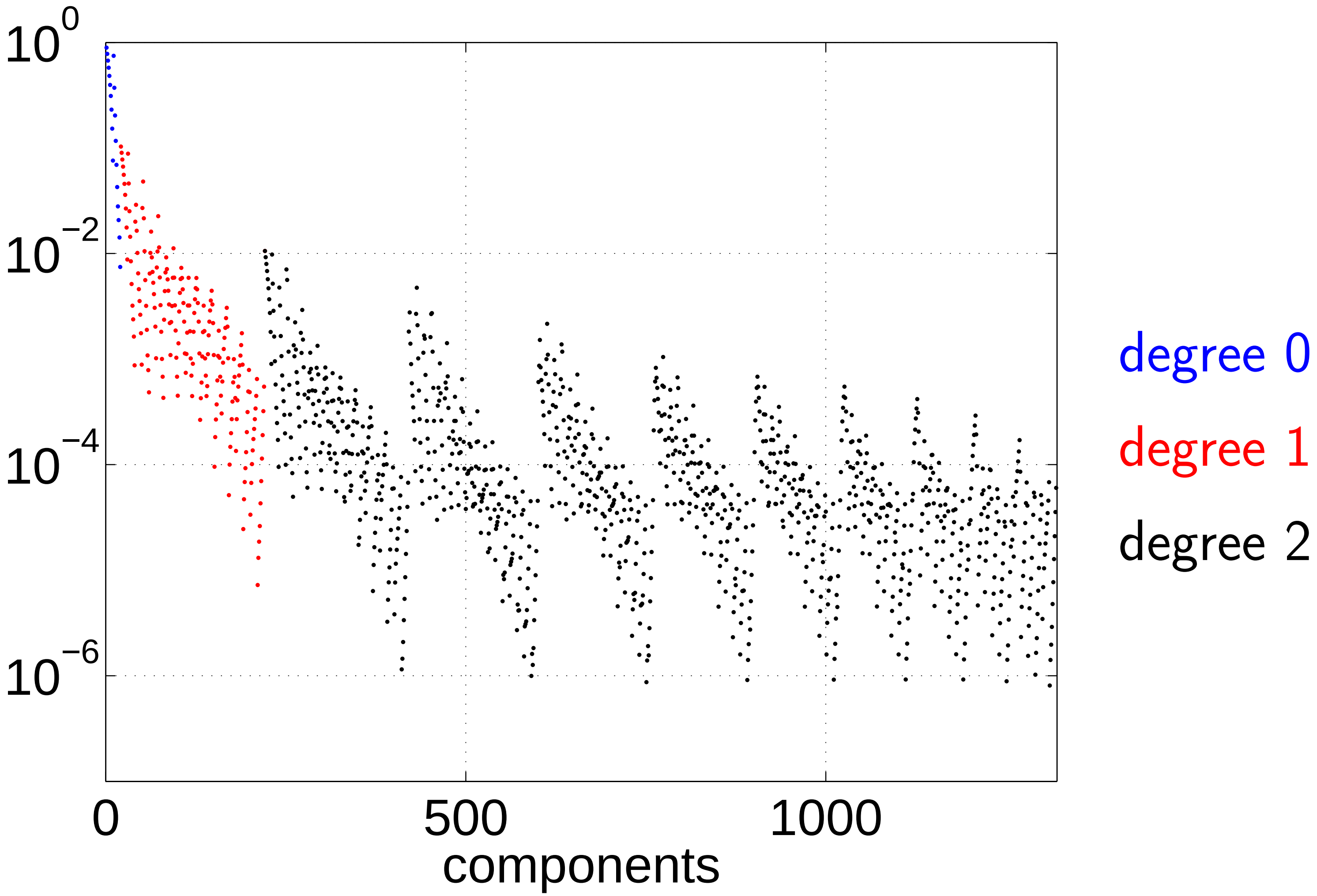


original system single-output (M outputs)

complete output (MN outputs)

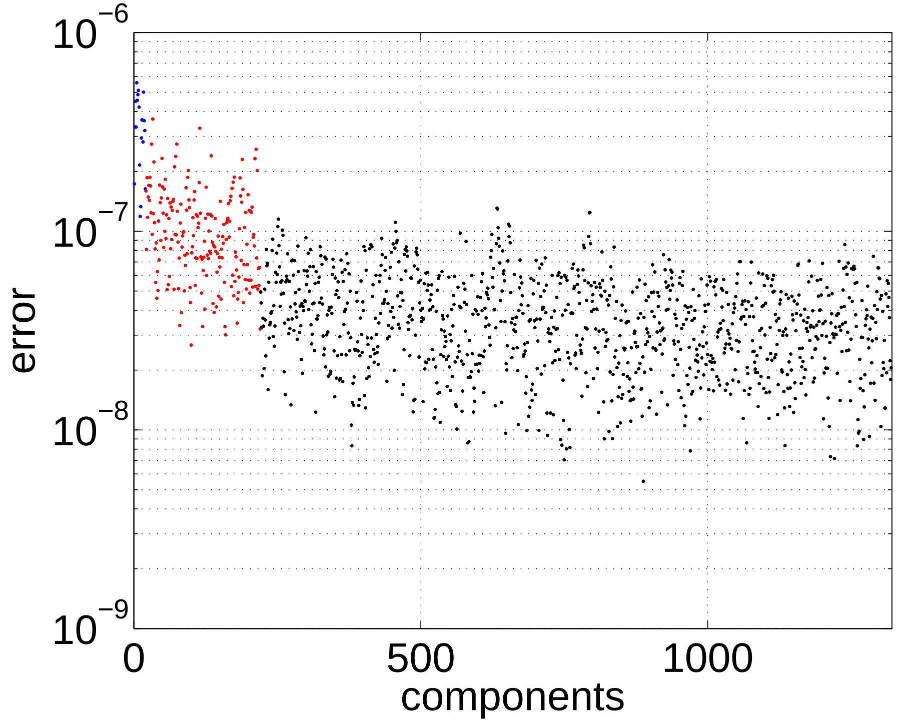
complete output ($\widehat{C} = I_{MN}$)

$$\max_{\omega \in [10^{-4}, 10^{1}]} |\widehat{H}_j(\mathrm{i}\omega)| \quad \text{for } j = 1, \ldots, 1320$$

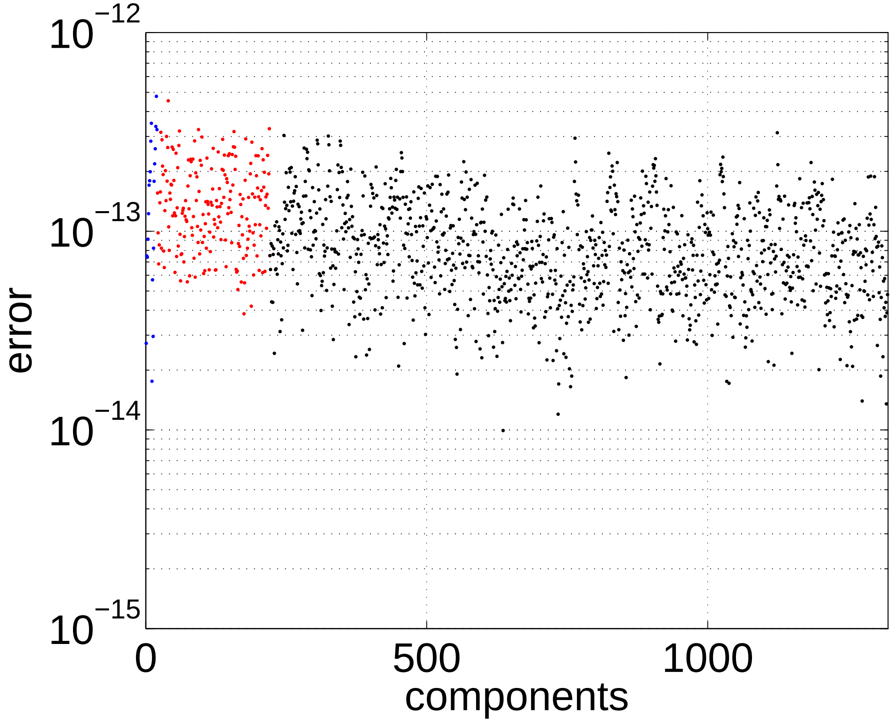


$$\max_{\omega\in[10^{-4},10^{1}]} \left| \hat{H}_j(\mathrm{i}\omega) - \hat{H}_j^{\mathrm{red}}(\mathrm{i}\omega) \right| \quad \text{for} \quad j = 1,\ldots,1320$$

reduction to $N_{\mathrm{red}} = 25$



reduction to $N_{\mathrm{red}} = 50$



Remark : $\left| \|\hat{H}_j\|_\infty - \|\hat{H}_j^{\mathrm{red}}\|_\infty \right| \leq \|\hat{H}_j - \hat{H}_j^{\mathrm{red}}\|_\infty$

Summary

- Uncertainty quantification via stochastic modelling
- Analysis of input-output/state behaviour by Galerkin system
- Analysis of input-output/state behaviour by reduced system
- Next step: Use reduced system in transient simulation

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