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OPTIMIZATION WITH PARTIAL DIFFERENTIAL EQUATIONS
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Optimal control-based feedback stabilization of multi-field flow problems (Continuation)

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Objectives

- Derive and investigate numerical algorithms for optimal control-based boundary feedback stabilization of multi-field flow problems.
- Explore the potentials and limitations of feedback-based (Riccati) stabilization techniques.
- Employ recent advances in reducing complexity of Riccati solvers to achieve stabilization cost proportional to the simulation of the forward problem.
- Extend current methods for flow described by Navier-Stokes equations to flow problems coupled with other field equations of increasing complexity.

Context

- Method-oriented, numerical analysis and implementation of structure-exploiting algorithms.
- Riccati-based stabilization so far mostly used for smaller scales — here, new algorithms will allow application to discretized problems of complexity 10^5–10^7 ($\rightsquigarrow$ nonlinear systems of order 10^{10}–10^{15}).
- Optimal control for coupled PDEs (relevant model problems of exemplary character for several engineering applications).

Previous work by others

- Stabilization of flows (with velocity field v and pressure p), described by Navier-Stokes equations (NSE)

$$\partial_t v + v \cdot \nabla v - \frac{1}{Re}\Delta v + \nabla p = f \text{ in } \Omega \times (0,\infty)$$
$$\operatorname{div} v = 0,$$

  $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, to steady-state solutions.
- Existence of stabilizing feedback control proved in 2D [Fernández-Cara et al 2004] and 3D [Fursikov 2004].
- Construction based on associated lq optimal control problem/algebraic Riccati equation (ARE):
  - for distributed control, see [Barbu 2003, Barbu/Triggiani 2004, Barbu/Lasiecka/Triggiani 2007];
  - for boundary control, see [Raymond 2005, Barbu/Lasiecka/Triggiani 2006, Raymond 2007].

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Methodological Tasks

1. Derive linearized equation for controlled system. ☑
2. Set up associated linear-quadratic optimal control problem and associated operator Riccati equation. ☑
3. Find suitable spatial Galerkin FEM discretization of linearized control system and associated ARE. ☑
4. Derive algorithmic framework for solving the resulting descriptor ARE. ☑
5. Problem-dependent (structure exploiting) implementation of Newton-ADI requires efficient solvers for the associated stationary PDE/saddle-point problems. (Emphasis on efficient preconditioners.) ☐
6. Apply the resulting finite-dimensional feedback control to the original nonlinear PDE problems and verify its stabilization properties. ☐

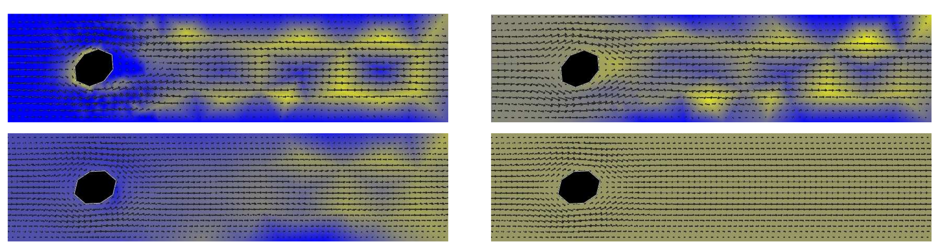
Extensive computational studies for each of the scenarios will be performed with the goal to obtain numerical evidence of the stabilization properties of the feedback law as well as of the convergence properties to the infinite-dimensional control.

Achievements

- Standard FE discretisation of linearized NSE using Taylor-Hood elements yields DAE system:

$$\begin{aligned} E_{11}\dot{z}_h(t) &= A_{11}z_h(t) + A_{12}p_h(t) + B_1u(t) \\ 0 &= A_{12}^T z_h(t) + B_2u(t), \end{aligned} \qquad + I.C. \quad (1)$$

  with $E_{11} \in \mathbb{R}^{n_v \times n_v}$ s.p.d., $A_{12}^T \in \mathbb{R}^{n_p \times n_v}$ of rank n_p. $z_h \in \mathbb{R}^{n_v}$ and $p_h \in \mathbb{R}^{n_p}$ are the states corresponding to velocities and pressure, and $u \in \mathbb{R}^{n_g}$ is the system input from the Dirichlet boundary control.
- ARE associated to (1) does in general not yield stabilizing feedback.
- Explicit Helmholtz projection yields ODE, which can be treated by ARE approach.
  Initial results for von Kármán vortex street at $Re = 500$, using approx. 5.000 degrees of freedom:



von Kármán vortex street, controlled, at $t = 1$ (top left), $t = 5$ (top right), $t = 8$ (bottom left), and $t = 10$ (bottom right).

But: too expensive in general! (non-sparse system)

- Newton for projected system requires solution of Lyapunov equation

$$A_j^T Z_{j+1} Z_{j+1}^T P_h E_{11} P_h^T + P_h E_{11} P_h^T Z_{j+1} Z_{j+1}^T A_j = -W_j W_j^T,$$

  where
$$\begin{aligned} P_h &:= I_{n_v} - A_{12}(A_{12}^T E_{11}^{-1} A_{12})^{-1}, \\ A_j &:= P_h(A_{11} - B_1 B_1^T P_h Z_j Z_j^T P_h E_{11}) P_h, \\ W_j &:= \left[\, P_h C^T \;\; P_h E_{11} P_h Z_j Z_j^T P_h B_1 \,\right]. \end{aligned}$$

- Explicit projection can be avoided using ADI-version for descriptor Lyapunov equations: solve sequence of saddle point problems involving only data from (1) [Heinkenschloss et al 2008]:
  $X_{j+1} \approx Z_{j+1} Z_{j+1}^T$ can be computed as

$$Z_{j+1} = \sqrt{\mu}\left[B_{j,\mu},\ A_{j,\mu}B_{j,\mu},\ A_{j,\mu}^2 B_{j,\mu},\ \ldots,\ A_{j,\mu}^j B_{j,\mu}\right],$$

  where $B_{j,\mu}$ solves the saddle point problem

$$\begin{bmatrix} E_{11} + \mu(A_{11} - B_1 B_1^T Z_j Z_j^T E_{11}) & A_{12} \\ A_{12}^T & 0 \end{bmatrix} \begin{bmatrix} B_{j,\mu} \\ * \end{bmatrix} = \begin{bmatrix} C^T & E_{11} Z_j Z_j^T B_1 \\ 0 & 0 \end{bmatrix}.$$

  and $A_{j,\mu}$ is obtained analogously.
- New methods to compute Z_0 [Benner 2008], Z_1.

References

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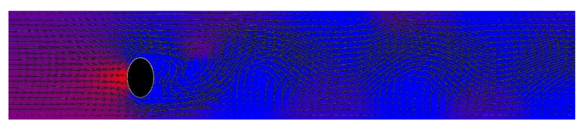
Application Tasks

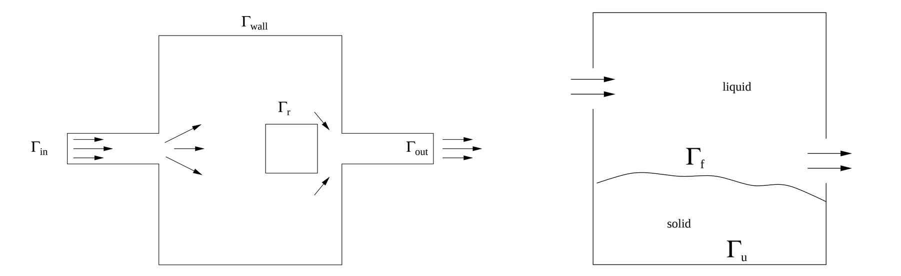
0. NSE with normal boundary control.



Setup: Flow past cylinder/von Kármán vortex street.
Aim: Suppress vortex shedding.
Control: Blowing and suction at cylinder.
$\rightsquigarrow$ benchmark problem to test algorithms.

1. NSE with passive transport of a reactive species.

Setup: Reactive species is transported from Γ_{in} to reacting surface Γ_r.
Aim: Stabilize and control the reaction.
Control: Inflow velocity distribution v at Γ_{in}.
$\rightsquigarrow$ NSE coupled with linear convection-diffusion equation.

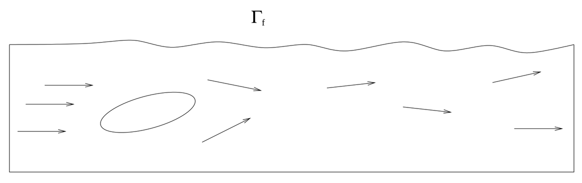


Settings for Examples 1 (left) and 2 (right).

2. Phase transition liquid/solid with convection.

Setup: Hot melt that solidifies while flowing through a mould.
Aim: Control the phase boundary Γ_f between the liquid and solid parts.
Control: Temperature distribution on parts of boundary.
$\rightsquigarrow$ NSE coupled with Stefan problem for free boundary.

3. Stabilization of a flow with a free capillary surface.



Setup: The flow past the obstacle is oscillatory due to Karman vortex shedding.
Aim: Stabilize the capillary free surfaces Γ_f.
Control: Inflow velocity distribution.
$\rightsquigarrow$ NSE coupled with highly nonlinear (boundary) curvature operator.

4. Control for electrically conducting fluids in presence of outer magnetic fields.

Setup: Hartmann flow (model problem for crystal growth/metallurgic processing) subject to outer magnetic field. Induced secondary magnetic field in streamwise direction can lead to instabilities in the flow and to transition from laminar to turbulent.
Aim: Stabilize flow and delay transition to oscillatory or turbulent character.
Control: Either blowing/suction of flow field or magnetic control.
$\rightsquigarrow$ MHD equations/NSE coupled with Maxwell's eqns.

Envisioned cooperation

Within SPP1253:

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