Numerical Methods for linear quadratic regulator systems with parabolic PDEs

Jens Saak
joint work with
Hermann Mena (EPN Quito Ecuador)
Peter Benner and Sabine Hein (née Görner) (MiIT)

Mathematik in Industrie und Technik
Fakultät für Mathematik
TU Chemnitz

Uni Trier
03.03.2009

Outline

Feedback-Control of Linear Parabolic PDEs

Feedback-Control of Linear Parabolic PDEs
Parabolic PDEs and Abstract Cauchy-Problems

Consider a control problem for a

parabolic partial differential equation

$$\frac{\partial \mathbf{x}}{\partial t} + \nabla \cdot (\mathbf{c}(\mathbf{x}) - \mathbf{k}(\nabla \mathbf{x})) + \mathbf{q}(\mathbf{x}) = \mathbf{v}(\xi, t), \quad t \in [0, T_f], \quad \text{(PDE)}$$

on a domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$.

Here:

- $\mathbf{q}$ uncontrolled sink or source
- $\mathbf{k}$ diffusive part
- $\mathbf{c}$ convection part

For ease of presentation we consider $T_f = \infty$.

Feedback-Control of Linear Parabolic PDEs
Parabolic PDEs and Abstract Cauchy-Problems

Consider a control problem for a

parabolic partial differential equation

$$\frac{\partial \mathbf{x}}{\partial t} + \nabla \cdot (\mathbf{c}(\mathbf{x}) - \mathbf{k}(\nabla \mathbf{x})) + \mathbf{q}(\mathbf{x}) = \mathbf{v}(\xi, t), \quad t \in [0, T_f], \quad \text{(PDE)}$$

on a domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$.

Here $\mathbf{v}(\xi, t) = \mathcal{B}(\xi)\mathbf{u}(t)$

- $\mathbf{u}$ control
- $\mathcal{B}$ input operator

Feedback-Control of Linear Parabolic PDEs
Parabolic PDEs and Abstract Cauchy-Problems

Consider a control problem for a

parabolic partial differential equation

$$\frac{\partial \mathbf{x}}{\partial t} + \nabla \cdot (\mathbf{c}(\mathbf{x}) - \mathbf{k}(\nabla \mathbf{x})) + \mathbf{q}(\mathbf{x}) = \mathbf{v}(\xi, t), \quad t \in [0, T_f], \quad \text{(PDE)}$$

on a domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$.

If (PDE) is linear, then a variational formulation leads to a Cauchy problem for the

linear evolution equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X}.$$

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Formulation)

lineare evolution equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X}, \quad \text{(Cauchy)}$$

with linear operators

$$\mathbf{A} : \mathrm{dom}(\mathbf{A}) \subset \mathcal{X} \to \mathcal{X}, \qquad \mathbf{B} : \mathcal{U} \to \mathcal{X},$$

on separable Hilbert spaces $\mathcal{X}$ (state space), $\mathcal{U} = \mathbb{R}^k$ (i.e., $\mathcal{U}$ is finite dim.).

Feedback-Control of Linear Parabolic PDEs
LQR Design for Abstract Cauchy Problems (Formulation)

lineare evolution equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X}, \quad \text{(Cauchy)}$$

output equation

$$\mathbf{y} = \mathbf{C}\mathbf{x}, \quad \text{(output)}$$

with linear operators

$$\mathbf{A} : \mathrm{dom}(\mathbf{A}) \subset \mathcal{X} \to \mathcal{X}, \qquad \mathbf{B} : \mathcal{U} \to \mathcal{X}, \qquad \mathbf{C} : \mathcal{X} \to \mathcal{Y},$$

on separable Hilbert spaces $\mathcal{X}$ (state space), $\mathcal{U} = \mathbb{R}^k$ (i.e., $\mathcal{U}$ is finite dim.) and $\mathcal{Y}$ (observation space).

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Formulation)

lineare evolution equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X}, \quad \text{(Cauchy)}$$

output equation

$$\mathbf{y} = \mathbf{C}\mathbf{x}, \quad \text{(output)}$$

Defining $\mathbf{Q} := \mathbf{C}^*\hat{\mathbf{Q}}\mathbf{C}$ with $\hat{\mathbf{Q}} = \hat{\mathbf{Q}}^* \geq 0$, and $\mathbf{R} = \mathbf{R}^* > 0$ we state the

cost function

$$\mathcal{J}(\mathbf{u}) = \frac{1}{2}\int_0^\infty < \hat{\mathbf{Q}}\mathbf{y}, \mathbf{y} > + < \mathbf{R}\mathbf{u}, \mathbf{u} > dt. \qquad \text{(cost)}$$

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Formulation)

lineare evolution equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X}, \quad \text{(Cauchy)}$$

output equation

$$\mathbf{y} = \mathbf{C}\mathbf{x}, \quad \text{(output)}$$

Defining $\mathbf{Q} := \mathbf{C}^*\hat{\mathbf{Q}}\mathbf{C}$ with $\hat{\mathbf{Q}} = \hat{\mathbf{Q}}^* \geq 0$, and $\mathbf{R} = \mathbf{R}^* > 0$ we state the

cost function

$$\mathcal{J}(\mathbf{u}) = \frac{1}{2}\int_0^\infty < \mathbf{Q}\mathbf{x}, \mathbf{x} > + < \mathbf{R}\mathbf{u}, \mathbf{u} > dt. \qquad \text{(cost)}$$

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Formulation)

lineare evolution equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X}, \quad \text{(Cauchy)}$$

output equation

$$\mathbf{y} = \mathbf{C}\mathbf{x}, \quad \text{(output)}$$

cost function

$$\mathcal{J}(\mathbf{u}) = \frac{1}{2}\int_0^\infty < \mathbf{Q}\mathbf{x}, \mathbf{x} > + < \mathbf{R}\mathbf{u}, \mathbf{u} > dt. \qquad \text{(cost)}$$

We can now formulate the

LQR–problem.

Minimize (cost) with respect to (Cauchy).

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Solution)

Well understood in the open literature:
Analogously to ODE systems case we find the

optimal state feedback

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^{*}\mathbf{X}_{\infty}\mathbf{x}.$$

Here $\mathbf{X}_{\infty}$ is the stabilizing, positive semidefinite, selfadjoint solution to the

Operator–Algebraic–Riccati–equation

$$0 = \mathcal{R}(\mathbf{X}) := \mathbf{Q} + \mathbf{A}^{*}\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{*}\mathbf{X}. \qquad \text{(O-ARE)}$$

---

e.g. [Lions ‘71; Lasiecka/Triggiani ‘00; Bensoussan et al. ‘92/‘06; Pritchard/Salamon ‘87; Curtain/Zwart ’95]

Feedback-Control of Linear Parabolic PDEs
LQR Design for Abstract Cauchy Problems (Solution)

(Cauchy) can now be rewritten as

closed loop system

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{X}_\infty)\mathbf{x},$$

and the

optimal solution trajectory

is given as

$$\mathbf{x}(t) = \mathbf{S}(t)\mathbf{x}_0,$$

where $\mathbf{S}(t)$ is the operator semigroup generated by $\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{X}_\infty$.

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Let $(\mathcal{X}_n)_{n\in\mathbb{N}}$ a Galerkin scheme for $\mathcal{X}$. We formulate the

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Let $(\mathcal{X}_n)_{n\in\mathbb{N}}$ a Galerkin scheme for $\mathcal{X}$. We formulate the

n-d evolution equation

$$\dot{x} = A_n x + B_n \mathbf{u}, \qquad \mathcal{X}_n \ni x_n(0) = \mathbf{P}_n \mathbf{x}_0,$$
(n-d Cauchy)

output equation

$$y_n = C_n x_n,$$
(n-d output)

with linear operators

$$A_n : \mathrm{dom}(A_n) \subset \mathcal{X}_n \to \mathcal{X}_n, \qquad B_n : \mathcal{U} \to \mathcal{X}_n, \qquad C_n : \mathcal{X}_n \to \mathcal{Y}_n,$$

on n-d Hilbert spaces $\mathcal{X}_n$ (state space) and $\mathcal{Y}_n$ (observation space), but still $\mathcal{U} = \mathbb{R}^k$.
$\mathbf{P}_n : \mathcal{X} \to \mathcal{X}_n$ the canonical orthogonal projection.

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Let $(\mathcal{X}_n)_{n\in\mathbb{N}}$ a Galerkin scheme for $\mathcal{X}$. We formulate the

n-d evolution equation

$$\dot{x} = A_n x + B_n \mathbf{u}, \qquad \mathcal{X}_n \ni x_n(0) = \mathbf{P}_n \mathbf{x}_0,$$
(n-d Cauchy)

output equation

$$y_n = C_n x_n,$$
(n-d output)

Defining $Q_n := C_n^* \hat{Q}_n C_n$ with $\hat{Q}_n = \hat{Q}_n^* \geq 0$, and $\mathbf{R} = \mathbf{R}^* > 0$ we formulate

cost function

$$\mathcal{J}_n(\mathbf{u}) = \frac{1}{2} \int_0^\infty < \hat{Q}_n y_n, y_n > + < \mathbf{Ru}, \mathbf{u} > dt. \qquad \text{(n-d Cost)}$$

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Let $(\mathcal{X}_n)_{n\in\mathbb{N}}$ a Galerkin scheme for $\mathcal{X}$. We formulate the

n-d evolution equation

$$\dot{x} = A_n x + B_n \mathbf{u}, \qquad \mathcal{X}_n \ni x_n(0) = \mathbf{P}_n \mathbf{x}_0, \qquad \text{(n-d Cauchy)}$$

output equation

$$y_n = C_n x_n, \qquad \text{(n-d output)}$$

Defining $Q_n := C_n^* \hat{Q}_n C_n$ with $\hat{Q}_n = \hat{Q}_n^* \geq 0$, and $\mathbf{R} = \mathbf{R}^* > 0$ we formulate

cost function

$$\mathcal{J}_n(\mathbf{u}) = \frac{1}{2} \int_0^\infty < Q_n x_n, x_n > + < \mathbf{Ru}, \mathbf{u} > dt. \qquad \text{(n-d Cost)}$$

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

n-d evolution equation

$$\dot{x} = A_n x + B_n \mathbf{u}, \qquad \mathcal{X}_n \ni x_n(0) = \mathbf{P}_n \mathbf{x}_0,$$

(n-d Cauchy)

output equation

$$y_n = C_n x_n,$$

(n-d output)

cost function

$$\mathcal{J}(\mathbf{u}) = \frac{1}{2} \int_0^\infty < Q_n x_n, x_n > + < \mathbf{R}\mathbf{u}, \mathbf{u} > dt. \qquad \text{(n-d cost)}$$

and state the

n-d LQR–problem.

Minimize (n-d Cost) with respect to (n-d Cauchy).

Feedback-Control of Linear Parabolic PDEs
LQR Design for Abstract Cauchy Problems (Approximation)

Analogously to the ∞-dim. case we now find:

optimal state feedback

$$\mathbf{u} = -\mathbf{R}^{-1} B_n^* X_n x_n,$$

where X_n is the stabilizing, positive semidefinite, selfadjoint solution to the

n-d Operator-Algebraic-Riccati-Equation

$$0 = \mathcal{R}_n(X) := Q_n + A_n^* X + X A_n - X B_n \mathbf{R}^{-1} B_n^* X. \quad \text{(n-d O-ARE)}$$

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

As above we can write (n-d Cauchy) as

closed loop system

$$\dot{x}_n = (A_n - B_n \mathbf{R}^{-1} B_n^* X_n) x_n,$$

and the

optimal solution

is given as

$$x_n(t) = S_n(t) \mathbf{P}_n \mathbf{x}_0,$$

also again $S_n(t)$ is the operator semigroup generated by $A_n - B_n \mathbf{R}^{-1} B_n^* X_n$.

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Approximation

The n-d LQR–problems approximate the ∞-dim LQR–problem in the following sense

- $X_n \mathbf{P}_n \mathbf{v} \rightarrow \mathbf{X}\mathbf{v}$ for $n \rightarrow \infty$ and any $\mathbf{v} \in \mathcal{X}$,
- $S_n(t)\mathbf{P}_n \mathbf{v} \rightarrow \mathbf{S}(t)\mathbf{v}$ for $n \rightarrow \infty$ and any $\mathbf{v} \in \mathcal{X}$,

Feedback-Control of Linear Parabolic PDEs
LQR Design for Abstract Cauchy Problems (Approximation)

Approximation

The n-d LQR–problems approximate the ∞-dim LQR–problem in the following sense

- $X_n \mathbf{P}_n \mathbf{v} \to \mathbf{X}\mathbf{v}$ for $n \to \infty$ and any $\mathbf{v} \in \mathcal{X}$,
- $S_n(t)\mathbf{P}_n \mathbf{v} \to \mathbf{S}(t)\mathbf{v}$ for $n \to \infty$ and any $\mathbf{v} \in \mathcal{X}$,

that means in the strong operator-topology.

[Banks/Kunisch'84] distributed control of parabolic PDEs
[Benner/S.'05] boundary control with mixed boundary conditions
[Lasiecka/Triggiani'00] weakens regularity conditions on (Cauchy), also has convergence rates
[Ito'87/'90; Morris'94] general Cauchy problems
[...] many more

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Approximation

The n-d LQR–problems approximate the ∞-dim LQR–problem in the following sense

- $X_n \mathbf{P}_n \mathbf{v} \to \mathbf{X}\mathbf{v}$ for $n \to \infty$ and any $\mathbf{v} \in \mathcal{X}$,
- $S_n(t)\mathbf{P}_n \mathbf{v} \to \mathbf{S}(t)\mathbf{v}$ for $n \to \infty$ and any $\mathbf{v} \in \mathcal{X}$,

that means in the strong operator-topology.

Remarks:

- For a chosen basis (e.g. from spatial FDM/FEM discretization) all n-d operators have matrix representations and $S(t)$ coincides with the matrix-exponential $e^{(A-BR^{-1}B^TX)t}$.

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Approximation

The n-d LQR–problems approximate the ∞-dim LQR–problem in the following sense

- $X_n \mathbf{P}_n \mathbf{v} \rightarrow \mathbf{X}\mathbf{v}$ for $n \rightarrow \infty$ and any $\mathbf{v} \in \mathcal{X}$,
- $S_n(t)\mathbf{P}_n \mathbf{v} \rightarrow \mathbf{S}(t)\mathbf{v}$ for $n \rightarrow \infty$ and any $\mathbf{v} \in \mathcal{X}$,

that means in the strong operator-topology.

Remarks:

- For a chosen basis (e.g. from spatial FDM/FEM discretization) all n-d operators have matrix representations and $S(t)$ coincides with the matrix-exponential $e^{(A-BR^{-1}B^TX)t}$.
- $\mathbf{u}$ and $\mathbf{R}$ are always kept fixed, i.e., $\mathbf{u}$ from computations for an n-d problem can directly be applied in the ∞-d problem.

Feedback-Control of Linear Parabolic PDEs
LQR Design for Abstract Cauchy Problems (Numerics)

Main task in numerical solution

Efficient solution of the large sparse matrix–Riccati–equations

$$0 = \mathcal{R}_h(X) := Q_h + A_h^* X + X A_h - X B_h \mathbf{R}^{-1} B_h^* X, \qquad \text{(M-ARE)}$$

with regard to both memory and CPU usage.

Classical methods are not applicable due to their cubic complexity.

Feedback-Control of Linear Parabolic PDEs
LQR Design for Abstract Cauchy Problems (Numerics)

(M-ARE) is non-linear $\Rightarrow$

Newton's method for the ARE

$$\mathcal{R}'_h|_X(N_\ell) = -\mathcal{R}_h(X_\ell), \qquad X_{\ell+1} = X_\ell + N_\ell.$$

The Frechét derivative of $\mathcal{R}_h$ at X is given as the Lyapunov operator

$$\mathcal{R}'_h|_X : \quad Z \mapsto (A_h - B_h R^{-1} B_h^T X)^T Z + Z(A_h - B_h R^{-1} B_h^T X).$$

Thus we find the

one step Newton iteration

$$(A_h - B_h R^{-1} B_h^T X_\ell)^T X_{\ell+1} + X_{\ell+1}(A_h - B_h R^{-1} B_h^T X_\ell) = -C_h^T Q_h C_h - X_\ell B_h R^{-1} B_h^T X_\ell.$$

Feedback-Control of Linear Parabolic PDEs
LQR Design for Abstract Cauchy Problems (Numerics)

In every Newton step we solve a

Lyapunov equation

$$F^T X + XF = -GG^T. \qquad \text{(Lyapunov)}$$

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Numerics)

In every Newton step we solve a

Lyapunov equation

$$F^T X + XF = -GG^T. \qquad \text{(Lyapunov)}$$

Available solvers for large sparse Lyapunov equations (Lyapunov)

ADI [Wachspress'88; Penzl'99; Benner/Li/Penzl'08; Li/White'02];

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Numerics)

In every Newton step we solve a

Lyapunov equation

$$F^T X + XF = -GG^T. \qquad \text{(Lyapunov)}$$

Available solvers for large sparse Lyapunov equations (Lyapunov)

- **ADI** [Wachspress'88; Penzl'99; Benner/Li/Penzl'08; Li/White'02];
- **Krylov** [Kasenally/Jaimoukha'94; Jbilou/Riquet'06; Simoncini'06]
- **Smith** [Penzl'99; Gugercin/Sorensen/Antoulas'03]
- **...** many more

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Numerics)

In every Newton step we solve a

Lyapunov equation

$$F^T X + XF = -GG^T. \qquad \text{(Lyapunov)}$$

Available solvers for large sparse Lyapunov equations (Lyapunov)

- **ADI** [Wachspress'88; Penzl'99; Benner/Li/Penzl'08; Li/White'02];
- **Krylov** [Kasenally/Jaimoukha'94; Jbilou/Riquet'06; Simoncini'06]
- **Smith** [Penzl'99; Gugercin/Sorensen/Antoulas'03]
- **...** many more

ADI needs shift–paramets; Choice of shifts: [Ellner/Wachspress'91; Penzl'00; Benner/Mena/S.'06; Starke '89; Sabino'06; Wachspress '08]

Feedback-Control of Linear Parabolic PDEs
LQR Design for Abstract Cauchy Problems (Numerics)

In every Newton step we solve a

Lyapunov equation

$$F^T X + XF = -GG^T. \qquad \text{(Lyapunov)}$$

Available solvers for large sparse Lyapunov equations (Lyapunov)

- **ADI** [Wachspress'88; Penzl'99; Benner/Li/Penzl'08; Li/White'02];
- **Krylov** [Kasenally/Jaimoukha'94; Jbilou/Riquet'06; Simoncini'06]
- **Smith** [Penzl'99; Gugercin/Sorensen/Antoulas'03]
- **...** many more

For Systems with very few inputs Newton-ADI and Newton-Smith can iterate on the feedback $K_h := \mathbf{R}^{-1} B_h^T X$ directly[Penzl'00;Banks/Ito'91].

Tracking Control

LQR Design for Abstract Cauchy Problems

Tracking Control
Linear Systems with Inhomogenities

Reminder for systems with

linear inhomogeneous evolution equations

$$\dot{x} = Ax + Bu + f.$$

Let $\hat{x}$ solve the uncontrolled system $\dot{x} = Ax + f$, then

$$f = \dot{\hat{x}} - A\hat{x},$$

and

$$\dot{x} - \dot{\hat{x}} = A(x - \hat{x}) + Bu.$$

We can solve the system

$$\dot{z} = Az + Bu$$

for $z = x - \hat{x}$ to compute the control u.

e.g., [Godunov'97]

Tracking Control
Application in Tracking Control of Parabolic PDEs

Consider $\tilde{\mathbf{x}}$ the state we want to track and the

tracking problem

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v}, \qquad \mathbf{y} = \mathbf{C}(\mathbf{x} - \tilde{\mathbf{x}}),$$
$$\mathcal{J}(\mathbf{u}) = \frac{1}{2}\int_0^\infty < \mathbf{Q}(\mathbf{x} - \tilde{\mathbf{x}}), \mathbf{x} - \tilde{\mathbf{x}} > + < \mathbf{R}\mathbf{u}, \mathbf{u} > dt. \tag{tracking}$$

Define $\mathbf{z} := \mathbf{x} - \tilde{\mathbf{x}}$ and the Cauchy problem

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{v}, \qquad \mathbf{y} = \mathbf{C}\mathbf{z}. \tag{1}$$

The optimal control then is given as $\mathbf{v} = -\mathbf{K}\mathbf{z}$ as above and (1) is equivalent to

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} + \dot{\tilde{\mathbf{x}}} - \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\mathbf{K}\tilde{\mathbf{x}}.$$

Tracking Control
Application in Tracking Control of Parabolic PDEs

Consider $\tilde{\mathbf{x}}$ the state we want to track and the

tracking problem

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v}, \qquad \mathbf{y} = \mathbf{C}(\mathbf{x} - \tilde{\mathbf{x}}),$$
$$\mathcal{J}(\mathbf{u}) = \frac{1}{2}\int_0^\infty < \mathbf{Q}(\mathbf{x} - \tilde{\mathbf{x}}), \mathbf{x} - \tilde{\mathbf{x}} > + < \mathbf{R}\mathbf{u}, \mathbf{u} > dt. \qquad \text{(tracking)}$$

- $\mathbf{f} := \dot{\tilde{\mathbf{x}}} - \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\mathbf{K}\tilde{\mathbf{x}}$ is a known inhomogenity when solving the closed loop system.
- Equations (tracking) and (1) require the same algebraic Riccati equation.

Tracking Control
Application in Tracking Control of Parabolic PDEs

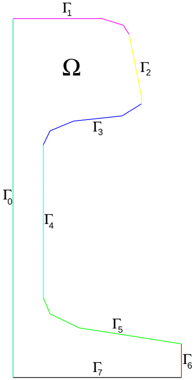
Consider $\tilde{\mathbf{x}}$ the state we want to track and the

tracking problem

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v}, \qquad \mathbf{y} = \mathbf{C}(\mathbf{x} - \tilde{\mathbf{x}}),$$
$$\mathcal{J}(\mathbf{u}) = \frac{1}{2}\int_0^\infty < \mathbf{Q}(\mathbf{x} - \tilde{\mathbf{x}}), \mathbf{x} - \tilde{\mathbf{x}} > + < \mathbf{R}\mathbf{u}, \mathbf{u} > dt. \qquad \text{(tracking)}$$

- We can compute the Feedback for (tracking) with the above technique for (1) and afterwards solve the inhomogeneous closed loop system.
- Method also works for a reference pair $(\tilde{\mathbf{x}}, \tilde{\mathbf{u}})$
  [Benner/Görner/S.'06]

Non-linear Systems
Heat Distribution in Steel Profiles, a Model Problem



Non-linear Systems

Heat Distribution in Steel Profiles, a Model Problem

The active cooling of steel profiles in a rolling facility serves as a model problem. We consider the in stationary heat equation

$$
\begin{aligned}
c(x)\rho(x)\tfrac{\partial}{\partial t}x(\xi,t) &= \nabla.(\lambda(x)\nabla x(\xi,t)) && \text{on } \Omega\times(0,T),\\
-\lambda(x)\tfrac{\partial}{\partial \nu}x(\xi,t) &= \kappa_i(x(\xi,t)-u_i(t)) && \text{on } \Gamma_i\times(0,T),\\
x(\xi,0) &= x_0(\xi) && \text{on } \Omega,
\end{aligned}
\tag{heat}
$$

x	state, temperature	$c(x)$	specific heat capacity
u	control	$\varrho(x)$	density
$T \in \mathbb{R}\cup\{\infty\}$	final time	$\lambda(x)$	heat conductivity

Non-linear Systems
Heat Distribution in Steel Profiles, a Model Problem

The active cooling of steel profiles in a rolling facility serves as a model problem. We consider the in stationary heat equation

$$\begin{aligned} c(x)\rho(x)\frac{\partial}{\partial t}x(\xi,t) &= \nabla.(\lambda(x)\nabla x(\xi,t)) && \text{on } \Omega\times(0,T),\\ -\lambda(x)\frac{\partial}{\partial \nu}x(\xi,t) &= \kappa_i(x(\xi,t)-u_i(t)) && \text{on } \Gamma_i\times(0,T),\\ x(\xi,0) &= x_0(\xi) && \text{on } \Omega, \end{aligned} \tag{heat}$$

(heat) obviously is non-linear due to c, ϱ and λ depending on the temperature x.

Idea

Freeze the material parameters for one or more time steps. $\Rightarrow$ Linearization $\Rightarrow$ method from the introduction can be applied.

Non-linear Systems

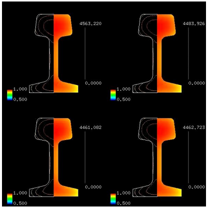
Linearization and Results

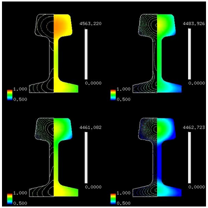
Idea

Freeze the material parameters for one or more time steps. ⇒ Linearization ⇒ method from the introduction can be applied.

Numerics semi-implicit discretization

Theory embeds to model predictive control. [Benner/S.'07]





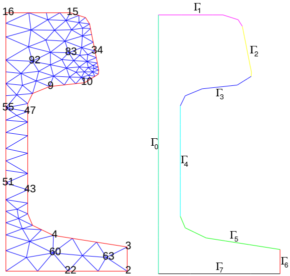
The End

Thank you for your attention!

- FEM: ALBERTA
- Graphics:
  Grape/MATLAB
- AREs: LYAPACK

The End

Thank you for your attention!



- FEM: ALBERTA
- Graphics: Grape/MATLAB
- AREs: LYAPACK