Winterterm 2012/2013

Otto-von-Guericke-University Magdeburg
Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods for Systems and Control Theory

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Website: http://www.mpi-magdeburg.mpg.de/mpcsc/lehre/2012_WS_SC/

Scientific Computing 1
12th Homework

Handout: 12/20/2012 **Return:** 01/10/2013

Exercise 1: (3 Points)

Let $V, W \in \mathbb{R}^{n \times k}$, $k < n$, be two matrices of rank k. Show that

$$A = VW^T \quad \in \mathbb{R}^{n \times n}$$

has exactly rank k.

Exercise 2: (10 Points)

The outer product Gaussian elimination defines one possible way to compute an LU decomposition of a matrix.

a.) Implement this algorithm as a C function with the following header:

```
void LU(struct my_matrix_st A);
```

The input `A` should be overwritten by its LU decomposition as shown in the lecture. Use the `scale_col` and the `r1_update` functions from Homework 11/Exercise 3 to perform the necessary operations.

b.) Write a solver function that takes the LU decomposed matrix from a and a right hand side $b \in \mathbb{R}^n$ as inputs and overwrites b with the solution of $LUx = b$. Use the BLAS function `DTRSV` for this purpose. The function header should be

```
void LU_solve(struct my_matrix_st LU, double *b);
```

c.) Solve the following linear system to check your code:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} \end{pmatrix} x = \begin{pmatrix} \frac{49}{20} \\ \frac{223}{140} \\ \frac{341}{280} \\ \frac{2509}{2520} \\ \frac{2131}{2520} \\ \frac{20417}{27720} \end{pmatrix}$$

What can you recognize? Is the solution sufficiently accurate when you think about the possibilities of double precision floating point numbers?

Hint: A skeleton code with a reference implementation of `scale_col` and `r1_update` is available on the web page.

Exercise 3: **(8 Points)**

The accuracy of the solution of a linear system can be improved using iterative refinement. Implement this procedure as a C function

```
int refine(struct my_matrix_st A, double *b, int maxiter, double tol);
```

where the right hand side `b` is overwritten with the solution. The `maxiter` parameter defines the maximum number of iterations. The `tol` argument stops the iteration when

$$||b - Ax_i||_2 < \texttt{tol}.$$

The return value of the function is the number of actually performed iteration steps.
Solve the linear system from Exercise 2 with this function and compare the results.

Hints:

- Use `DGESV` from LAPACK to solve the linear system.
- `DGESV` overwrites its input matrix.
- The 2-norm of a vector is computed by the BLAS function `DNRM2`

Exercise 4: **(4 Points)**

The Jacobi method is a classical splitting technique to solve linear systems iteratively. It splits a matrix $A \in \mathbb{R}^{n \times n}$ into the diagonal $D = \mathrm{diag}\{a_{11}, a_{22}, \ldots, a_{nn}\}$ and off-diagonal elements and repeatedly solves the equation

$$x_{i+1} = M_J x_i + D^{-1} b$$

where the iteration matrix is

$$M_J = D^{-1}(D - A).$$

Show that the Jacobi method converges to the solution $x = A^{-1}b$ if the matrix A is strictly diagonal-dominant, i.e.,

$$\sum_{j=1,\ j \neq i}^{n} |a_{ij}| < |a_{ii}|, \qquad \forall i = 1, \ldots, n.$$

Hint: Use Theorem 5.24.

Overall Points: 25