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Nonlinear Model Reduction Based on Generalized Moment-Matching

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Computational Methods in Systems and Control Theory

NETWORK THEORY

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Motivation

Given a large-scale state-nonlinear control system of the form

$$\boxed{\Sigma : \begin{cases} \dot{x}(t) = f(x(t)) + bu(t), \\ y(t) = cx(t), \quad x(0) = x_0, \end{cases}}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ nonlinear and $b, c^T \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$.

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Outline

1. Quadratic-Bilinear Differential-Algebraic Equations
2. Model Reduction via Moment-Matching
3. Numerical Examples

Quadratic-Bilinear DAEs

State-Space Representation

We will consider quadratic-bilinear SISO systems of the form

$$E\,\dot{x} = A_1\,x + A_2\,x \otimes x + N\,xu + b\,u$$

$$y = c\,x$$

where $E, A_1, N \in \mathbb{R}^{n\times n}, A_2 \in \mathbb{R}^{n\times n^2}$ (Hessian tensor), $b, c^T \in \mathbb{R}^n$.

- A large class of nonlinear control-affine systems can be transformed into the above type of control system.
- The transformation is exact, but a slight increase of the state dimension has to be accepted.
- Input-output behavior can be characterized by generalized transfer functions.

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$$x \otimes x := \begin{pmatrix} x_1 x_1 \\ x_1 x_2 \\ \vdots \\ x_1 x_n \\ \vdots \\ x_n x_1 \\ \vdots \\ x_n x_n \end{pmatrix}$$

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McCormick Relaxation

Example

Let us consider the following nonlinear control system:

$$\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1}, \quad \dot{x}_2 = \sin(x_2) + u, \quad \dot{x}_3 = -x_3^4.$$

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$$
\begin{aligned}
H_2(s_1, s_2) = \frac{1}{2!} c\left((s_1 + s_2)E - A_1\right)^{-1} \big[& N\left(G_1(s_1) + G_1(s_2)\right) \\
& + A_2\left(G_1(s_1) \otimes G_1(s_2) + G_1(s_2) \otimes G_1(s_1)\right)\big].
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The Galerkin Projection

Let us again consider the quadratic-bilinear system from the beginning

$$E\,\dot{x} = A_1\,x + A_2\,x \otimes x + N\,xu + b\,u$$

$$y = c\,x$$

where $E, A_1, N \in \mathbb{R}^{n\times n}, A_2 \in \mathbb{R}^{n\times n^2}, b, c^T \in \mathbb{R}^n$.

- How do we reduce the above system?

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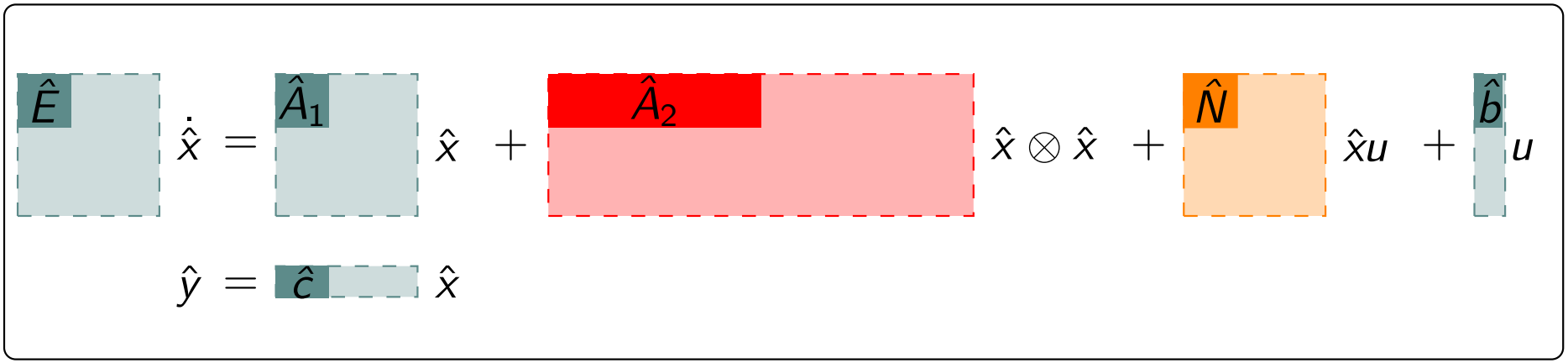

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  $\rightsquigarrow$ Galerkin projection $\mathcal{P} = VV^T,\ V \in \mathbb{R}^{n\times\hat{n}},\ V^TV = I$:

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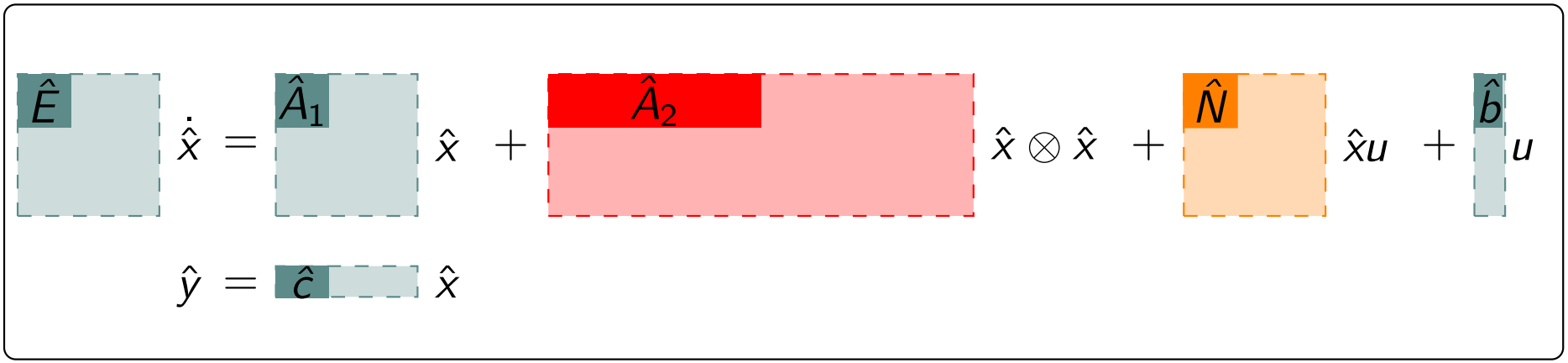

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- What is a good choice for the projection matrix V?

Model Reduction via Moment-Matching

The Moments of a Function

Definition

Let $H : \mathbb{C} \to \mathbb{C},\ s \mapsto H(s)$ be a meromorphic function. Then the k-th moment of H at $s_0 \in \mathbb{C}$ is defined as

$$m_k(s_0) = (-1)^k \frac{\partial^k}{\partial s^k} H(s) \bigg|_{s=s_0} .$$

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Moments locally specify H in the neighborhood of s_0, i.e.

$$\begin{aligned} H(s) &= H(s_0) + H^{(1)}(s_0)\frac{(s-s_0)}{1!} + \cdots + H^{(k)}(s_0)\frac{(s-s_0)^k}{k!} + \ldots \\ &= m_0(s_0) - m_1(s_0)\frac{(s-s_0)}{1!} + \cdots \pm m_k(s_0)\frac{(s-s_0)^k}{k!} \mp \ldots \end{aligned}$$

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For example, consider the (simplified) linear subsystem

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + bu(t) \\ y_1(t) &= cx_1(t) \end{aligned}$$

and its transfer function

$$H_1(s_1) = c(s_1 I - A_1)^{-1} b.$$

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Here, the moments can be computed as:

$$m_0(s_0) = c(s_0 I - A_1)^{-1} b, \quad m_k(s_0) = c(s_0 I - A_1)^{-(k+1)} b.$$

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$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + bu(t), \\ y_1(t) &= cx_1(t) \end{aligned}$$

and its transfer function

$$H_1(s_1) = c(s_1 I - A_1)^{-1} b.$$

Similarly, H_1 can be expanded at ∞, with Markov parameters:

$$m_0(\infty) = 0, \quad m_k(\infty) = cA_1^{k-1} b.$$

Model Reduction via Moment-Matching

The Concept of Moment-Matching

For the reduced (simplified) linear subsystem

$$\begin{aligned}\dot{\hat{x}}_1(t) &= \hat{A}_1\hat{x}_1(t) + \hat{b}u(t)\\ \hat{y}_1(t) &= \hat{c}\hat{x}_1(t)\end{aligned}$$

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$$\operatorname{span}\{V\} = \mathcal{K}_q\left((A_1 - s_0 I)^{-1}, (A_1 - s_0 I)^{-1} b\right).$$

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The Krylov subspace $\mathcal{K}_q(A, b)$ is defined as:

$$\mathcal{K}_q(A, b) = \operatorname{span}\{b, Ab, \ldots, A^{q-1}b\}$$

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Model Reduction via Moment-Matching

Generalized Moments for Multivariable Functions

We can easily extend these concepts to the multivariable case.

Definition

Let $H : \mathbb{C}^2 \to \mathbb{C}$, $(s_1, s_2) \mapsto H(s_1, s_2)$ be a sufficiently smooth complex function. Then

$$m_{k_1,k_2}(s_{1_0}, s_{2_0}) = (-1)^{k_1+k_2} \frac{\partial^{k_1+k_2}}{\partial s_1^{k_1} \partial s_2^{k_2}} H(s_1, s_2) \Bigg|_{s_1 = s_{1_0},\ s_2 = s_{2_0}}$$

is called multimoment of order $k_1 + k_2$.

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$$m_{k_1,\ldots,k_j}(s_{1_0}, \ldots, s_{j_0}) = (-1)^{\mathbf{k}} \frac{\partial^{\mathbf{k}}}{\partial s_1^{k_1} \cdots \partial s_j^{k_j}} H(s_1, \ldots, s_j) \Big|_{s_i = s_{i_0}}$$

is called multimoment of order $\mathbf{k} = k_1 + \cdots + k_j$.

Model Reduction via Moment-Matching

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For the approximation of higher transfer functions, we aim at matching some of these multimoments by the reduced system:

$$m_{k_1,\ldots,k_j}(s_{1_0}, \ldots, s_{j_0}) = \hat{m}_{k_1,\ldots,k_j}(s_{1_0}, \ldots, s_{j_0}).$$

Numerical Examples

The FitzHugh-Nagumo System

- FitzHugh-Nagumo system modeling a neuron

[Chaturantabut, Sorensen '09]

$$\begin{aligned} \epsilon v_t(x,t) &= \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g, \\ w_t(x,t) &= hv(x,t) - \gamma w(x,t) + g, \end{aligned}$$

with $f(v) = v(v-0.1)(1-v)$ and initial and boundary conditions

$$\begin{aligned} v(x,0) &= 0, & w(x,0) &= 0, & x &\in [0,1], \\ v_x(0,t) &= -i_0(t), & v_x(1,t) &= 0, & t &\geq 0, \end{aligned}$$

where
$\epsilon = 0.015,\ h = 0.5,\ \gamma = 2,\ g = 0.05,\ i_0(t) = 5 \cdot 10^4 t^3 \exp(-15t)$

- original state dimension $n = 2 \cdot 400$, QBDAE dimension $N = 3 \cdot 400$, reduced QBDAE dimension $r = 26$, chosen expansion point $\sigma = 1$
- 3D phase space

Numerical Examples

Jet Diffusion Flame Model

Next, let us focus on a nonlinear PDE arising in jet diffusion flame models [Galbally, Willcox '09]

$$\frac{\partial w}{\partial t} + U \cdot \nabla w - \nabla(\kappa \nabla w) + f(w) = 0, \quad (x,t) \in (0,1) \times (0,T),$$

with Arrhenius type term $f(w) = Aw(c-w)e^{-\frac{E}{d-w}}$ and constant parameters U, A, E, c, d, κ. Again define initial and boundary conditions:

$$\begin{aligned} w(x,0) &= 0, \quad x \in [0,1], \\ w(0,t) &= u(t), \quad t \geq 0, \\ w(1,t) &= 0, \quad t \geq 0, \\ w_{center} &= \begin{bmatrix} \mathbf{0} & 1 & \mathbf{0} \end{bmatrix}. \end{aligned}$$

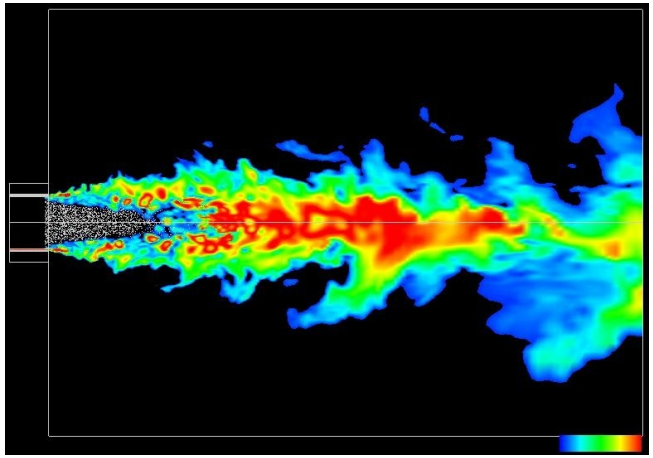
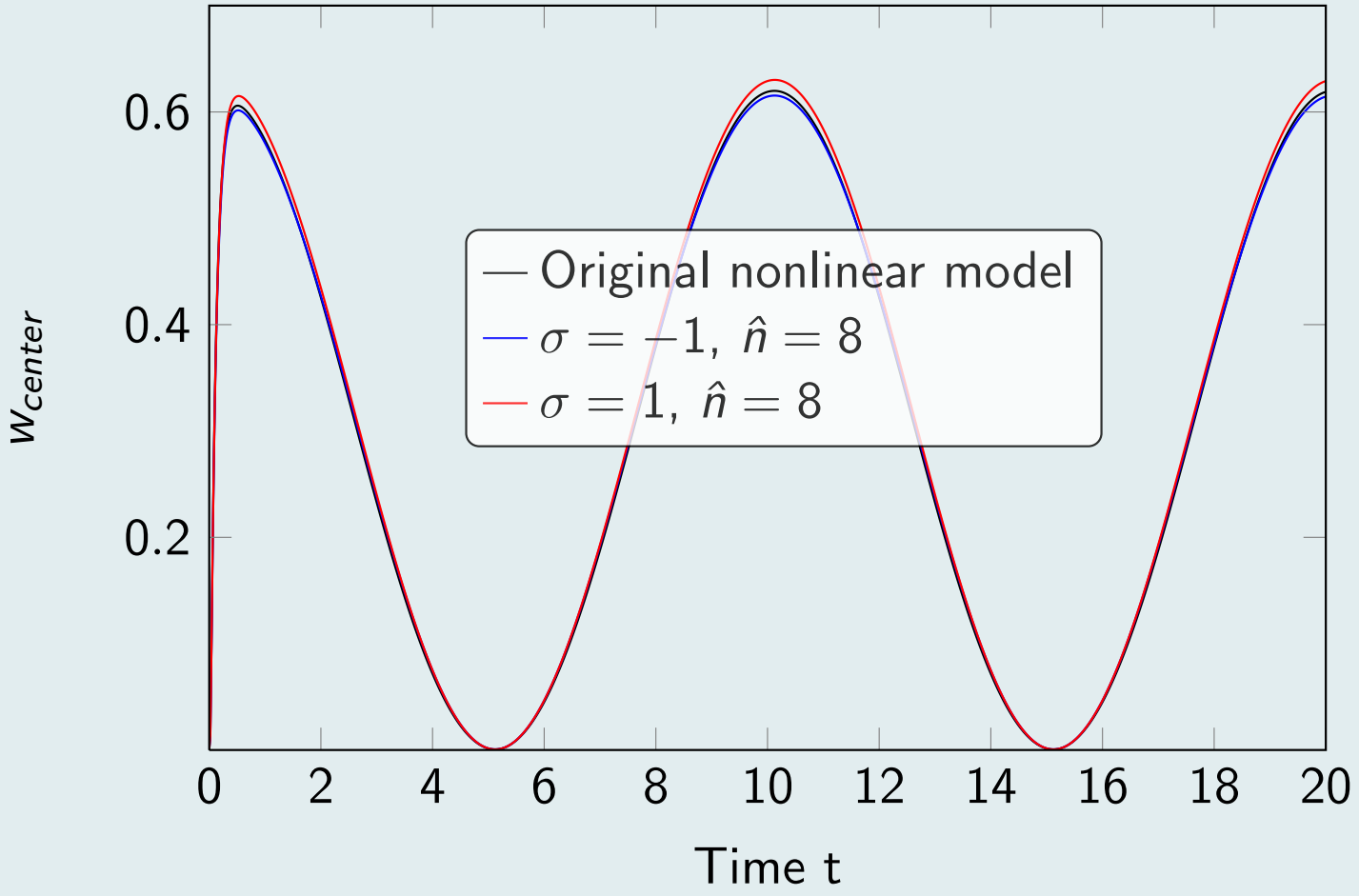


Figure: [Kurose]

Numerical Examples

Jet Diffusion Flame Model

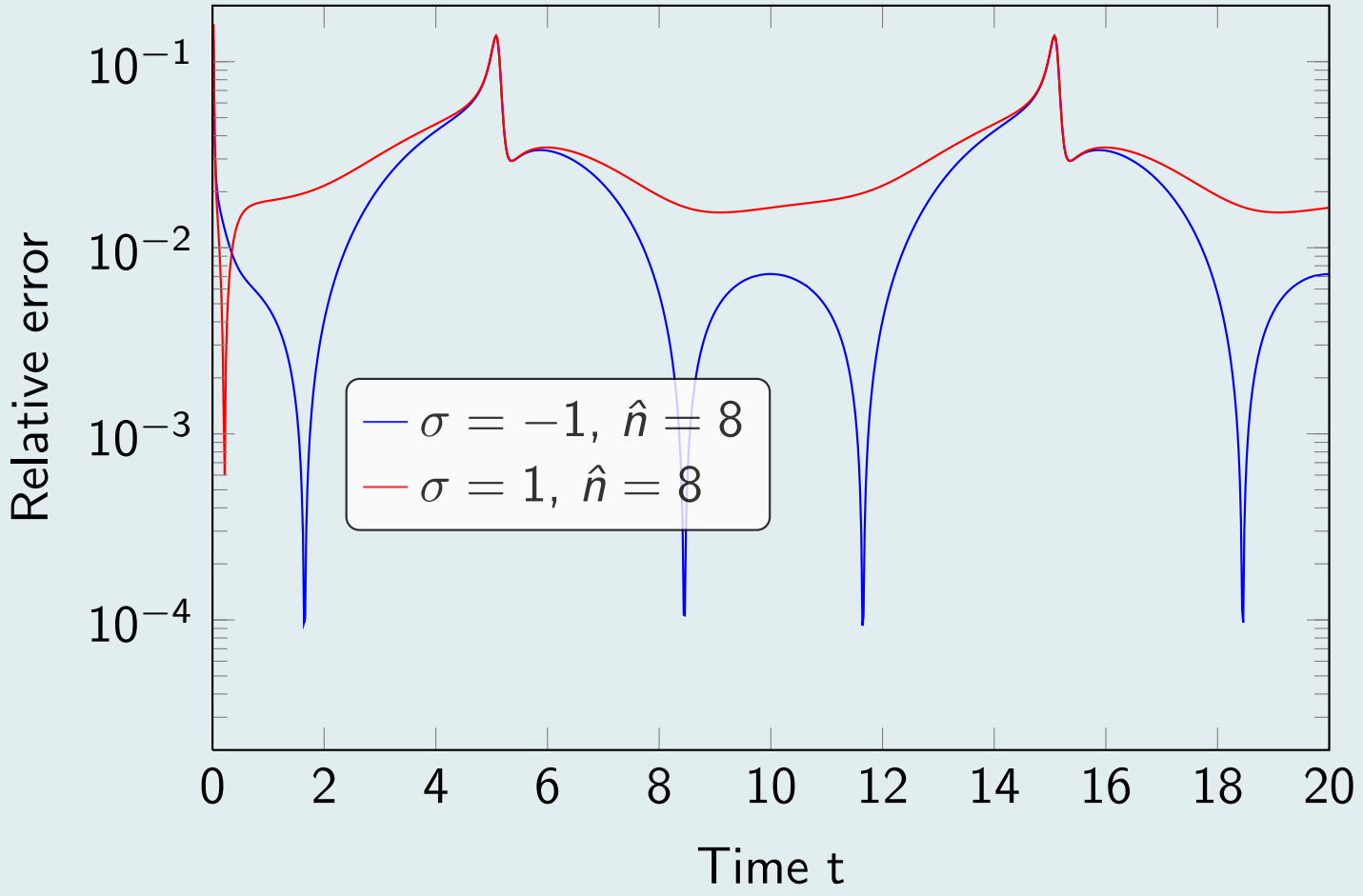
Transient responses for $k = 1500$ and $u(t) = \frac{1}{2}\cos(\frac{\pi t}{5} + 1)$



Numerical Examples

Jet Diffusion Flame Model

Relative errors for $k = 1500$ and $u(t) = \frac{1}{2}\cos(\frac{\pi t}{5} + 1)$



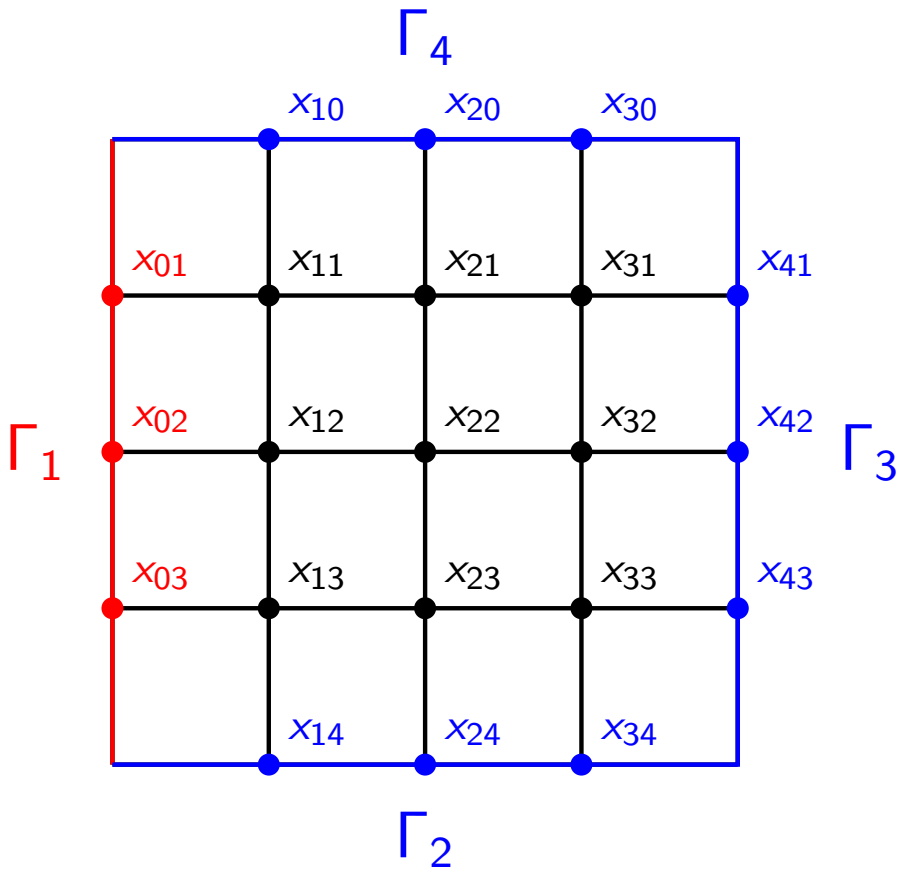
Numerical Examples

A Heat Transfer Model

- 2-dimensional heat distribution [Benner, Saak '05]
- boundary control by spraying intensities of a cooling fluid

$$\begin{aligned} \Omega &= (0,1) \times (0,1) \\ x_t &= \Delta x && \text{in } \Omega \\ n \cdot \nabla x &= u(x-1) && \text{on } \Gamma_1 \\ x &= 0 && \text{on } \Gamma_2, \Gamma_3, \Gamma_4 \end{aligned}$$

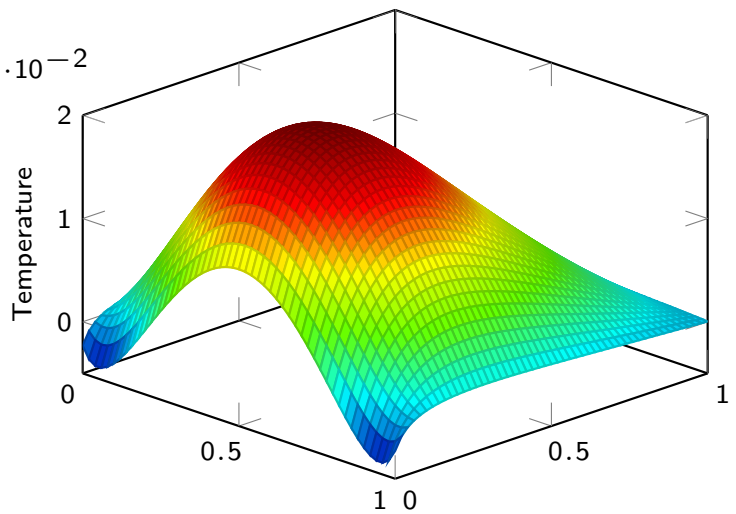
- spatial discretization $r \times r$-grid
  $\Rightarrow \dot{x} \approx A_1 x + Nxu + bu$
  $\Rightarrow A_2 = 0$
- state approximation $x \approx V\hat{x}$



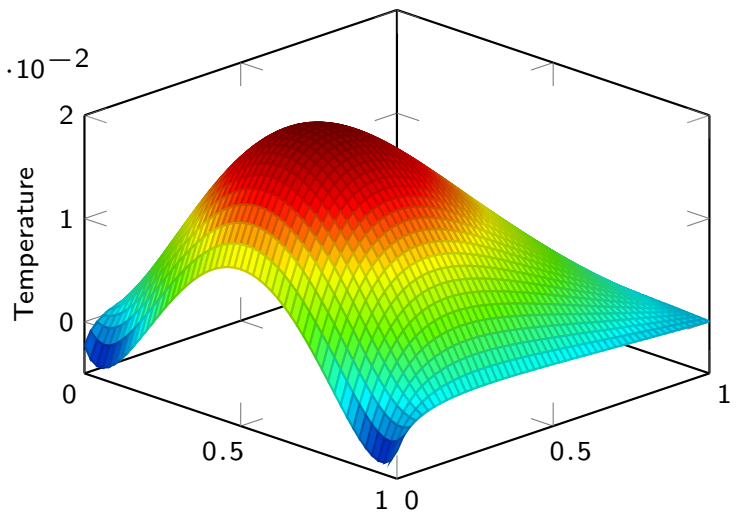
Numerical Examples

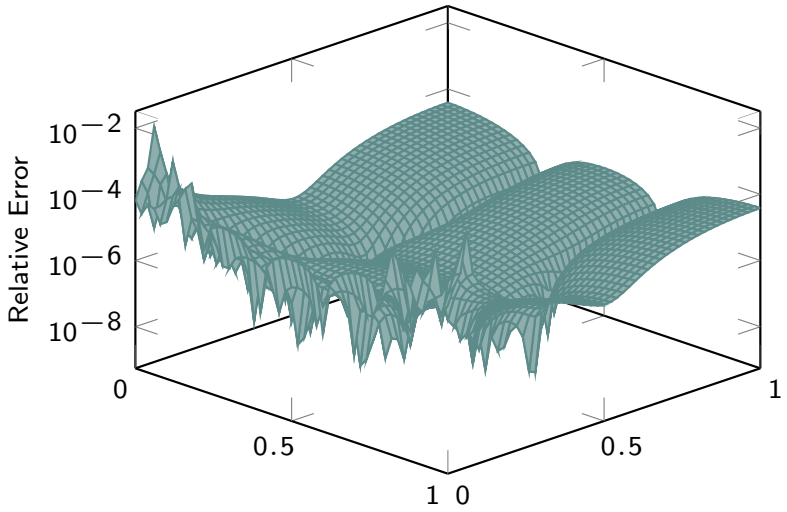
A Heat Transfer Model

Original Model, $t = 4.55s$, $n = 2601$, $u(t) = cos(\pi t)$



Reduced Model, $t = 4.55s$, $\hat{n} = 20$, $u(t) = cos(\pi t)$





Numerical Examples

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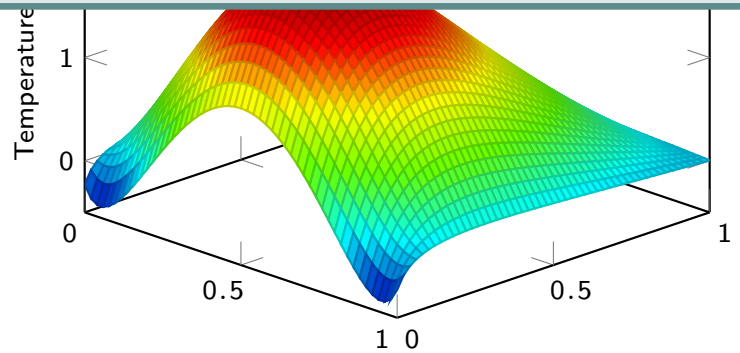
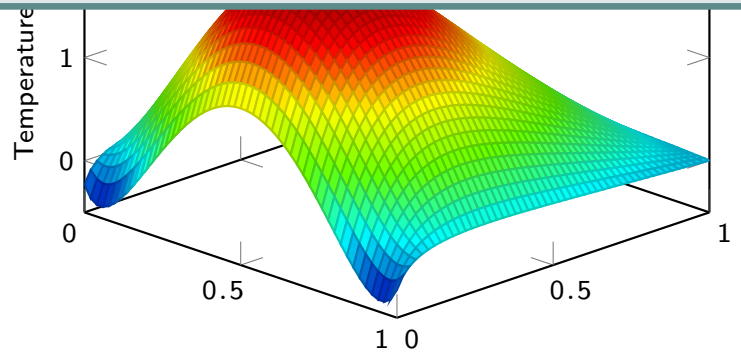
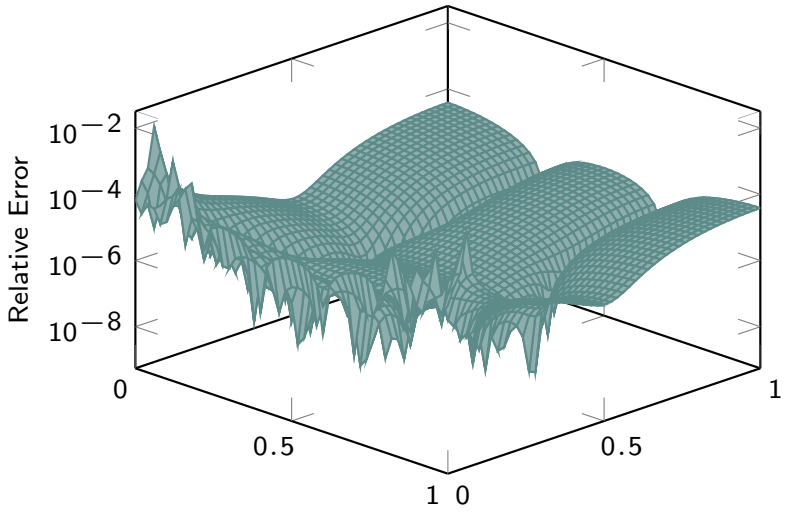
Thank you for your attention...







Numerical Examples

A Heat Transfer Model

Thank you for your attention...

... and enjoy the coffee break!