



Basics of Systems and Control Theory for pyMOR

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pyMOR Online Course 2020



$$\begin{aligned}
 E \dot{x}(t) &= A x(t) + B u(t) \\
 y(t) &= C x(t) + D u(t)
 \end{aligned}$$

MOR

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 \hat{E} \dot{\hat{x}}(t) &= \hat{A} \hat{x}(t) + \hat{B} u(t) \\
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 \end{aligned}$$

- 1 Linear Time-Invariant (LTI) Systems
- 2 Transfer Function and Realizations
- 3 System Analysis
- 4 A Selection of MOR Methods

- Only continuous-time systems

Discrete-time is treated in [1]

- No differential-algebraic systems

For DAE aspects see [6, 3, 4, 5]

- No non-linearities

- No parameter dependencies

- 1 Linear Time-Invariant (LTI) Systems
 - Setting for this course
 - Examples
- 2 Transfer Function and Realizations
- 3 System Analysis
- 4 A Selection of MOR Methods

First-order State-space Systems

(PYMOR: LTIModel)

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t).\end{aligned}\tag{\Sigma}$$

Here

- $\mathbf{x}(t) \in \mathbb{R}^n$ is called the **state**,
- $\mathbf{u}(t) \in \mathbb{R}^m$ is called the **input**,
- $\mathbf{y}(t) \in \mathbb{R}^p$ is called the **output**

of the LTI system. Correspondingly, we have

$$\mathbf{A} \in \mathbb{R}^{n \times n}, \quad \mathbf{B} \in \mathbb{R}^{n \times m}, \quad \mathbf{C} \in \mathbb{R}^{p \times n} \quad \text{and} \quad \mathbf{D} \in \mathbb{R}^{p \times m}.$$

We assume $t \in [0, \infty)$, $\mathbf{x}(0) = \mathbf{0}$.

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$$\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n \times n}, \quad \mathbf{B} \in \mathbb{R}^{n \times m}, \quad \mathbf{C} \in \mathbb{R}^{p \times n}.$$

We assume $t \in [0, \infty)$, $\mathbf{x}(0) = \mathbf{0}$ and \mathbf{E} invertible.

Second-order State-space Systems

(PYMOR: SecondOrderModel)

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{E}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) &= \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}_v\dot{\mathbf{x}}(t) + \mathbf{C}_p\mathbf{x}(t). \end{aligned}$$

Here

- $\mathbf{x}(t) \in \mathbb{R}^n$ is called the **position**,
- $\dot{\mathbf{x}}(t) \in \mathbb{R}^n$ is called the **velocity**,
- $\mathbf{u}(t) \in \mathbb{R}^m$ is called the **input**,
- $\mathbf{y}(t) \in \mathbb{R}^p$ is called the **output**

of the LTI system. Correspondingly, we have

$$\mathbf{M}, \mathbf{E}, \mathbf{K} \in \mathbb{R}^{n \times n}, \quad \mathbf{B} \in \mathbb{R}^{n \times m}, \quad \mathbf{C}_v, \mathbf{C}_p \in \mathbb{R}^{p \times n}.$$

Heat Equation [MORWiki thermal block] I

For $t \in (0, T)$, $\xi \in \Omega$ and initial values

$$\theta(0, \xi) = 0, \text{ for } \xi \in \Omega,$$

consider

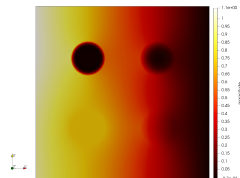
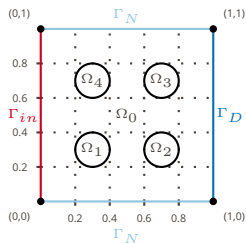
$$\partial_t \theta(t, \xi) + \nabla \cdot (-\sigma(\xi) \nabla \theta(t, \xi)) = 0,$$

with boundary conditions

$$\sigma(\xi) \nabla \theta(t, \xi) \cdot n(\xi) = u(t) \quad t \in (0, T), \xi \in \Gamma_{in},$$

$$\sigma(\xi) \nabla \theta(t, \xi) \cdot n(\xi) = 0 \quad t \in (0, T), \xi \in \Gamma_N,$$

$$\theta(t, \xi) = 0 \quad t \in (0, T), \xi \in \Gamma_D.$$



Heat Equation [MORWiki thermal block] II

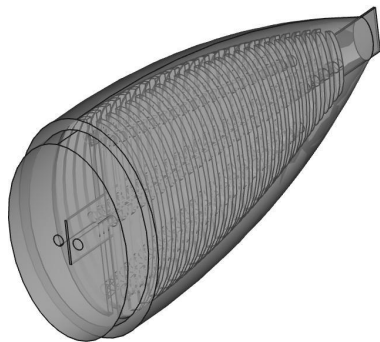
Finite element semi-discretization in space

- pairwise inner products of ansatz functions $\rightsquigarrow \mathbf{E}$
- discretized spatial operator + Dirichlet boundary condition $\rightsquigarrow \mathbf{A}$
- discretized non-zero Neumann boundary condition $\rightsquigarrow \mathbf{B}$
- average temperatures on the inclusions $\rightsquigarrow \mathbf{C}$

- $n = 7488$
- $m = 1$
- $p = 4$

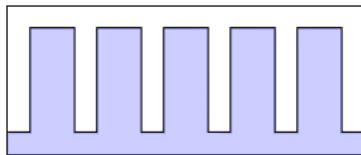
An Artificial Fishtail [MORWiki Artificial Fishtail] I

Construction:

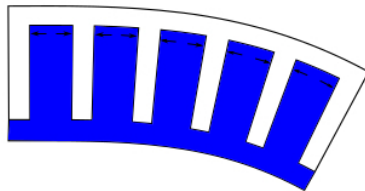


Fluid Elastomer Actuation:

no pressure



under pressure



An Artificial Fishtail [MORWiki Artificial Fishtail] II

Variables:

- displacement $\vec{s}(t, \vec{z})$
- strain $\underline{\underline{\epsilon}}(\vec{s}(t, \vec{z}))$
- stress $\underline{\underline{\sigma}}(\vec{s}(t, \vec{z}))$

Material parameters:

- density ρ
- Lamé parameters λ, μ

Basic principle:

$$\underline{\underline{\epsilon}}(\vec{s}(t, \vec{z})) = \frac{1}{2} (\nabla \vec{s}(t, \vec{z}) + \nabla^T \vec{s}(t, \vec{z})) \quad \text{(kinematic equation)}$$

$$\underline{\underline{\sigma}}(\vec{s}(t, \vec{z})) = \lambda \operatorname{tr}(\underline{\underline{\epsilon}}(\vec{s}(t, \vec{z}))) \underline{\underline{I}} + 2\mu \underline{\underline{\epsilon}}(\vec{s}(t, \vec{z})) \quad \text{(material equation)}$$

$$\rho \frac{\partial^2 \vec{s}(t, \vec{z})}{\partial t^2} = \nabla \cdot \underline{\underline{\sigma}}(\vec{s}(t, \vec{z})) + \vec{f}(t, \vec{z}) \quad \text{(equation of motion)}$$

+ initial and boundary conditions

An Artificial Fishtail [MORWiki Artificial Fishtail] III

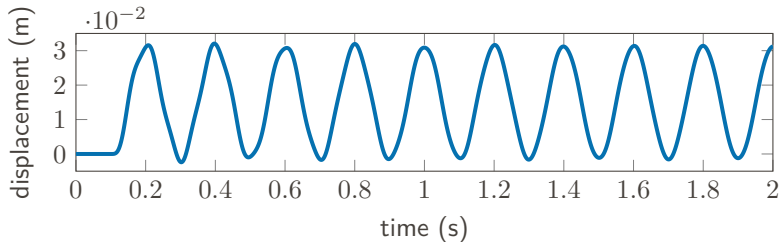
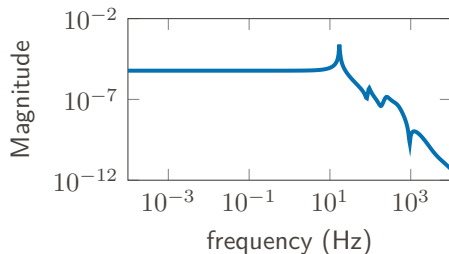
FEM semi-discretization:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{E}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t),$$

$$\mathbf{y}(t) = \mathbf{C}_p\mathbf{x}(t),$$

with

- $\mathbf{M}, \mathbf{E}, \mathbf{K} > 0, \mathbf{C}_v = 0,$
- $n = 779\,232, m = 1, p = 3.$



- 1 Linear Time-Invariant (LTI) Systems
- 2 Transfer Function and Realizations
 - Laplace Transform
 - Transfer Function
 - Realizations
 - Projection-based MOR
- 3 System Analysis
- 4 A Selection of MOR Methods

Definition

Let $f : [0, \infty) \rightarrow \mathbb{R}^n$ be exponentially bounded with bounding exponent α . Then

$$\mathcal{L}\{f\}(s) := \int_0^{\infty} f(\tau)e^{-s\tau} d\tau$$

for $\operatorname{Re}(s) > \alpha$ is called the **Laplace transform** of f . The process of forming the Laplace transform is called **Laplace transformation**.

It can be shown that the integral converges uniformly in a domain with $\operatorname{Re}(s) \geq \beta$ for all $\beta > \alpha$.

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Allows us to map time signals to frequency signals.

Theorem

Let $f, g, h : [0, \infty) \rightarrow \mathbb{R}^n$ be given. Then the following two statements hold true:

- a) The Laplace transformation is linear, i. e., if f and g are exponentially bounded, then $h := \gamma f + \delta g$ is also exponentially bounded and

$$\mathcal{L}\{h\} = \gamma\mathcal{L}\{f\} + \delta\mathcal{L}\{g\}$$

holds for all $\gamma, \delta \in \mathbb{C}$.

- b) If $f \in \mathcal{PC}^1([0, \infty), \mathbb{R}^n)$ and \dot{f} is exponentially bounded, then f is exponentially bounded and

$$\mathcal{L}\{\dot{f}\}(s) = s\mathcal{L}\{f\}(s) - f(0).$$

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$$\mathcal{L}\{\dot{f}\}(s) = s\mathcal{L}\{f\}(s) - f(0).$$

- $X(s) := \mathcal{L}\{\mathbf{x}\}(s)$, $U(s) := \mathcal{L}\{\mathbf{u}\}(s)$, and $Y(s) := \mathcal{L}\{\mathbf{y}\}(s)$
- $\mathbf{Ax}(t) + \mathbf{Bu}(t) \rightsquigarrow \mathbf{AX}(s) + \mathbf{BU}(s)$
- $\mathbf{y}(t) = \mathbf{Cx}(t) \rightsquigarrow Y(s) = \mathbf{CX}(s)$
- $sX(s) := \mathcal{L}\{\dot{\mathbf{x}}\}(s)$ since $\mathbf{x}(0) = 0$

Rational Matrix Function Representation

In summary we have:

- $s\mathbf{E}X(s) = \mathbf{A}X(s) + \mathbf{B}U(s)$
- $Y(s) = \mathbf{C}X(s)$

Thus the mapping from inputs to outputs in frequency domain can be expressed as

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}.$$

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Analogously, for second-order systems we get

$$\mathbf{H}(s) = (s\mathbf{C}_v + \mathbf{C}_p) (s^2\mathbf{M} + s\mathbf{E} + \mathbf{K})^{-1}\mathbf{B}.$$

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$$\mathbf{H}(s) = (s\mathbf{C}_v + \mathbf{C}_p) (s^2\mathbf{M} + s\mathbf{E} + \mathbf{K})^{-1}\mathbf{B}.$$

\mathbf{H} is analytic in $\mathbb{C} \setminus \Lambda(\mathbf{E}, \mathbf{A})$, or $\mathbb{C} \setminus \Lambda(\mathbf{M}, \mathbf{E}, \mathbf{K})$, respectively

Important Representations of $\hat{H}(s)$

(Laurent) series expansion

$$\mathbf{H}(s) = \sum_{k=0}^{\infty} (s - s_0)^k M_k(s_0) \quad \mathbf{H}(s) = \sum_{k=0}^{\infty} s^{-k} M_k(\infty)$$

The matrices $M_k(s_0)$ are called **moments** of \mathbf{H} . At infinity they are also referred to as **Markov parameters**.

Pole Residue Form

Let (λ_i, w_i, v_i) be the eigentriplets of the pair (\mathbf{A}, \mathbf{E}) with no degenerate eigenspaces. Then we have

$$\mathbf{H}(s) = \sum_{i=1}^n \frac{R_i}{s - \lambda_i},$$

where $R_i = (\mathbf{C}v_i)(w_i^H \mathbf{B})$, assuming $w_i^H v_i = 1$.

The **representation** of \mathbf{H} using $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is **not unique**.

In fact for any invertible matrix $\mathbf{T} \in \mathbb{R}^{n \times n}$, we have

$$\begin{aligned} \mathbf{H}(s) &= \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \\ &= \mathbf{C}\mathbf{T}^{-1}\mathbf{T}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{T}^{-1}\mathbf{T}\mathbf{B} \\ &= \mathbf{C}\mathbf{T}^{-1}(s\mathbf{T}\mathbf{E}\mathbf{T}^{-1} - \mathbf{T}\mathbf{A}\mathbf{T}^{-1})^{-1}\mathbf{T}\mathbf{B} \end{aligned}$$

and thus a system given, by $(\mathbf{T}\mathbf{E}\mathbf{T}^{-1}, \mathbf{T}\mathbf{A}\mathbf{T}^{-1}, \mathbf{T}\mathbf{B}, \mathbf{C}\mathbf{T}^{-1})$ realizes the exact same input/output behavior.

Definition

- All sets of matrices leading to the same function \mathbf{H} are called its **realizations**.
- The matrix \mathbf{T} above is called **state-space transformation**.

Important Realizations

- Minimal Realizations

Can we realize \mathbf{H} with less equations?

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- Balanced Realizations

Can we find state coordinates that allow us to decide what is important?

▶ see here

McMillan Degree and Minimal Realization

Example

Realizations can even be of different dimensions. Take for example:

$$\mathbf{E} = \mathbf{I} \text{ the identity, } \mathbf{A} = \begin{bmatrix} -11 & 0 \\ 0 & -5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{C} = [1 \quad 0].$$

Truncating the second state component does not change \mathbf{H} .

Definition

There exists a minimum number of equations necessary to describe \mathbf{H} . The state dimension n of this minimal set of equations is called **McMillan degree** of the system. A realization of \mathbf{H} with this dimension is called **minimal realization**.

Truncated Realizations via Ritz/Petrov-Galerkin Projection

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{u}(t) &= 0, \\ \mathbf{y}(t) - \mathbf{C}\mathbf{x}(t) - \mathbf{D}\mathbf{u}(t) &= 0. \end{aligned}$$

Truncated Realizations via Ritz/Petrov-Galerkin Projection

$$\begin{aligned} \mathbf{E}\mathbf{V}\dot{\hat{\mathbf{x}}}(t) - \mathbf{A}\mathbf{V}\hat{\mathbf{x}}(t) - \mathbf{B}\mathbf{u}(t) &= e_{\text{res}}(t), \\ \mathbf{y}(t) - \mathbf{C}\mathbf{V}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t) &= e_{\text{output}}(t). \end{aligned}$$

Step I: Use truncated state transformation

Replace

$$\mathbf{x}(t) \approx \mathbf{V}\hat{\mathbf{x}}(t)$$

with $\mathbf{V} \in \mathbb{R}^{n \times r}$ and $\hat{\mathbf{x}}(t) \in \mathbb{R}^r$.

Truncated Realizations via Ritz/Petrov-Galerkin Projection

$$\begin{aligned} \mathbf{V}^T \mathbf{E} \mathbf{V} \dot{\hat{\mathbf{x}}}(t) - \mathbf{V}^T \mathbf{A} \mathbf{V} \hat{\mathbf{x}}(t) - \mathbf{V}^T \mathbf{B} \mathbf{u}(t) &= 0, \\ \mathbf{y}(t) - \mathbf{C} \mathbf{V} \hat{\mathbf{x}}(t) - \mathbf{D} \mathbf{u}(t) &= e_{\text{output}}(t). \end{aligned}$$

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Step II: Mitigate transformation error

Suppress truncation residual through left projection.

- one-sided method: use \mathbf{V} again.

Truncated Realizations via Ritz/Petrov-Galerkin Projection

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Step II: Mitigate transformation error

Suppress truncation residual through left projection.

- one-sided method: use \mathbf{V} again.
- two-sided method: find $\mathbf{W} \in \mathbb{R}^{n \times r}$.

\hat{A}

\approx

W^T

A

V

Reduced order model (ROM)

(PYMOR: LTIPGReductor)

Define $\hat{\mathbf{E}} = \mathbf{W}^T \mathbf{E} \mathbf{V}$, $\hat{\mathbf{A}} = \mathbf{W}^T \mathbf{A} \mathbf{V} \in \mathbb{R}^{r \times r}$, $\hat{\mathbf{B}} = \mathbf{W}^T \mathbf{B} \in \mathbb{R}^{r \times m}$ and $\hat{\mathbf{C}} = \mathbf{C} \mathbf{V} \in \mathbb{R}^{p \times r}$. Then

$$\begin{aligned} \hat{\mathbf{E}} \dot{\hat{\mathbf{x}}}(t) &= \hat{\mathbf{A}} \hat{\mathbf{x}}(t) + \hat{\mathbf{B}} \mathbf{u}(t), \\ \hat{\mathbf{y}}(t) &= \hat{\mathbf{C}} \hat{\mathbf{x}}(t) + \mathbf{D} \mathbf{u}(t) \end{aligned} \tag{ROM}$$

approximates the dynamics of the full-order model (Σ) with output error

$$\mathbf{y}(t) - \hat{\mathbf{y}}(t) = e_{\text{output}}(t).$$

- We call the corresponding transfer function $\hat{\mathbf{H}}$.
- Model order reduction (MOR) \rightsquigarrow Find \mathbf{W} , $\mathbf{V} \in \mathbb{R}^{n \times r}$ such that $e_{\text{output}}(t)$ is small in a suitable sense.
- We will see energy-based and interpolation-based methods today and tomorrow.

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- 3 System Analysis
 - System Norms and Hardy Spaces
 - Frequency-Domain Analysis
- 4 A Selection of MOR Methods

We have

$$Y(s) = \mathbf{H}(s)U(s)$$

and

$$\hat{Y}(s) = \hat{\mathbf{H}}(s)U(s).$$

Question

What are suitable norms such that

$$\|y - \hat{y}\| \leq \left\| \mathbf{H} - \hat{\mathbf{H}} \right\| \|u\|?$$

The Banach Space $\mathcal{H}_\infty^{p \times m}$

$$\mathcal{H}_\infty^{p \times m} := \left\{ G : \mathbb{C}^+ \rightarrow \mathbb{C}^{p \times m} : G \text{ is analytic in } \mathbb{C}^+ \text{ and } \sup_{s \in \mathbb{C}^+} \|G(s)\|_2 < \infty \right\}.$$

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$\mathcal{H}_\infty^{p \times m}$ is a Banach space equipped with the \mathcal{H}_∞ -norm

$$\|G\|_{\mathcal{H}_\infty} := \sup_{\omega \in \mathbb{R}} \|G(i\omega)\|_2.$$

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Can show:
$$\|\mathbf{y} - \hat{\mathbf{y}}\|_{\mathcal{L}_2} \leq \left\| \mathbf{H} - \hat{\mathbf{H}} \right\|_{\mathcal{H}_\infty} \|\mathbf{u}\|_{\mathcal{L}_2}.$$

This bound can even be shown to be sharp.

The Hilbert Space $\mathcal{H}_2^{p \times m}$

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$\mathcal{H}_2^{p \times m}$ is a Hilbert space with the inner product

$$\langle F, G \rangle_{\mathcal{H}_2} := \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \left(F(i\omega)^{\text{H}} G(i\omega) \right) d\omega$$

and induced norm

$$\|G\|_{\mathcal{H}_2} := \langle G, G \rangle_{\mathcal{H}_2}^{1/2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(i\omega)\|_{\text{F}}^2 d\omega \right)^{1/2}.$$

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Can show:

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System Gramians and \mathcal{H}_2 -trace-formula

A system (Σ) with $\Lambda(\mathbf{E}, \mathbf{A}) \subset \mathbb{C}^-$ is called **asymptotically stable**. Then, all state trajectories decay exponentially as $t \rightarrow \infty$ and

- a) the infinite controllability and observability **Gramians** exist:

$$\mathbf{P} = \int_0^{\infty} e^{\mathbf{E}^{-1}\mathbf{A}t} \mathbf{E}^{-1} \mathbf{B} \mathbf{B}^T \mathbf{E}^{-T} e^{\mathbf{A}^T \mathbf{E}^{-T} t} dt$$

$$\mathbf{E}^T \mathbf{Q} \mathbf{E} = \int_0^{\infty} e^{\mathbf{A}^T \mathbf{E}^{-T} t} \mathbf{C}^T \mathbf{C} e^{\mathbf{E}^{-1} \mathbf{A} t} dt.$$

- b) \mathbf{P}, \mathbf{Q} solve the two **Lyapunov equations**

$$\mathbf{A} \mathbf{P} \mathbf{E}^T + \mathbf{E} \mathbf{P} \mathbf{A}^T = -\mathbf{B} \mathbf{B}^T, \quad \mathbf{A}^T \mathbf{Q} \mathbf{E} + \mathbf{E}^T \mathbf{Q} \mathbf{A} = -\mathbf{C}^T \mathbf{C}$$

- c) the \mathcal{H}_2 -norm can be expressed as

$$\|\mathbf{H}\|_{\mathcal{H}_2}^2 = \text{tr}(\mathbf{C} \mathbf{P} \mathbf{C}^T) = \text{tr}(\mathbf{B}^T \mathbf{Q} \mathbf{B}).$$

Bode Plots

The Bode plot for \mathbf{H} consists of a **magnitude plot** and a **phase plot**.

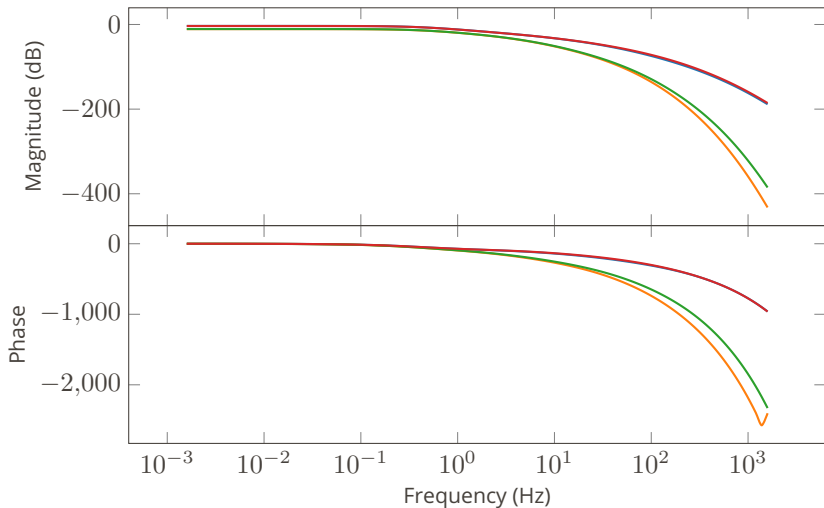
Bode magnitude plot

- component-wise graph of the function $|\mathbf{H}(i\omega)|$ for frequencies $\omega \in [\omega_{\min}, \omega_{\max}] \subset \mathbb{R}$.
- ω -axis is logarithmic.
- magnitude is given in decibels, i.e., $|\mathbf{H}(i.)|$ is plotted as $20 \log_{10}(|\mathbf{H}(i.)|)$.

Bode phase plot

- component-wise graph of the function $\arg \mathbf{H}(i\omega)$ for frequencies $\omega \in [\omega_{\min}, \omega_{\max}] \subset \mathbb{R}$.
- ω -axis is logarithmic.
- phase is given in degrees on a linear scale.

Bode Plot for the Thermal Block Example



(Sigma) Magnitude Plots

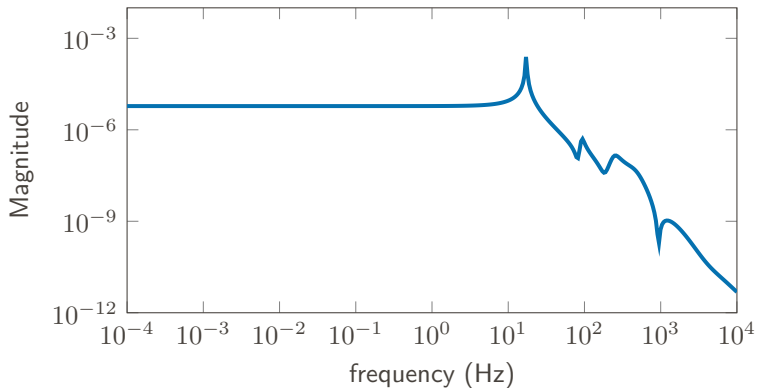
Sigma magnitude plot

- 2-norm-wise graph of the function $\mathbf{H}(i\omega)$ for frequencies $\omega \in [\omega_{\min}, \omega_{\max}] \subset \mathbb{R}$.
- ω -axis is logarithmic.

The name is due to the fact that for a given matrix \mathbf{M} the norm $\|\mathbf{M}\|_2$ is given by its largest singular value.

The real sigma magnitude plot depicts all singular values as functions of ω .

Sigma Magnitude Plot for the Artificial Fishtail



- 1 Linear Time-Invariant (LTI) Systems
- 2 Transfer Function and Realizations
- 3 System Analysis
- 4 A Selection of MOR Methods
 - Balancing Based MOR
 - Moments and Interpolation

Balanced Truncation aka. Lyapunov Balancing

Idea:

- The system (Σ) , in realization $(\mathbf{E} = \mathbf{I}, \mathbf{A}, \mathbf{B}, \mathbf{C})$, is called **balanced**, if the solutions \mathbf{P} , \mathbf{Q} of the **Lyapunov equations**

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0, \quad \mathbf{A}^T\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C} = 0,$$

satisfy: $\mathbf{P} = \mathbf{Q} = \text{diag}(\sigma_1, \dots, \sigma_n)$ where $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$.

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- $\{\sigma_1, \dots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ .
- A balanced realization is computed via **state space transformation**

$$\begin{aligned} \mathcal{T} : (\mathbf{I}, \mathbf{A}, \mathbf{B}, \mathbf{C}) &\mapsto (\mathbf{I}, \mathbf{T}\mathbf{A}\mathbf{T}^{-1}, \mathbf{T}\mathbf{B}, \mathbf{C}\mathbf{T}^{-1}) \\ &= \left(\left[\begin{array}{cc} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{array} \right], \left[\begin{array}{c} \mathbf{B}_1 \\ \mathbf{B}_2 \end{array} \right], \left[\begin{array}{cc} \mathbf{C}_1 & \mathbf{C}_2 \end{array} \right] \right). \end{aligned}$$

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- Truncation \rightsquigarrow reduced order model: $(\mathbf{I}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}) = (\mathbf{I}, \mathbf{A}_{11}, \mathbf{B}_1, \mathbf{C}_1)$.

Implementation: The Square Root Method

The SR Method

(PYMOR: BTReductor)

1. Compute (Cholesky) factors of the solutions to the Lyapunov equation,

$$\mathbf{P} = \mathbf{S}^T \mathbf{S}, \quad \mathbf{Q} = \mathbf{R}^T \mathbf{R}.$$

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2. Compute singular value decomposition

$$\mathbf{S} \mathbf{R}^T = [\mathbf{U}_1, \mathbf{U}_2] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^T \\ \mathbf{V}_2^T \end{bmatrix}.$$

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3. Define

$$\mathbf{W} := \mathbf{R}^T \mathbf{V}_1 \Sigma_1^{-1/2}, \quad \mathbf{V} := \mathbf{S}^T \mathbf{U}_1 \Sigma_1^{-1/2}.$$

4. Then the reduced order model is $(\mathbf{W}^T \mathbf{A} \mathbf{V}, \mathbf{W}^T \mathbf{B}, \mathbf{C} \mathbf{V})$.

Properties

- Lyapunov balancing **preserves asymptotic stability**.

- We have the a priori error bound: $\left\| \mathbf{H} - \hat{\mathbf{H}} \right\|_{\mathcal{H}_\infty} \leq 2 \sum_{k=r+1}^n \sigma_k$

Variants

(**PY MOR**: `BRBTReductor`, `LQGBTReductor`)

Other versions for special classes of systems or applications exist, such as

- **positive-real balancing**, (passivity-preserving)
- **bounded-real balancing**, (contractivity-preserving)
- **linear-quadratic Gaussian balancing**. (stability preserving)
(aims at low-order output feedback controllers)

The given ones all compute \mathbf{P} , \mathbf{Q} as solutions of **algebraic Riccati equations** of the form:

$$\begin{aligned} 0 &= \tilde{A}\mathbf{P}\tilde{E}^T + \tilde{E}\mathbf{P}\tilde{A}^T + \tilde{B}\tilde{B}^T \pm \tilde{E}\mathbf{P}\tilde{C}^T\tilde{C}\mathbf{P}\tilde{E}^T \\ 0 &= \tilde{A}^T\mathbf{Q}\tilde{E} + \tilde{E}^T\mathbf{Q}\tilde{A} + \tilde{C}^T\tilde{C} \pm \tilde{E}^T\mathbf{Q}\tilde{B}\tilde{B}^T\mathbf{Q}\tilde{E}. \end{aligned}$$

Tools I

Lemma (Neumann series)

Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ with spectral radius $\rho(\mathbf{A}) < 1$ be given. Then $\mathbf{I} - \mathbf{A}$ is invertible and it holds that

$$(\mathbf{I} - \mathbf{A})^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k.$$

Will be important to identify the actual shape of Markov parameters and system moments.

Tools II

Definition ((polynomial) Krylov subspace)

Given an invertible matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ and a vector $\mathbf{b} \in \mathbb{R}^n$ the k -dimensional **(polynomial) Krylov subspace** is defined as

$$\mathcal{K}_k(\mathbf{A}, \mathbf{b}) := \text{span}\{\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}, \dots, \mathbf{A}^{k-1}\mathbf{b}\}.$$

Definition (rational Krylov subspace)

Given an invertible matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ a vector $\mathbf{b} \in \mathbb{R}^n$ and a vector of shifts $s \in \mathbb{R}^k$ the k -dimensional **rational Krylov subspace** is defined as

$$\mathcal{K}_k(\mathbf{A}, \mathbf{b}, s) := \text{span}\left\{(s_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, (s_2\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \dots, (s_k\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\right\}.$$

Orthonormal bases of these spaces should be computed via the **Arnoldi iteration**.

Padé-type approximations

Goal

Match the coefficients $\mathbf{M}_k(s_0)$ or $\mathbf{M}_k(\infty)$ in

$$\mathbf{H}(s) = \sum_{k=0}^{\infty} (s - s_0)^k M_k(s_0) \quad \mathbf{H}(s) = \sum_{k=0}^{\infty} s^{-k} M_k(\infty)$$

Motivation

(assume: $m = p = 1$, s large enough)

$$\begin{aligned} \mathbf{H}(s) &= \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} = \frac{1}{s}\mathbf{C} \underbrace{\left(\mathbf{I} - \frac{1}{s}\mathbf{E}^{-1}\mathbf{A}\right)^{-1}}_{=\sum_{k=0}^{\infty} \frac{1}{s^k}(\mathbf{E}^{-1}\mathbf{A})^k} \mathbf{E}^{-1}\mathbf{B} \\ &= \sum_{k=1}^{\infty} \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})^{k-1} \mathbf{E}^{-1}\mathbf{B} \frac{1}{s^k}. \end{aligned}$$

Padé-type approximations

Motivation

 (assume: $m = p = 1$, s large enough)

$$\begin{aligned}
 \mathbf{H}(s) &= \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} = \frac{1}{s}\mathbf{C} \underbrace{\left(\mathbf{I} - \frac{1}{s}\mathbf{E}^{-1}\mathbf{A}\right)^{-1}}_{=\sum_{k=0}^{\infty} \frac{1}{s^k}(\mathbf{E}^{-1}\mathbf{A})^k} \mathbf{E}^{-1}\mathbf{B} \\
 &= \sum_{k=1}^{\infty} \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})^{k-1}\mathbf{E}^{-1}\mathbf{B} \frac{1}{s^k}.
 \end{aligned}$$

Therefore, we have

$$M_k(\infty) = \begin{cases} 0, & \text{if } k = 0, \\ \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})^{k-1}\mathbf{E}^{-1}\mathbf{B}, & \text{if } k \geq 1. \end{cases} \rightsquigarrow \text{use } \mathbf{V} = \mathcal{K}_r(\mathbf{E}^{-1}\mathbf{A}, \mathbf{E}^{-1}\mathbf{B})$$

Padé-type approximations

Approximation at ∞

$$\mathbf{V} = \mathcal{K}_r(\mathbf{E}^{-1}\mathbf{A}, \mathbf{E}^{-1}\mathbf{B}), \quad \mathbf{W} = \mathbf{V} \text{ or } \mathbf{W} = \mathcal{K}_r(\mathbf{A}^T\mathbf{E}^{-T}, \mathbf{C}^T)$$

Approximation at $s_0 = 0$

$$\mathbf{V} = \mathcal{K}_r(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B}), \quad \mathbf{W} = \mathbf{V} \text{ or } \mathbf{W} = \mathcal{K}_r(\mathbf{E}^T\mathbf{A}^{-T}, \mathbf{C}^T)$$

Approximation at $s_0 \in (0, \infty)$

$$\mathbf{V} = \mathcal{K}_r((s_0\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}, (s_0\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}), \quad \mathbf{W} = \mathbf{V}$$

or

$$\mathbf{W} = \mathcal{K}_r(\mathbf{E}^T(s_0\mathbf{E}^T - \mathbf{A}^T)^{-1}, \mathbf{C}^T)$$

Multi-point Moment Matching, Interpolation and IRKA/TSIA

Approximation at s_1, \dots, s_r

$$\mathbf{V} = \mathcal{K}_r(s, \mathbf{E}^{-1}\mathbf{A}, \mathbf{E}^{-1}\mathbf{B}), \quad \mathbf{W} = \mathbf{V} \text{ or } \mathbf{W} = \mathcal{K}_r(s, \mathbf{A}^T\mathbf{E}^{-T}, \mathbf{C}^T).$$

- $\mathbf{W} = \mathbf{V}$ as above matches first r moments of (Σ) .
- $\mathbf{W} \neq \mathbf{V}$ as above matches first $2r$ moments of (Σ) .
- $\mathbf{W} \neq \mathbf{V}$ as above actually achieves Hermite interpolation of \mathbf{H} , see, e.g., [2].

How do we choose s_1, \dots, s_r ?

\mathcal{H}_2 -optimal MOR

Find $\mathbf{s} = [s_1, \dots, s_r]^T$, such that $\left\| \mathbf{H} - \hat{\mathbf{H}} \right\|_{\mathcal{H}_2}$ is minimized.

IRKA iterative improvement of \mathbf{s} using $\Lambda(\hat{\mathbf{E}}_j, \hat{\mathbf{A}}_j)$.

(**PYMOR**: IRKAReducator)

TSIA run a fixed point iteration on the first order necessary conditions.

(**PYMOR**: TSIAReducator)

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Questions?