Optimal Control-Based Feedback Stabilization in Multi-Field Flow Problems

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Overview

Project Summary

Scientific goals of the project:

- derive and investigate numerical algorithms for optimal control-based (normal and tangential) boundary feedback stabilization of multi-field flow problems;
- explore the potentials and limitations of feedback-based (Riccati) stabilization techniques;
- extend current methods for flow described by Navier-Stokes equations to flow problems coupled with other field equations of increasing complexity.

Major challenge:

Numerical solution of algebraic Riccati equations associated to special LQR problem for linearized Navier-Stokes/Oseen-like equations.

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Optimal Control-Based Stabilization for NSEs

Analytical solution [RAYMOND '05–'07]

Linearized Navier-Stokes control system:

$$\partial_t \mathbf{z} + (\mathbf{z}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{z} - \frac{1}{Re}\Delta\mathbf{z} - \omega\mathbf{z} + \nabla\mathbf{p} = 0 \quad \text{in } Q_\infty \tag{1a}$$

$$\operatorname{div}\mathbf{z} = 0 \quad \text{in } Q_\infty \tag{1b}$$

$$\mathbf{z} = \mathbf{b}\mathbf{u} \text{ in } \Sigma_\infty \tag{1c}$$

$$\mathbf{z}(0) = \mathbf{z}_0 \quad \text{in } \Omega, \tag{1d}$$

$\omega\mathbf{z}$ with $\omega > 0$ de-stabilizes the system further, needed to guarantee exponential stabilization, ω controls decay rate!

Cost functional (with $\mathbf{P}$ = Helmholtz projector)

$$J(z,u) = \frac{1}{2}\int_0^\infty \langle \mathbf{P}\mathbf{z}, \mathbf{P}\mathbf{z}\rangle_{L_2(\Omega)} + \rho\mathbf{u}(t)^2\, dt, \tag{2}$$

the linear-quadratic optimal control problem associated to (1) becomes

$$\inf\left\{J(\mathbf{z},\mathbf{u}) \mid (\mathbf{z},\mathbf{u}) \text{ satisfies } (1),\ \mathbf{u}\in L_2(0,\infty)\right\}. \tag{3}$$

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Proposition [Raymond '05, Bahdra '09]

The solution to the instationary Navier-Stokes equations with perturbed initial data is exponentially controlled to the steady-state solution w by the feedback law

$$\mathbf{u} = -\rho^{-1}\mathbf{B}^*\mathbf{X}z_H,$$

where

- $\mathbf{z}_H := \mathbf{P}\mathbf{z}$, with $\mathbf{P} : L_2(\Omega) \mapsto V_n^0(\Omega)$ being the Helmholtz projector ($\rightsquigarrow \operatorname{div} \mathbf{z}_H \equiv 0$);
- $\mathbf{X} = \mathbf{X}^* \in \mathcal{L}(V_n^0(\Omega))$ is the unique nonnegative semidefinite weak solution of the operator Riccati equation

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$\mathbf{A}$ is the linearized Navier-Stokes operator restricted to V_n^0;
$\mathbf{B}_\tau$ and $\mathbf{B}_n$ correspond to the projection of the control action in the tangential and normal directions.

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- **Proof of concept:** Navier-Stokes with **normal** boundary control for model problem (von Kármán vortex shedding).
- Navier-Stokes coupled with (passive) transport of (reactive) species.
- Phase transition liquid/solid with convection.
- Stabilization of a flow with a free capillary surface.
- Control for electrically conducting fluids in presence of outer magnetic fields (MHD).

All scenarios require

- formulation as abstract parabolic Cauchy problem,
- definition of quadratic cost functional,
- formulation of corresponding ARE,
- spatial discretization (FEM),
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Solving Large-Scale Algebraic Riccati Equations

Low-Rank Newton-ADI for AREs

Consider

$$0 = \mathcal{R}(X) := C^T C + A^T X + XA - XBB^T X$$

Re-write Newton's method for AREs $(A_j := A - BB^T X_j)$

$$D\mathcal{R}(X_j)\,(N_j) = -\mathcal{R}(X_j)$$

$$\Longleftrightarrow$$

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + \underbrace{(X_j + N_j)}_{=X_{j+1}} A_j = \underbrace{-C^T C - X_j BB^T X_j}_{=:-W_j W_j^T}$$

Set $X_j = Z_j Z_j^T$ for $\operatorname{rank}(Z_j) \ll n \Longrightarrow$

$$A_j^T \left(Z_{j+1} Z_{j+1}^T\right) + \left(Z_{j+1} Z_{j+1}^T\right) A_j = -W_j W_j^T$$

Factored Newton Iteration [B./Li/Penzl '99/'08]

Solve Lyapunov equations for Z_{j+1} directly by factored ADI iteration and exploit 'sparse + low-rank' structure of A_j.

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Feedback Iteration

$Z_{j,k} = [Z_{j,k-1}, V_{j,k}]$,

j: Newton index,

k: ADI index.

Optimal feedback

$$K_* = B^T X_* = B^T Z_* Z_*^T$$

can be computed by direct feedback iteration:

- jth Newton iteration:

$$K_j = B^T Z_j Z_j^T = \sum_{k=1}^{k_{\max}} (B^T V_{j,k}) V_{j,k}^T \xrightarrow{j \to \infty} K_* = B^T Z_* Z_*^T$$

- K_j can be updated in ADI iteration, $A_j = BK_j$
  $\Rightarrow$ no need to form Z_j, need only fixed workspace for $K_j \in \mathbb{R}^{m \times n}$!

Related to earlier work by [Banks/Ito '91].

Solving the Helmholtz-projected Navier-Stokes ARE

$0 = M + (A + \omega M)^T X + X(A + \omega M) - XBB^T X$

Problems with Newton-Kleinman

1. Discretization of Helmholtz-projected linearized Navier-Stokes equations would need divergence-free finite elements.

   Here, we want to use standard discretization

   (Taylor-Hood elements available in flow solver NAVIER).

   Explicit projection of ansatz functions possible using application of Helmholtz projection, but too expensive in general.

2. Each step of Newton-Kleinman iteration: solve

$$A_j^T Z_{j+1} Z_{j+1}^T + Z_{j+1} Z_{j+1}^T A_j = -M - K_j^T K_j$$

   $n_v := \operatorname{rank}(M) =$ dim of ansatz space for velocities.

   $\rightsquigarrow$ need to solve $n_v + m$ linear systems of equations in each step of Newton-ADI iteration!

3. Linearized system (i.e., $A + \omega M$) is unstable in general.

   But to start Newton iteration, a stabilizing initial guess is needed!

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   annual meeting Banz 2009

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   Newton [illegible] iteration!

[BENNER '08-'10] *Partial Stabilization of Descriptor Systems Using Spectral Projectors*; to appear in V. Olshevsky et al (eds.), Numerical Linear Algebra in Signals, Systems and Control, Lecture Notes in Electrical Engineering, Springer-Verlag.

[HEIN '10] *MPC/LQG-Based Optimal Control of Nonlinear Parabolic PDEs*; PhD thesis Chemnitz UT.

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Solving AREs for Linearized Navier-Stokes Eqns.

Solution to 1. Problem/no need for divergence free FE

- incompressible Navier-Stokes-Equations

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} - \frac{1}{\mathrm{Re}}\Delta \mathbf{v} + \mathbf{v}.\nabla \mathbf{v} + \nabla \mathbf{p} &= 0 \quad + \text{B.C.} \\ \nabla.\mathbf{v} &= 0 \end{aligned} \tag{NSE}$$

- Spatial FE discretization

$$\begin{aligned} M\dot{v}(t) &= K(v)v(t) - Gp(t) + B_1\mathbf{u}(t) \\ 0 &= G^T v(t) \end{aligned} \tag{SNSE}$$

- Linearization and change of notation

$$\begin{aligned} E_{11}\dot{v}(t) &= A_{11}v(t) + A_{12}p(t) + B_1\mathbf{u}(t) \\ 0 &= A_{12}^T v(t) \end{aligned} \tag{DANSE}$$

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Multiplication of line one from the left by $A_{12}^T E_{11}^{-1}$ together with

$0 = A_{12}^T v(t) \Rightarrow 0 = A_{12}^T \dot{v}(t)$ reveals the

hidden manifold

$$0 = A_{12}^T E_{11}^{-1} A_{11} v(t) + A_{12}^T E_{11}^{-1} A_{12} p(t) + A_{12}^T E_{11}^{-1} B_1 \mathbf{u}(t),$$

Solving AREs for Linearized Navier-Stokes Eqns.

Solution to 1. Problem/no need for divergence free FE

$$\begin{aligned} E_{11}\dot{v}(t) &= A_{11}v(t) + A_{12}p(t) + B_1\mathbf{u}(t) \\ 0 &= A_{12}^T v(t) \\ y(t) &= C_v v(t) + C_p p(t) + D\mathbf{u}(t) \end{aligned}$$

Multiplication of line one from the left by $A_{12}^T E_{11}^{-1}$ together with

$0 = A_{12}^T v(t) \Rightarrow 0 = A_{12}^T \dot{v}(t)$ reveals the

hidden manifold

$$0 = A_{12}^T E_{11}^{-1} A_{11} v(t) + A_{12}^T E_{11}^{-1} A_{12} p(t) + A_{12}^T E_{11}^{-1} B_1 \mathbf{u}(t),$$

which implies

$$p(t) = -\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1} A_{11} v(t) - \left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1} B_1 \mathbf{u}(t).$$

Solving AREs for Linearized Navier-Stokes Eqns.

Solution to 1. Problem/no need for divergence free FE

Inserting p we find

$$
\begin{aligned}
E_{11}\dot{v}(t) = {} & \left(I - A_{12}\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1}\right) A_{11} v(t) \\
& + \left(I - A_{12}\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1}\right) B_1 \mathbf{u}(t)
\end{aligned}
$$

Definition [Heinkenschloss/Sorensen/Sun '08]

$$
\Pi := I - A_{12}\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1}
$$

Solving AREs for Linearized Navier-Stokes Eqns.

Solution to 1. Problem/no need for divergence free FE

Inserting p we find

$$E_{11}\dot{v}(t) = \left(I - A_{12}\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1}\right) A_{11} v(t) + \left(I - A_{12}\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1}\right) B_1 \mathbf{u}(t)$$

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Solving AREs for Linearized Navier-Stokes Eqns.

Derivation of the Projected State Space System and Matrix Equations

Definition [Heinkenschloss/Sorensen/Sun '08]

$$\Pi := I - A_{12}\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1}$$

Properties

- $\Pi^2 = \Pi$
- $\Pi E_{11} = E_{11}\Pi^T$
- $null(\Pi) = range(A_{12})$
- $range(\Pi) = null(A_{12}^T E_{11}^{-1})$

Which imply

Lemma 1 [Heinkenschloss/Sorensen/Sun '08]

- Π is an oblique projector
- $A_{12}^T z = 0 \Leftrightarrow \Pi^T z = z$
- $\Rightarrow \Pi^T v(t) = v(t)$

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Solving AREs for Linearized Navier-Stokes Eqns.

Derivation of the Projected State Space System and Matrix Equations

Thus (DANSE) is equivalent to the

projected state space system

$$\Pi E_{11}\Pi^T \frac{d}{dt}v(t) = \Pi A_{11}\Pi^T v(t) + \Pi B_1 \mathbf{u}(t)$$
$$y(t) = C\Pi^T v(t) + \left(D - C_p \left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1} B_1\right) \mathbf{u}(t),$$

where $C = C_v - C_p \left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1} A_{11}.$

Solving AREs for Linearized Navier-Stokes Eqns.

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Simplification: $D = 0,\ C_p = 0.$

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$$y(t) = C \Pi^T v(t),$$

where $C = C_v$.

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If necessary p can be determined from

$$p(t) = -\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1} A_{11} v(t) - \left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1} B_1 \mathbf{u}(t).$$

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where $C = C_v$.

Leads to

projected Riccati equation

$$\begin{aligned}\Pi C^T C \Pi^T + \Pi A_{11}^T \Pi^T X \Pi E_{11} \Pi^T + \Pi E_{11}^T \Pi^T X \Pi A_{11} \Pi^T&\\ -\Pi E_{11}^T \Pi^T X \Pi B_1 B_1^T \Pi^T X \Pi E_{11} \Pi^T &= 0\\ \Pi^T X \Pi &= X.\end{aligned}$$

Solving AREs for Linearized Navier-Stokes Eqns.

Solving the Projected Matrix Equations

Apply factored-Newton-ADI

Central question

How do we solve systems of equations $(A_i := A_{11} + BK_i)$

$$Z = \Pi^T Z, \qquad \Pi\,(E_{11} + p_i A_i)\,\Pi^T Z = \Pi \tilde{G}$$

in the (inner) ADI steps avoiding the computation of Π?

For $A_i = A_{11}$

Lemma [Heinkenschloss/Sorensen/Sun '08]

$$\begin{aligned} Z &= \Pi^T Z \\ \Pi\,(E_{11} + p_i A_{11})\,\Pi^T Z &= \Pi \tilde{G} \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} E_{11} + p_i A_{11} & A_{12} \\ A_{12}^T & 0 \end{bmatrix} \begin{bmatrix} Z \\ \Lambda \end{bmatrix} = \begin{bmatrix} \tilde{G} \\ 0 \end{bmatrix}$$

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in the (inner) ADI steps avoiding the computation of Π.

tasks

- exploit "sparse + low rank" structure of A_i,
- precondition our saddle point problem.
  (joint work with A. Wathen/M. Stoll)

For $A_i = A_{11}$

Lemma [Heinkenschloss/Sorensen/Sun '08]

$$\begin{aligned} Z &= \Pi^T Z \\ \Pi\,(E_{11} + p_i A_{11})\,\Pi^T Z &= \Pi \tilde{G} \end{aligned} \quad \Leftrightarrow \begin{bmatrix} E_{11} + p_i A_{11} & A_{12} \\ A_{12}^T & 0 \end{bmatrix} \begin{bmatrix} Z \\ \Lambda \end{bmatrix} = \begin{bmatrix} \tilde{G} \\ 0 \end{bmatrix}$$

First Results for Scenario 1

Navier-Stokes Coupled with (Passive) Transport of (Reactive) Species

Goal: stabilize concentration at certain level

Model equations:

$$\begin{aligned} \partial_t \mathbf{v} - \frac{1}{Re}\Delta \mathbf{v} + \mathbf{v}.\nabla \mathbf{v} + \nabla \mathbf{p} &= \mathbf{f} \\ \operatorname{div} \mathbf{v} &= 0 \\ \partial_t \mathbf{c} + \mathbf{v}.\nabla \mathbf{c} - \frac{1}{Re \cdot Sc}\Delta \mathbf{c} &= 0 \end{aligned}$$

with boundary conditions:

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_0 & \mathbf{c} &= \mathbf{c}_0 = const & &\text{on } \Gamma_{in} \\ \mathbf{v} &= 0 & \partial_\nu \mathbf{c} &= 0 & &\text{on } \Gamma_{wall} \\ \mathbf{v} &= 0 & \mathbf{c} &= 0 & &\text{on } \Gamma_r, \end{aligned}$$

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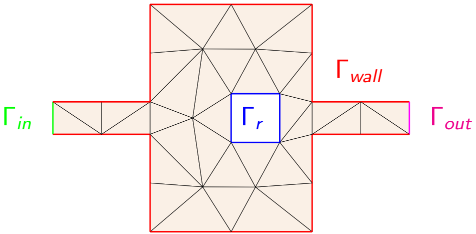
$$\operatorname{div} \mathbf{v} = 0$$

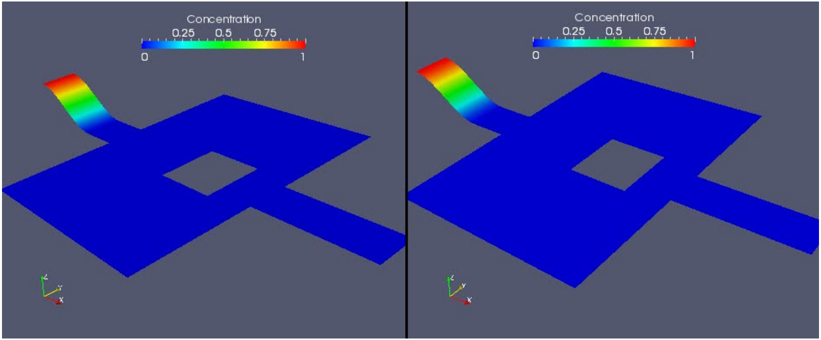
$$\partial_t \mathbf{c} + \mathbf{v}.\nabla \mathbf{c} - \frac{1}{Re \cdot Sc}\Delta \mathbf{c} = 0$$

Domain:



First Results for Scenario 1

Results for $Re = 10$, $Sc = 10$ shown at $3\times$speed



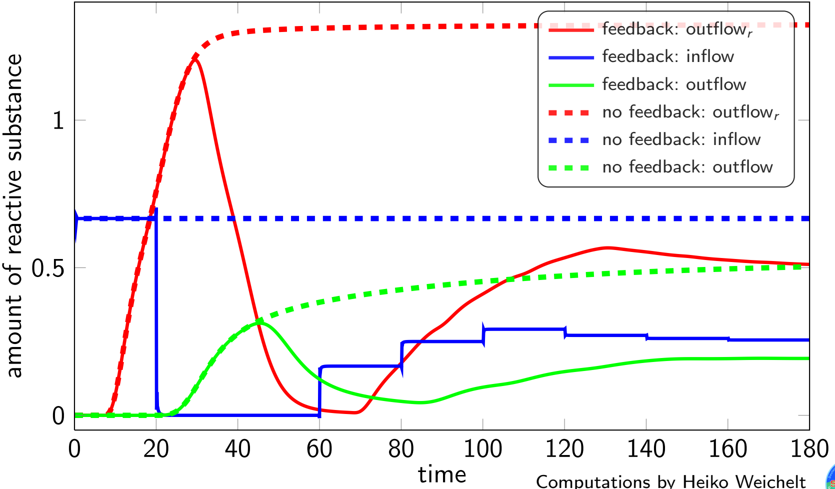
no control

piecewise constant feedback

Computations by Heiko Weichelt

First Results for Scenario 1

Results for $Re = 10$, $Sc = 10$



Computations by Heiko Weichelt

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