Abstract

Riccati-Based Boundary Feedback Stabilization of Multi-Field Flow Problems

We present recent progress in the numerical realization for boundary feedback stabilization for Navier-Stokes equations. Therefor, we will explain how the projection idea of [Heinkenschloss/Sorensen/Sun '05] can be used to apply the approach of Raymond [Raymond '06] for boundary feedback stabilization of (Navier-)Stokes equations. To this end, we will illustrate details about the arising nested iteration and how we treat the connected technical difficulties.
Furthermore, we will introduce a more complicated application, namely the boundary feedback stabilization of a coupled system of Navier-Stokes equations and a diffusion-convection equation that describes the spread of a concentration within the flow field. We will point out the system structure and the analogies to the pure Navier-Stokes system.

Riccati-Based Boundary Feedback Stabilization of Multi-Field Flow Problems

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Research group Mathematics in Industry and Technology

OCIP 2012
Munich, March 11-13 2013

Overview

Problem Setting

Test-Scenarios: NSE on von Kármán Vortex Street

PDE: NSE

Goal: $\vec{z} = \vec{v} - \vec{w} \to 0$

$\rightsquigarrow$ Linearized Navier-Stokes equations:

$$\frac{\partial \vec{z}}{\partial t} - \frac{1}{\text{Re}} \Delta \vec{z} + (\vec{z} \cdot \nabla)\vec{w} + (\vec{w} \cdot \nabla)\vec{z} + \nabla p = 0$$

$$\operatorname{div} \vec{z} = 0$$

defined for $t \in (0, \infty)$ and $\vec{x} \in \Omega$

+ boundary and initial conditions

LQR

Minimize

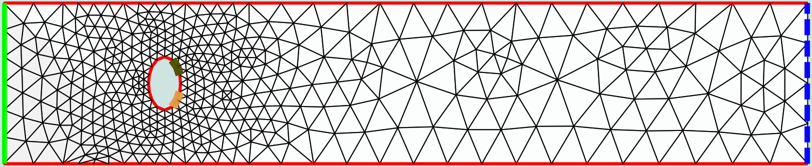
$$\mathcal{J}(\mathbf{y}, \mathbf{u}) = \frac{1}{2} \int_0^\infty \lambda ||\mathbf{y}||^2 + \frac{1}{\rho} ||\mathbf{u}||^2 \, \mathrm{dt}$$

s.t.

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{z} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} A & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y}(t) = C\mathbf{z}(t)$$

Domain Ω: *von Kármán vortex street*



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NSE

stationary NSE

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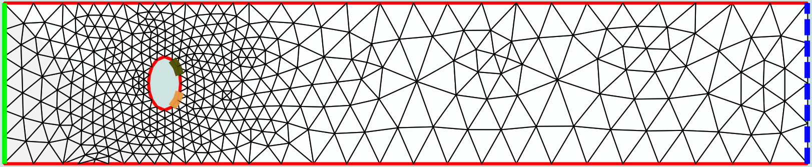

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s.t.

$$\mathcal{M} \frac{d}{dt} \tilde{\mathbf{z}} = \mathcal{A} \tilde{\mathbf{z}} + \mathcal{B} \mathbf{u}$$

$$\mathbf{y}(t) = \mathcal{C} \tilde{\mathbf{z}}$$

[Heinkenschloss/Sorensen/Sun '08]

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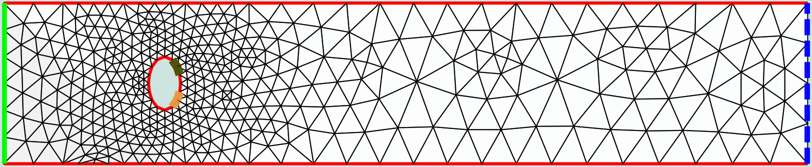

Problem Setting

Test-Scenarios: NSE Coupled with DCE in Reactor Model

PDE: NSE+DCE

Goal: $\vec{z} = \vec{v} - \vec{w} \rightarrow 0, c = c^{(\vec{v})} - c^{(\vec{w})} \rightarrow 0$

$\rightsquigarrow$ Linearized coupled system:

$$\frac{\partial \vec{z}}{\partial t} - \frac{1}{\text{Re}}\Delta\vec{z} + (\vec{z}\cdot\nabla)\vec{w} + (\vec{w}\cdot\nabla)\vec{z} + \nabla p = 0$$

$$\frac{\partial c}{\partial t} - \frac{1}{\text{Re}\,\text{Sc}}\Delta c + (\vec{w}\cdot\nabla)c + (\vec{z}\cdot\nabla)c^{(\vec{w})} = 0$$

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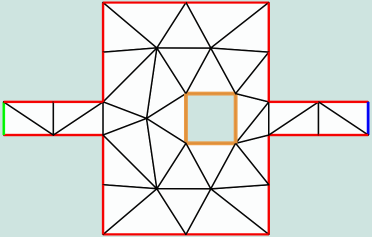
defined for $t \in (0,\infty), \vec{x} \in \Omega$ and some BC,IC

Domain Ω: Reactor Model



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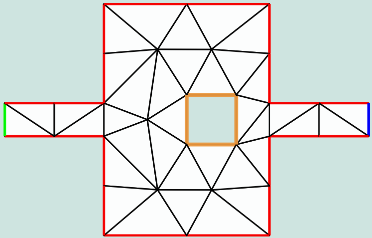

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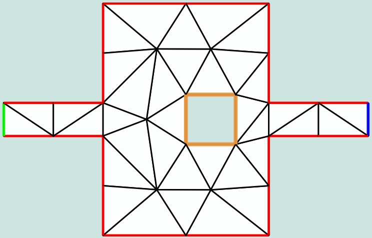

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[Heinkenschloss/Sorensen/Sun '08]

Workflow

LQR for Nonlinear PDEs with Algebraic Constraints

Continuous Level

- Linearize around a given stationary trajectory.
- Index reduction via projection method. [Heinkenschloss/Sorensen/Sun '08]
- Formulate stabilization problem for the perturbation.

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- Optimize cost functional design to improve simulation results.

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Workflow

Leray Projection

Continuous Level

$P : (L^2(\mathbb{R}^d)^d) \rightarrow \mathbf{H}_{\mathrm{div}\,0}(\mathbb{R}^d)$

To compute $\vec{v}_{div}$, solve for given $\vec{v}$

$$\vec{v}_{div} + \nabla p = \vec{v},$$
$$\mathrm{div}\,\vec{v}_{div} = 0.$$

Semi-Discretized Level

OCIP 2012

[Heinkenschloss/Sorensen/Sun '08]

Index reduction idea for Lyapunov-solver:

$$\Pi^T := I - M^{-1}G(G^TM^{-1}G)^{-1}G^T.$$

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$$M\mathbf{w} + \qquad G\mathbf{p} = \qquad M\mathbf{v}$$

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$$\mathbf{p} = (G^TM^{-1}G)^{-1}G^T\mathbf{v}$$

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$$\mathbf{p} = (G^T M^{-1} G)^{-1} G^T \mathbf{v}$$

$$M\mathbf{w} = M(I - M^{-1}G(G^T M^{-1} G)^{-1} G^T)\mathbf{v}$$

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$$\mathbf{p} = (G^TM^{-1}G)^{-1}G^T\mathbf{v}$$
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$$\mathbf{w} = (I - M^{-1}G(G^TM^{-1}G)^{-1}G^T)\mathbf{v}$$

Leray vs. Π^T

$$\begin{array}{r} \vec{w} = P(\vec{v}) \\ 0 = \mathrm{div}\,\vec{w} \end{array} \Rightarrow \begin{array}{r} \mathbf{w} = \Pi^T\mathbf{v} \\ 0 = G^T\mathbf{w} \end{array}$$

Workflow

Nested Iteration

Compute feedback matrix $\mathcal{K} = \mathcal{B}^T X \mathcal{M}$ with X solves:

$$\mathcal{R}(X) = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T X \mathcal{M} + \mathcal{M}^T X \mathcal{A} - \mathcal{M}^T X \mathcal{B}\mathcal{B}^T X \mathcal{M} = 0$$

Newton Kleinman method

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Step m + 1: solve Lyapunov equation

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)})^T X^{(m+1)} \mathcal{M} + \mathcal{M}^T X^{(m+1)} (\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)}) = -(\mathcal{W}^{(m)})^T \mathcal{W}^{(m)}$$

Newton Kleinman method

low rank ADI method

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Step i: solve the projected linear system

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)} + q_i \mathcal{M})^T \mathcal{V}_i = \mathcal{Y} \qquad (1)$$

Newton Kleinman method

low rank ADI method

Krylov solver

Workflow

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Avoid explicit projection using [Heinkenschloss/Sorensen/Sun '08]:

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Step i: solve the projected linear system

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Krylov solver

Avoid explicit projection using [Heinkenschloss/Sorensen/Sun '08]:
Replace (1) and **solve instead** the saddle point system (SPS)

$$\begin{bmatrix} A^T - (K^{(m)})^T B^T + q_i M^T & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} V_i \\ * \end{bmatrix} = \begin{bmatrix} Y \\ 0 \end{bmatrix}$$

for different ADI shifts q_i for a couple of rhs Y.

Workflow

Additional Tasks

- Compute initial feedback for unstable systems.
  - ↪ Determine the invariant unstable subspace $\mathcal{U}$.
  - ↪ Solve Bernoulli equation on $\mathcal{U}$ [Benner '11, Amodei/Buchot '12].

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SPP1253

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OCIP 2012

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SPP1253 | OCIP 2012 | work in progress

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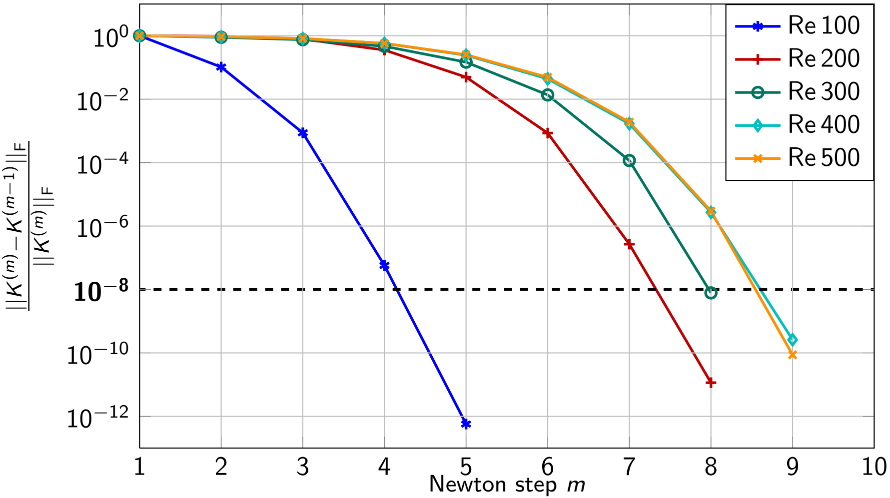
Numerical Examples

Newton-ADI: NSE on von Kármán Vortex Street



Relative change of feedback matrix K for different Reynolds numbers ($\lambda = \rho = 1$).

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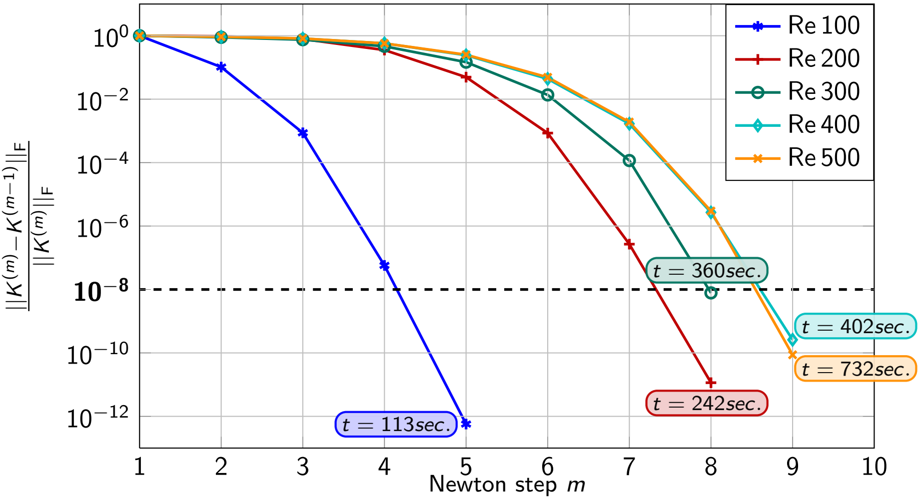

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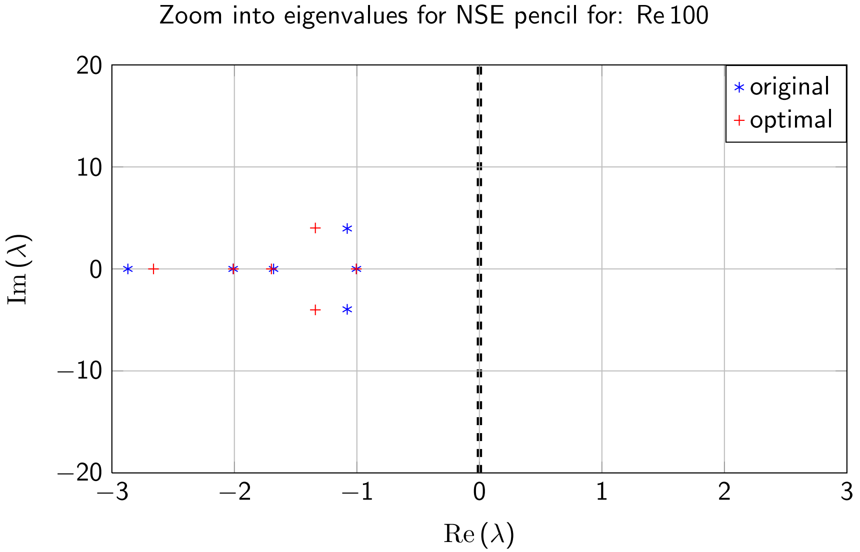
Eigenvalue Behavior of NSE on von Kármán Vortex Street



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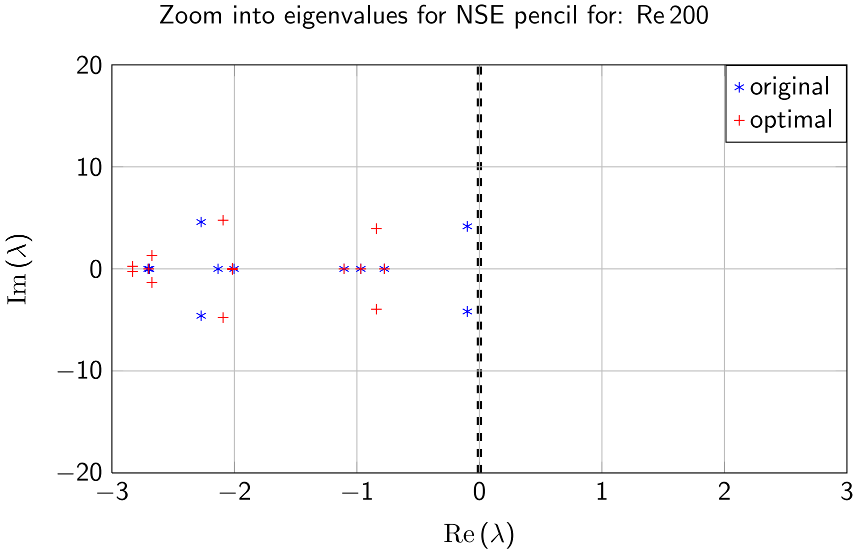

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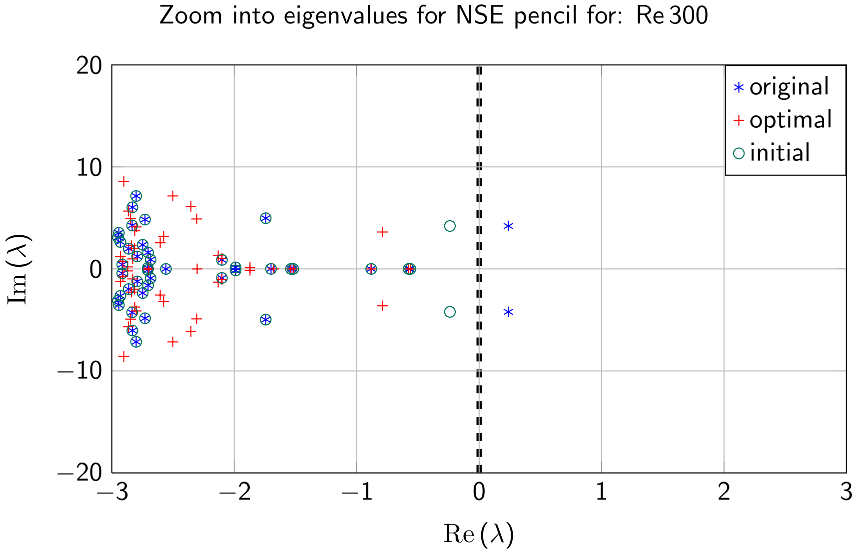

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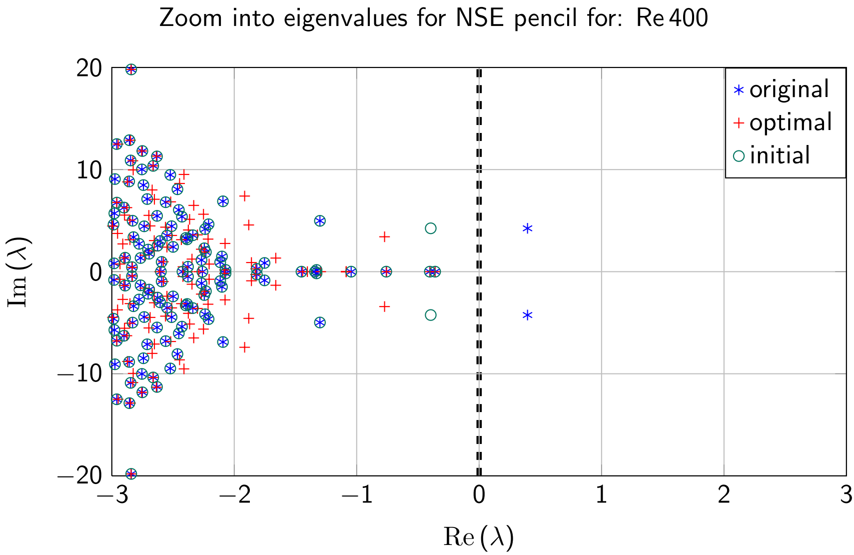
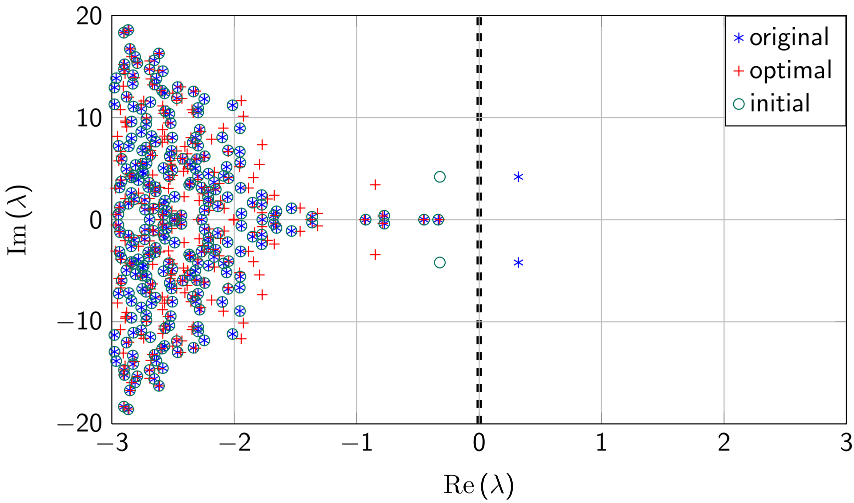


Numerical Examples

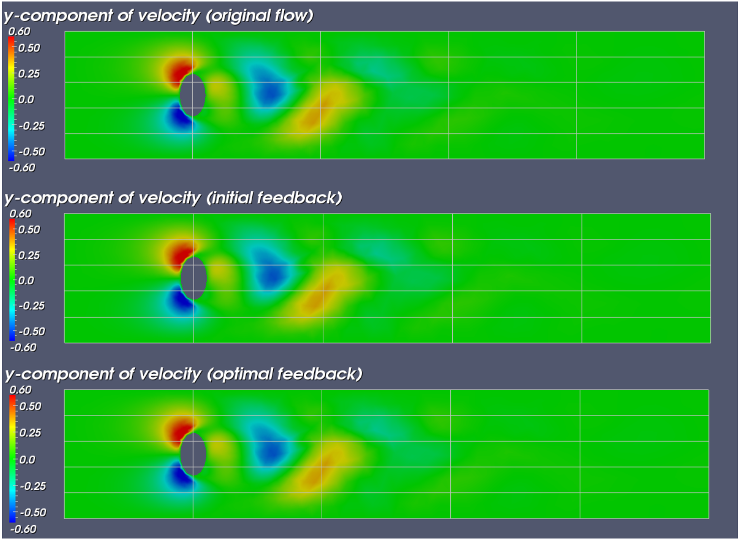
Eigenvalue Behavior of NSE on von Kármán Vortex Street

Zoom into eigenvalues for NSE pencil for: Re 500



Numerical Examples

Closed-Loop Simulation of NSE on von Kármán Vortex Street for Re = 300



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Conclusion

Review

- Adapt Newton-ADI algorithm and ADI shift determination for flow problems (DAE structure).
- Improved input matrix B w.r.t. Raymond's analytic approach.
- Closed-loop simulation for NSE.
- Adapted method to coupled flow in reactor model.
- Developed suitable output matrix C for reactor model.

Outlook

- Improve idea of *inexact Newton* to threefold nested iteration.
- Residual based stopping criteria for feedback computation.
- Closed-loop simulation of coupled flow in reactor model.
- Improve Krylov solver via the use of recycling or block techniques.
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Many thanks for your attention!

Literature

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