Non-Conforming Finite Elements and Riccati-Based Feedback Stabilization of the Stokes Equations

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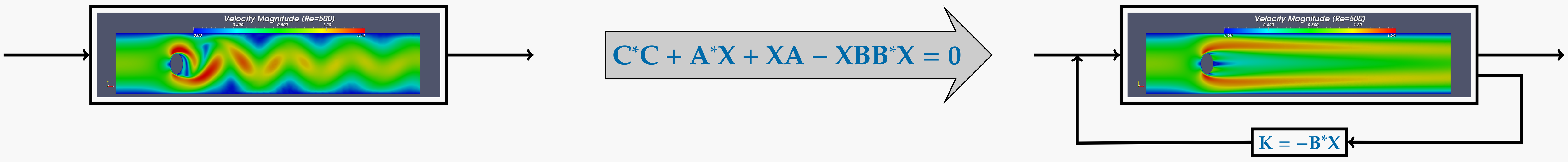
Goal



Problem Setting

Abstract Problem Setting

Motivation:

- Stabilization of flows described by Navier-Stokes equations (NSE)

$$\left.\begin{aligned}\frac{\partial}{\partial t}\mathbf{v} - \frac{1}{\mathrm{Re}}\Delta\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p &= \mathbf{0}\\ \operatorname{div}\mathbf{v} &= \mathbf{0}\end{aligned}\right\}\text{ in } (0,\infty)\times\Omega, \qquad (1)$$

  to steady-state solution, with $\Omega \subset \mathbb{R}^d, d = 2,3$, the velocity field $\mathbf{v}(t,\mathbf{x}) \in \mathbb{R}^d$, the pressure $p(t,\mathbf{x}) \in \mathbb{R}$, the time $t \in (0,\infty)$, the spatial variable $\mathbf{x} \in \Omega$, and the Reynolds number $\mathrm{Re} \in \mathbb{R}^+$.
- Construction based on associated linear quadratic control problem (LQR) for boundary control [4].
- Numerical treatment for 2D case with linearized NSE described in [1].

Here: Stokes equations

$$\left.\begin{aligned}\frac{\partial}{\partial t}\mathbf{v} - \frac{1}{\mathrm{Re}}\Delta\mathbf{v} + \nabla p &= \mathbf{0}\\ \operatorname{div}\mathbf{v} &= \mathbf{0}\end{aligned}\right\}\text{ in } (0,\infty)\times\Omega. \qquad (2)$$

Semi Discretzed Problem Setting

Finite element discretization of (2) yields

$$\begin{aligned} M\dot{\mathbf{z}} &= A\mathbf{z} + G\mathbf{p} + B\mathbf{u},\\ \mathbf{0} &= G^T\mathbf{z},\\ \mathbf{y} &= C\mathbf{z}, \end{aligned} \qquad (3)$$

with

- discretized velocity $\mathbf{z}(t) \in \mathbb{R}^{n_v}$ and pressure $\mathbf{p}(t) \in \mathbb{R}^{n_p}$,
- symmetric positive definite mass matrix $M \in \mathbb{R}^{n_v\times n_v}$,
- system matrix $A \in \mathbb{R}^{n_v\times n_v}$ (symmetric for Stokes) and
- discretized gradient $G \in \mathbb{R}^{n_v\times n_p}$ of rank n_p.

In the context of an LQR problem one additionally gets

- the input matrix $B \in \mathbb{R}^{n_v\times n_r}$ and
- the input $\mathbf{u}(t) \in \mathbb{R}^{n_r}$,

which describe the boundary control. Partial observation furthermore leads to

- the output $\mathbf{y}(t) \in \mathbb{R}^{n_a}$ and
- the output matrix $C \in \mathbb{R}^{n_a\times n_v}$.

Implicit Index Reduction

To rewrite the DAE system (3) with differential index two as a generalized state space system, we use the projector

$$\Pi^T = I - M^{-T}G(G^TM^{-1}G)G^T,$$

defined in [3]. The projected ODE system is of the form

$$\begin{aligned} \mathcal{M}\dot{\tilde{\mathbf{z}}} &= \mathcal{A}\tilde{\mathbf{z}} + \mathcal{B}\mathbf{u},\\ \mathbf{y} &= \mathcal{C}\tilde{\mathbf{z}}, \end{aligned} \qquad (4)$$

with $\mathcal{M} = \mathcal{M}^T > 0$ and $\tilde{\mathbf{z}}(t) \in \mathbb{R}^{n_v - n_p}$.
To solve the algebraic Riccati equation associated to the system (4) we use a Newton-ADI-method. Instead of solving the projected dense Lyapunov equations in the innermost loop, we use [3, Lemma 5.2] and have to solve the saddle point system

$$\begin{bmatrix} A^T + \mu_i M^T & G\\ G^T & 0\end{bmatrix}\begin{bmatrix}\Lambda\\ *\end{bmatrix} = \begin{bmatrix} Y\\ 0\end{bmatrix}, \qquad (5)$$

for a couple of right hand sides Y and a different shift μ_i in each ADI step during each Newton step.

Contribution Details

Following [3] equation (4) is the semi discretized formulation of (2) including boundary data and projected to the manifold of divergence free discrete functions.

⇓

The pair $(\mathcal{A},\mathcal{E})$ then implements the semi discretized, projected spatial differential operator from (2).

⇓

For $i = 1$ and $X_0 = 0$ for every column in V_j equation (6), (or (5) respectively) corresponds to solving a modified stationary Stokes problem:

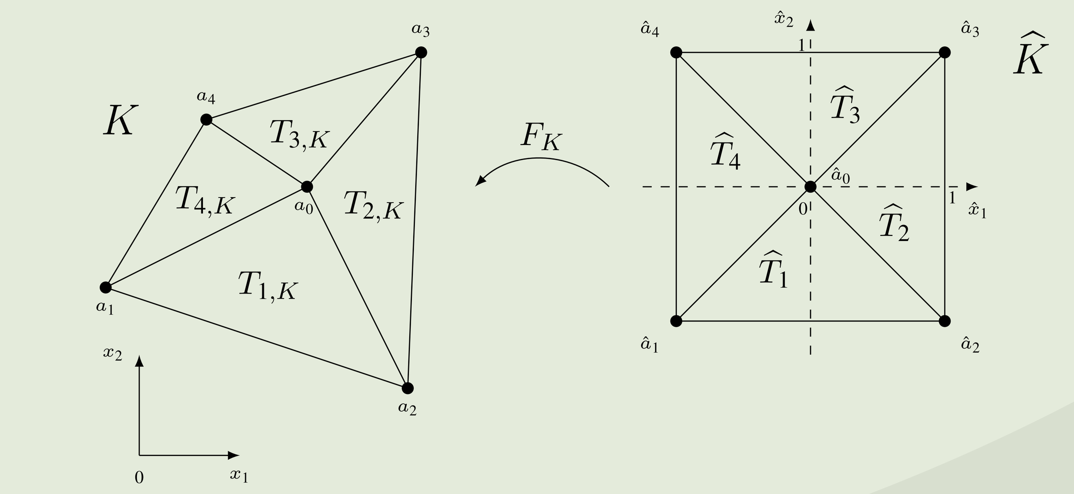
$$\begin{aligned} -\frac{1}{\mathrm{Re}}(\nabla\mathbf{v}_{j,k},\nabla\varphi) + (p,\operatorname{div}\varphi) + \mu_j(\mathbf{v}_{j,k},\varphi) &= (\mathbf{v}_{j-1,k},\varphi),\\ (\operatorname{div}\mathbf{v},\psi) &= 0, \end{aligned} \qquad (7)$$

for test functions $\varphi \in (H^1(\Omega))^2$ – respecting the boundary conditions – and $\psi \in L^2(\Omega)$, in the evaluation of the k-th column of (6)/(5).

Similarly applications of $\mathcal{A}$ and $\mathcal{M}$ can be pulled back to the weak formulation level.

Advantages:

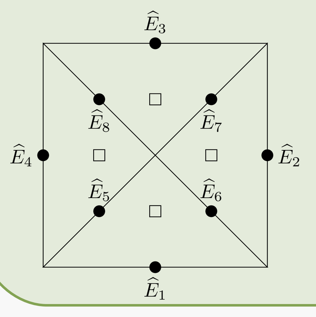
- (7) allows higher flexibility of formulation (e.g., adapting [5]),
- possibility to work matrix free,
- parallel implementations can exploit full FEM, PDE or domain features.



The composite cell $K = F_K(\widehat{K})$, where $F_K|_{\widehat{T}_i} \in [\mathbb{P}_1(\widehat{T}_i)]^2$.

Features of the composite non-conforming element [?]:

- inf-sup stable,
- low computational costs,
- pointwise mass-conservation within the son-triangles,
- L_2 orthogonal basis for velocity ⇒ diagonal mass matrix,
- after static condensation of interior dofs only $2\times4+1$ dofs per cell ⇒ produce a better stencil compared to the conforming case,
- optimal approximation order on general meshes,
- easy implementation.



Numbering of the edges of the son-triangles on the reference element and local degrees of freedom (dofs) of the composite $\mathbb{P}_1^{nc}(\widehat{K})$-element (marked by • for velocity and □ for pressure dofs)

Newton Kleinman Method
Approximate X solving:

$$\mathcal{C}^T\mathcal{C} + \mathcal{A}^TX\mathcal{M} + \mathcal{M}^TX\mathcal{A} - \mathcal{M}^TX\mathcal{B}\mathcal{B}^TX\mathcal{M} = 0$$

In step i solve the Lyapunov equation:

$$(\mathcal{A}^T - K_{i-1}^T\mathcal{B}^T)X_i\mathcal{M} + \mathcal{M}^TX_i(\mathcal{A} - \mathcal{B}K_{i-1}) = -G_{i-1}G_{i-1}^T,$$

where $K_{i-1} = \mathcal{B}^TX_{i-1}\mathcal{M}$ and $G_{i-1} = \left[\mathcal{C}^T,\ K_{i-1}\right]$.

Applying the low rank ADI algorithm requires to solve

$$(\mathcal{A}_i + \mu_j\mathcal{M})^TV_j = \mathcal{M}V_{j-1}, \qquad (6)$$

with $\mathcal{A}_i = \mathcal{A}^T - K_{i-1}^T\mathcal{B}^T$ for a possibly complex μ_j in each step.

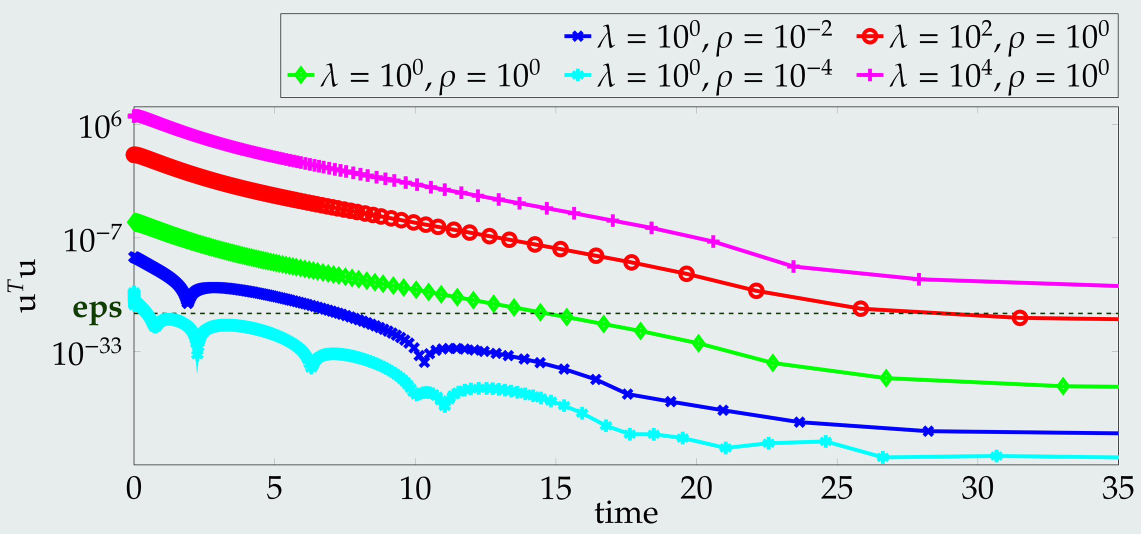
Solve (5) instead of (6) to increase efficiency. Requires:

- Sherman-Morrison-Woodbury formula,
- block preconditioning (e.g., [2]),
- investigation of required accuracies, i.e., inexact Newton-Kleinman-ADI.

New here:

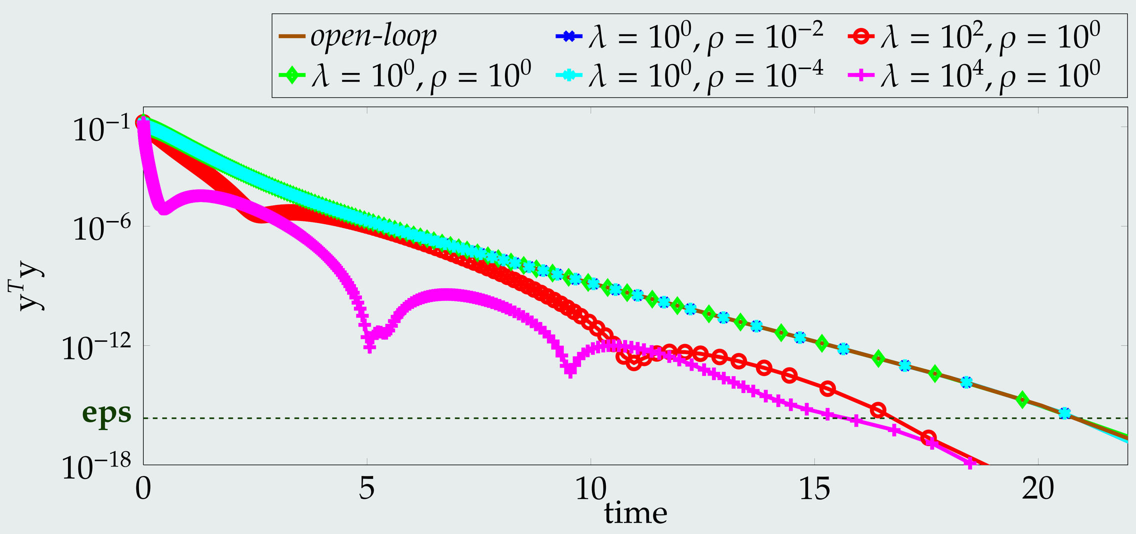
- Investigation of special finite elements that help ensuring "divergence free"-condition. for inexact solves.
- Interpretation of (5) in terms of the original PDE system.

Numerical Results



Evolution of
⇐ control
and
output ⇒
for the cost function

$$\mathcal{J}(y,u) = \frac{1}{2}\int_0^\infty \lambda\|y\|^2 + \frac{1}{\varrho}\|u\|^2 dt.$$



References

[1] E. Bänsch and P. Benner, *Stabilization of Incompressible Flow Problems by Riccati-Based Feedback*, in Constrained Optimization and Optimal Control for Partial Differential Equations, G. Leugering, S. Engell, A. Griewank, M. Hinze, R. Rannacher, V. Schulz, M. Ulbrich, and S. Ulbrich, eds., vol. 160 of International Series of Numerical Mathematics, Birkhäuser, 2012, pp. 5–20.

[2] P. Benner, J. Saak, M. Stoll, and H. K. Weichelt, *Efficient Solution of Large-Scale Saddle Point Systems Arising in Riccati-Based Boundary Feedback Stabilization of Incompressible Stokes Flow*, Preprint SPP1253-130, DFG-SPP1253, 2012. Submitted to SISC Copper Mountain Special Section 2012.

[3] M. Heinkenschloss, D. Sorensen, and K. Sun, *Balanced truncation model reduction for a class of descriptor systems with applications to the oseen equations*, SIAM J. Sci. Comput., 30 (2008), pp. 1038–1063.

[4] J.-P. Raymond, *Feedback boundary stabilization of the two-dimensional Navier-Stokes equations*, SIAM Journal on Control and Optimization, 45 (2006), pp. 790–828.

[5] T. Reis and W. Wollner, *Finite-Rank ADI Iteration for Operator Lyapunov Equations*, Preprint 2012-09, Hamburger Beiträge zur Angewandten Mathematik, 2012. Submitted to SICON.

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