Suboptimality Estimation for the Semi-Discretized LQR Problem for Parabolic PDEs

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Outline

1. Parabolic PDEs and Abstract Cauchy-Problems
2. LQR Design for Abstract Cauchy Problems
3. Suboptimality Estimation

Parabolic PDEs and Abstract Cauchy-Problems

Consider a control problem for a

parabolic partial differential equation

$$\frac{\partial \mathbf{x}}{\partial t} + \nabla \cdot (\mathbf{c}(\mathbf{x}) - \mathbf{k}(\nabla \mathbf{x})) + \mathbf{q}(\mathbf{x}) = \mathbf{v}(\xi, t), \quad t \in [0, T_f], \quad \text{(PDE)}$$

on a domain $\Omega \subset \mathbb{R}^d, d = 1, 2, 3.$

Here:

- $\mathbf{q}$ uncontrolled sink or source
- $\mathbf{k}$ diffusive part
- $\mathbf{c}$ convection part

For ease of presentation we consider $T_f = \infty$.

Parabolic PDEs and Abstract Cauchy-Problems

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on a domain $\Omega \subset \mathbb{R}^d, d = 1, 2, 3.$

Here $\mathbf{v}(\xi, t) = \mathcal{B}(\xi)\mathbf{u}(t)$

- $\mathbf{u}$ control
- $\mathcal{B}$ input operator

Parabolic PDEs and Abstract Cauchy-Problems

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on a domain $\Omega \subset \mathbb{R}^d, d = 1, 2, 3.$

If (PDE) is linear, then a variational formulation leads to a Cauchy problem for the

linear evolution equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X}.$$

LQR Design for Abstract Cauchy Problems
Formulation

lineare evolution equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X}, \quad \text{(Cauchy)}$$

output equation

$$\mathbf{y} = \mathbf{C}\mathbf{x}, \quad \text{(output)}$$

with linear operators

$$\mathbf{A} : \mathrm{dom}(\mathbf{A}) \subset \mathcal{X} \to \mathcal{X}, \qquad \mathbf{B} : \mathcal{U} \to \mathcal{X}, \qquad \mathbf{C} : \mathcal{X} \to \mathcal{Y},$$

on separable Hilbert spaces $\mathcal{X}$ (state space), $\mathcal{U} = \mathbb{R}^k$ (i.e., $\mathcal{U}$ is finite dim.).

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Defining $\mathbf{Q} := \mathbf{C}^*\hat{\mathbf{Q}}\mathbf{C}$ with $\hat{\mathbf{Q}} = \hat{\mathbf{Q}}^* \geq 0$, and $\mathbf{R} = \mathbf{R}^* > 0$ we state the

cost function

$$\mathcal{J}(\mathbf{u}) = \frac{1}{2}\int_0^\infty < \hat{\mathbf{Q}}\mathbf{y}, \mathbf{y} > + < \mathbf{R}\mathbf{u}, \mathbf{u} > dt. \qquad \text{(cost)}$$

LQR Design for Abstract Cauchy Problems

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We can now formulate the

LQR–problem.

Minimize (cost) with respect to (Cauchy).

LQR Design for Abstract Cauchy Problems

Solution

Well understood in the open literature:
Analogously to ODE systems case we find the

optimal state feedback

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^*\mathbf{X}_\infty\mathbf{x}.$$

Here $\mathbf{X}_\infty$ is the stabilizing, positive semidefinite, selfadjoint solution to the

Operator–**A**lgebraic–**R**iccati–**E**quation

$$0 = \mathcal{R}(\mathbf{X}) := \mathbf{Q} + \mathbf{A}^*\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{X}. \qquad \text{(O-ARE)}$$

---

e.g. [Lions '71; Lasiecka/Triggiani '00; Bensoussan et al. '92/'06; Pritchard/Salamon '87; Curtain/Zwart '95]

LQR Design for Abstract Cauchy Problems
Solution

(Cauchy) can now be rewritten as

closed loop system

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{X}_\infty)\mathbf{x},$$

and the

optimal solution trajectory

is given as

$$\mathbf{x}(t) = \mathbf{S}(t)\mathbf{x}_0,$$

where $\mathbf{S}(t)$ is the operator semigroup generated by $\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{X}_\infty$.

LQR Design for Abstract Cauchy Problems
Approximation

Let $(\mathcal{X}_n)_{n\in\mathbb{N}}$ a Galerkin scheme for $\mathcal{X}$. We formulate the

n-d evolution equation

$$\dot{x} = A_n x + B_n \mathbf{u}, \qquad \mathcal{X}_n \ni x_n(0) = \mathbf{P}_n \mathbf{x}_0, \qquad \text{(n-d Cauchy)}$$

output equation

$$y_n = C_n x_n, \qquad \text{(n-d output)}$$

with linear operators

$$A_n : \text{dom}(A_n) \subset \mathcal{X}_n \to \mathcal{X}_n, \qquad B_n : \mathcal{U} \to \mathcal{X}_n, \qquad C_n : \mathcal{X}_n \to \mathcal{Y}_n,$$

on n-d Hilbert spaces $\mathcal{X}_n$ (state space) and $\mathcal{Y}_n$ (observation space), but still $\mathcal{U} = \mathbb{R}^k$.

$\mathbf{P}_n : \mathcal{X} \to \mathcal{X}_n$ the canonical orthogonal projection.

LQR Design for Abstract Cauchy Problems

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Defining $Q_n := C_n^* \hat{Q}_n C_n$ with $\hat{Q}_n = \hat{Q}_n^* \geq 0$, and $\mathbf{R} = \mathbf{R}^* > 0$ we formulate

cost function

$$\mathcal{J}_n(\mathbf{u}) = \frac{1}{2}\int_0^\infty < \hat{Q}_n y_n, y_n > + < \mathbf{R}\mathbf{u}, \mathbf{u} > dt. \qquad \text{(n-d Cost)}$$

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and state the

n-d LQR–problem.

Minimize (n-d Cost) with respect to (n-d Cauchy).

LQR Design for Abstract Cauchy Problems

Approximation

Analogously to the ∞-dim. case we now find:

optimal state feedback

$$\mathbf{u} = -\mathbf{R}^{-1} B_n^* X_n x_n,$$

where X_n is the stabilizing, positive semidefinite, selfadjoint solution to the

n-d Operator-Algebraic-Riccati-Equation

$$0 = \mathcal{R}_n(X) := Q_n + A_n^* X + X A_n - X B_n \mathbf{R}^{-1} B_n^* X. \quad \text{(n-d O-ARE)}$$

LQR Design for Abstract Cauchy Problems

Approximation

As above we can write (n-d Cauchy) as

closed loop system

$$\dot{x}_n = (A_n - B_n\mathbf{R}^{-1}B_n^* X_n)x_n,$$

and the

optimal solution

is given as

$$x_n(t) = S_n(t)\mathbf{P}_n\mathbf{x}_0,$$

also again $S_n(t)$ is the operator semigroup generated by $A_n - B_n\mathbf{R}^{-1}B_n^* X_n$.

LQR Design for Abstract Cauchy Problems

Approximation

Approximation

The n-d LQR–problems approximate the ∞-dim LQR–problem in the following sense

- $X_n \mathbf{P}_n \mathbf{v} \rightarrow \mathbf{X} \mathbf{v}$ for $n \rightarrow \infty$ and any $\mathbf{v} \in \mathcal{X}$,
- $S_n(t) \mathbf{P}_n \mathbf{v} \rightarrow \mathbf{S}(t) \mathbf{v}$ for $n \rightarrow \infty$ and any $\mathbf{v} \in \mathcal{X}$,

that means in the strong operator-topology.

[Banks/Kunisch‘84] distributed control of parabolic PDEs

[Benner/S.‘05] boundary control with mixed boundary conditions

[Lasiecka/Triggiani‘00] weakens regularity conditions on (Cauchy), also has convergence rates

[Ito‘87/‘90; Morris‘94] general Cauchy problems

[...] many more

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that means in the strong operator-topology.

Remarks:

- For a chosen basis (e.g. from spatial FDM/FEM discretization) all n-d operators have matrix representations and $S(t)$ coincides with the matrix-exponential $e^{(A-BR^{-1}B^TX)t}$.
- dimensions of $\mathbf{u}$ and $\mathbf{R}$ are always kept fixed, i.e., $\mathbf{u}$ from computations for an n-d problem can directly be applied in the ∞-d problem.

LQR Design for Abstract Cauchy Problems

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Suboptimality Estimation
How to Measure Suboptimality

concepts of suboptimality

- deviation of states
- deviation of controls
- deviation of optimal cost

Measure suboptimality in terms of

$$s_{x,n} := \|\mathbf{x} - x_n\|_{\mathcal{X}}$$

Suboptimality Estimation

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Suboptimality Estimation

How to Measure Suboptimality

concepts of suboptimality

- deviation of states
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- deviation of optimal cost

Measure suboptimality in terms of

$$s_{\mathcal{J},n} := |\mathcal{J}(\mathbf{u}) - \mathcal{J}_n(\mathbf{u}_n)|$$

Note

$$\mathcal{J}(\mathbf{u}) = \frac{1}{2}\int_0^\infty < \mathbf{Qx}, \mathbf{x} > + < \mathbf{Ru}, \mathbf{u} > dt,$$

i.e., deviations in controls and states are measured simultaneously.

Suboptimality Estimation

How to Measure Suboptimality

concepts of suboptimality

- deviation of states
- deviation of controls
- **deviation of optimal cost**

Measure suboptimality in terms of

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i.e., deviations in controls and states are measured simultaneously.

Suboptimality Estimation

Representation of Optimal Costs

observation

Optimal costs are

$$\mathcal{J}(\mathbf{u}) = \mathbf{x}(t_0)^* \mathbf{X}_\infty \mathbf{x}(t_0), \quad \text{and} \quad \mathcal{J}_n(\mathbf{u}_n) = x_n(t_0)^* X_n x_n(t_0),$$

respectively.

Employing

$$\hat{X}_n := P_n^* X_n P_n$$

we find

$$\mathcal{J}_n(\mathbf{u}_n) = x_n(t_0)^* X_n x_n(t_0) = \mathbf{x}(t_0)^* \hat{X}_n \mathbf{x}(t_0) = \mathcal{J}(u_n)$$

and prove:

Suboptimality Estimation

Representation of Optimal Costs

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Suboptimality Estimation

Basic Result

Theorem (suboptimality estimation [S. '09])

Under assumptions fulfilled for standard FDM/FEM approximations, the suboptimality applying u_n instead of $\mathbf{u}$ can asymptotically be estimated by

$$|\mathcal{J}(\mathbf{u}) - \mathcal{J}(u_n)| \leq \zeta \left(\|\mathbf{x}^0 - x_n^0\|_{\mathcal{X}} + \|\mathbf{X}_\infty - \hat{X}_n\|_{\mathcal{X}} \right),$$

where

- $\zeta = \zeta(\|\mathbf{x}^0\|, \|\mathbf{X}_\infty\|)$,
- $\mathbf{x}^0 := x(t_0), \quad x_n^0 := x_n(t_0)$.

Suboptimality Estimation
Approximation of Riccati Solution Operator

For $\|\mathbf{x}^0 - x_n^0\|_{\mathcal{X}}$ standard estimates apply.

For $\|\mathbf{X}_\infty - \hat{X}_n\|_{\mathcal{X}}$:

[Ito '87]

$$\begin{aligned}\|\mathbf{X}_\infty - \hat{X}_n\| &\leq 2\,\|(\mathbf{A}^* - A_n^* P_n)\mathbf{X}_\infty\| \\ &\quad + 2c\,\|\mathbf{B}\|\,\|\left(B^* - B_n^* P_n\right)\mathbf{X}_\infty\| \\ &\leq \mathcal{O}(h)\end{aligned}$$

[Kroller/Kunisch '91]

$$\text{discretization error } \mathcal{O}(h^2) \Rightarrow \|\mathbf{X}_\infty - \hat{X}_n\| \leq \mathcal{O}\left(h^2 \ln \frac{1}{h}\right)$$

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Suboptimality Estimation

Refined Result

Corollary

For FDM/FEM approximations allowing h^2 error estimates, the suboptimality applying u_n instead of $\mathbf{u}$ is asymptotically bounded by

$$|\mathcal{J}(\mathbf{u}) - \mathcal{J}(u_n)| \leq \mathcal{O}\left(h^2 \ln \frac{1}{h}\right)$$

Conclusions

- Certain control problems for parabolic PDEs can be formulated as ∞-d LQR-problems
- spatial semi-discretization
  $\longrightarrow$ finite dimensional LQR systems
- n-d controls directly applicable in original setting
- convergence in strong operator topology
- suboptimality governed by data and solution approximation
- h^2 discretization error
  $\Rightarrow \mathcal{O}\left(h^2 \ln \frac{1}{h}\right)$ suboptimality estimate

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