MAX-PLANCK-GESELLSCHAFT

Peter Benner Matthias Voigt

Spectral Characterization and Enforcement of Negative Imaginariness for Descriptor Systems

MAX–PLANCK–INSTITUT
FÜR DYNAMIK KOMPLEXER
TECHNISCHER SYSTEME
MAGDEBURG

Max Planck Institute Magdeburg Preprints

MPIMD/12-09 April 12, 2012

Impressum:

Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg

Publisher:
Max Planck Institute for Dynamics of Complex Technical Systems

Address:
Max Planck Institute for Dynamics of Complex Technical Systems
Sandtorstr. 1
39106 Magdeburg

www.mpi-magdeburg.mpg.de/preprints

Spectral Characterization and Enforcement of Negative Imaginariness for Descriptor Systems

Peter Benner[a], Matthias Voigt[a,*]

[a] *Max Planck Institute for Dynamics of Complex Technical Systems, Sandtorstr. 1, 39106 Magdeburg, Germany*

Abstract

Systems with counterclockwise input-output dynamics (or negative imaginary transfer functions) arise in various applications such as the modeling of flexible mechanical structures or electrical circuits where certain kinds of measurements are taken. In this paper we introduce descriptor systems with such an additional structure. We state various of their properties and prove algebraic characterizations of negative imaginariness in terms of spectral conditions of certain structured matrix pencils. For this purpose we also analyze particular boundary cases which are characterized by properties of a structured Kronecker canonical form. Finally, we describe a method which can be used to restore the negative imaginary property in case that this is lost. This happens, e.g., when a system with theoretically negative imaginary transfer function is obtained by, e.g., model order reduction methods, linearization, or other approximations. The method is illustrated by numerical examples.

Keywords: Descriptor system, even matrix pencil, Kronecker canonical form, negative imaginariness, skew-Hamiltonian/Hamiltonian matrix pencil, structure enforcement
2010 MSC: 93B99, 65L80, 15A22, 15A23

1. Introduction and Preliminaries

Dynamical systems with additional structures such as passivity or contractivity play a great role in the modeling and analysis of, e.g., flexible mechanical structures or electrical circuits. They have found great interest in the literature such as [1, 16, 33]. A less known property of dynamical systems is counterclockwise input-output dynamics [2] (or equivalently a negative imaginary frequency response). This property often occurs in models corresponding to mechanical systems and electrical circuits provided that certain measurements are taken. For instance, mechanical structures with collocated force actuators and *position sensors* yield such systems [31]. Another application area where this theory is of practical importance is electrical networks. Suppose that each voltage source is connected in series with a capacitor and that the corresponding system output is the voltage across this capacitor divided by the capacitance. Also, suppose that each current source is connected in parallel with an inductor and that the corresponding system output is the inductor current divided by the inductance. It can be shown that in this situation the dynamical system has a counterclockwise input-output dynamics [31]. The goal of this paper is to generalize the negative imaginary theory [31, 44] to descriptor systems. Roughly speaking, a descriptor system is a dynamical system with constrained dynamics. In case of a mechanical system, we can constrain the dynamics by, e.g., interconnecting masses by rigid bars. Then, at least two masses cannot move independently from each other. In this case, the dynamics is said to be holonomically constrained. In case of electrical circuits, descriptor systems occur when resistors are present. By Ohm's law, the voltage and the current of a resistor fulfill an algebraic relation which prevents the network from attaining all possible values of voltages and currents at each component [43].

*Corresponding author, phone: +49 391 6110 464, fax: +49 391 6110 500
Email addresses: benner@mpi-magdeburg.mpg.de (Peter Benner), voigtm@mpi-magdeburg.mpg.de (Matthias Voigt)

Preprint submitted to Elsevier *April 12, 2012*

Linear models of the above types can be described as continuous-time linear time-invariant *descriptor systems* of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where E, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, $x(t) \in \mathbb{R}^n$ is the descriptor vector, $u(t) \in \mathbb{R}^m$ is the input vector, and $y(t) \in \mathbb{R}^m$ is the output vector. Here, E usually is a *singular* matrix. A common approach to display the relation between inputs and outputs of the system (1) is to work in the frequency domain. By Laplace transforming both equations of (1), subsequently inserting the one equation into the other one and assuming $Ex(0) = 0$, we obtain the *transfer function*

$$G(s) := C\,(sE - A)^{-1}\,B + D \tag{2}$$

of the descriptor system. The transfer function is often evaluated at purely imaginary values $\mathrm{i}\omega$. Then ω can be interpreted as a frequency (scaled by $1/2\pi$). In the following, we assume that the matrix pencil $\lambda E - A \in \mathbb{R}[\lambda]^{n\times n}$ is *regular*, i.e., $\det(\lambda E - A) \not\equiv 0$. Here, $\mathbb{K}[\lambda]^{p\times q}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ denotes the set of all polynomials with coefficients in $\mathbb{K}^{p\times q}$. A popular tool for the analysis of such systems is the *Weierstraß canonical form* [39], i.e., for every regular matrix pencil $\lambda E - A \in \mathbb{R}[\lambda]^{n\times n}$, there exist nonsingular matrices T, $W \in \mathbb{C}^{n\times n}$ such that

$$\lambda E - A = W\left(\lambda\begin{bmatrix} I_{n_\mathrm{f}} & 0 \\ 0 & N\end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty}\end{bmatrix}\right)T, \tag{3}$$

where J and N are in Jordan canonical form and N is nilpotent with index of nilpotency ν. The numbers n_f and n_∞ are the dimensions of the deflating subspaces of $\lambda E - A$ corresponding to the finite and infinite eigenvalues, respectively. A descriptor system is *(asymptotically) stable* if all finite eigenvalues of $\lambda E - A$ lie in the open left half-plane. By using the Weierstraß canonical form (3) and setting $B = W^{-1}\begin{bmatrix} B_1 \\ B_2\end{bmatrix}$, $C = \begin{bmatrix} C_1 & C_2\end{bmatrix}T^{-1}$, we realize a restricted equivalence transform of the system (1). Then we can decompose the transfer function (2) as

$$G(s) = \underbrace{C_1\,(sI_{n_\mathrm{f}} - J)^{-1}\,B_1}_{=:G_\mathrm{sp}(s)} + \underbrace{(D - C_2B_2)}_{=:M_0} \underbrace{- \sum_{k=1}^{\nu-1} C_2N^kB_2s^k}_{=:G_\mathrm{i}(s)}. \tag{4}$$

By $G_\mathrm{sp}(s)$ we denote the *strictly proper part* of the system, i.e., $\lim\limits_{\omega\to\infty} \|G_\mathrm{sp}(\mathrm{i}\omega)\| = 0$, where $\|\cdot\|$ denotes an arbitrary matrix norm. The *proper part* $G_\mathrm{p}(s) := G_\mathrm{sp}(s) + M_0$ fulfills $\lim\limits_{\omega\to\infty} \|G_\mathrm{p}(\mathrm{i}\omega)\| < \infty$. Finally, by $G_\mathrm{i}(s)$ we denote the *improper part*, i.e., $\lim\limits_{\omega\to\infty} \|G_\mathrm{i}(\mathrm{i}\omega)\| = \infty$. According to the above definitions, we call the transfer function $G(s)$ *strictly proper* if $M_0 = 0$ and $G_\mathrm{i}(s) \equiv 0$, *proper* if $G_\mathrm{i}(s) \equiv 0$, and *improper* otherwise. Furthermore, we denote the Banach space of all real-rational proper and stable $m \times m$-matrix valued functions by $\mathcal{RH}_\infty^{m\times m}$.

Finally, we need some concepts for controllability and observability [13, 33]. A descriptor system (1) is called *R-controllable* if $\operatorname{rank}\begin{bmatrix}\lambda E - A & B\end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$, and *R-observable* if $\operatorname{rank}\begin{bmatrix}\lambda E^T - A^T & C^T\end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$. These properties are analoguous to the usual controllability and observability concepts for standard state space systems. Note that other controllability and observability concepts for descriptor systems exist [12, 34, 42] but are not needed in this context.

The remainder of this article is structured as follows. In Section 2 we introduce negative imaginariness for descriptor systems and provide some of its properties. In Section 3 we derive the spectral characterizations of structured matrix pencils for negative imaginariness. In Section 4 we suggest an algorithm which can be used to restore the negative imaginary property of a system if it has been lost by approximating the system by, e.g., reducing the model order. Finally, in Section 5 we summarize this paper and point towards further possible research directions. Both, spectral characterizations and an enforcement procedure have already been analyzed in [28, 29] for standard state space systems. In this paper we give a generalizations of these concepts for descriptor systems by employing slightly different matrix structures and techniques.

2. Systems with Counterclockwise Input-Output Dynamics and Negative Imaginary Transfer Functions

First, we introduce the notion of a system with counterclockwise input-output dynamics and adapt the definition to systems of type (1) [2]. Here, $\mathcal{L}^2_{\text{loc}}(\mathcal{I},\mathcal{X})$ denotes the spaces of locally measurable and square integrable functions that map from the interval $\mathcal{I} \subset \mathbb{R}$ to the set $\mathcal{X}$. Furthermore we define the space $\mathcal{H}^1_{\text{loc}}(\mathcal{I},\mathcal{X}) := \left\{ f \in \mathcal{L}^2_{\text{loc}}(\mathcal{I},\mathcal{X}) : \dot{f} \in \mathcal{L}^2_{\text{loc}}(\mathcal{I},\mathcal{X}) \right\}$.

Definition 1. A descriptor system (1) has a *counterclockwise input-output dynamics* if

$$\liminf_{t\to\infty} \int_0^t \dot{y}(\tau)^T u(\tau)\mathrm{d}\tau > -\infty \tag{5}$$

for all $u \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}^+,\mathbb{R}^m)$ that are consistent with $Ex(0) = Ex_0$ and such that $y \in \mathcal{H}^1_{\text{loc}}(\mathbb{R}^+,\mathbb{R}^m)$.

A counterclockwise input-output dynamics is closely related to *passivity* of a system, that is

$$\liminf_{t\to\infty} \int_0^t y(\tau)^T u(\tau)\mathrm{d}\tau > -\infty$$

for all $u \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}^+,\mathbb{R}^m)$ that are consistent with $Ex(0) = Ex_0$ [27]. Often, for LTI systems one can also find the following definition. The descriptor system (1) is called passive if

$$\int_0^t y(\tau)^T u(\tau)\mathrm{d}\tau \geq 0$$

holds for all $u \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}^+,\mathbb{R}^m)$ that are consistent with $Ex(0) = 0$ and all $t \geq 0$. Note that a similarly fashioned definition for counterclockwise input-output dynamics of LTI systems cannot be given.

Roughly speaking, a counterclockwise input-output dynamics can be interpreted as passivity with respect to the derivative of the output (instead of the output itself). Mathematically there is the following relation.

Lemma 1. *Consider a descriptor system* (1) *with strictly proper transfer function* $G(s)$. *Assume furthermore that* $G(s)$ *has an equivalent state-space realization*

$$\begin{aligned}\dot{x}(t) &= Jx(t) + B_1 u(t),\\ y(t) &= C_1 x(t).\end{aligned}$$

Then (1) *has counterclockwise input-output dynamics if and only if the system*

$$\begin{aligned}\dot{x}(t) &= Jx(t) + B_1 u(t),\\ \tilde{y}(t) &= C_1 J x(t) + C_1 B_1 u(t)\end{aligned} \tag{6}$$

is passive.

Proof. Apply [2, Proposition III.3] to an LTI system. □

For the case of a system with (non-strictly) proper transfer function and $M_0 = M_0^T \succcurlyeq 0$, the passivity of (6) also implies that (1) has counterclockwise input-output dynamics. Furthermore it turns out that for stable systems (1) with proper transfer function, a counterclockwise input-output dynamics is the same as "negative imaginary frequency response" or a negative imaginary transfer function [9]. Now, we define negative imaginariness of a stable and proper transfer function. For convenience, we will call systems with counterclockwise input-output dynamics negative imaginary as this is directly related to the corresponding transfer functions.

Definition 2. A transfer function matrix $G \in \mathcal{RH}_\infty^{m\times m}$ is *negative imaginary* if $\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right) \succcurlyeq 0$ for all $\omega \geq 0$. Furthermore, it is called *strictly negative imaginary* if $\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right) \succ 0$ for all $\omega > 0$.

As for counterclockwise input-output dynamics and passivity there exists a relation between negative imaginariness and positive realness of transfer functions. We briefly define positive realness and also give some equivalent conditions for positive realness and negative imaginariness of particular transfer functions.

Definition 3. A square transfer function matrix $G(s)$ is called *positive real* if

(1) $G(s)$ has no poles in $\mathbb{C}^+ := \{s \in \mathbb{C} : \mathrm{Re}\,(s) > 0\}$,

(2) $G(\bar{s}) = \overline{G(s)}$ for all $s \in \mathbb{C}^+$,

(3) $G(s) + G^H(s) \succcurlyeq 0$ for all $s \in \mathbb{C}^+$.

For real-rational transfer functions there exist the following equivalent conditions.

Lemma 2. *[1] A square real-rational transfer function matrix $G(s)$ is positive real if and only if*

(1) $G(s)$ has no poles in $\mathbb{C}^+$,

(2) $G(\mathrm{i}\omega) + G^H(\mathrm{i}\omega) \succcurlyeq 0$ for all $\omega \in \mathbb{R}$ except values of ω where $\mathrm{i}\omega$ is a pole of $G(s)$,

(3) if $\mathrm{i}\omega_0$ is a pole of $G(s)$, it is at most a simple pole and the residue matrix $R_0 := \lim_{s \to \mathrm{i}\omega_0} (s - \mathrm{i}\omega_0)G(s)$ in case ω_0 is finite, and $R_\infty = \lim_{\omega \to \infty} (G(\mathrm{i}\omega)/(\mathrm{i}\omega))$ in case ω_0 is infinite, is positive semidefinite Hermitian.

Lemma 3. *[44, Lemma 3, Lemma 4]*

(a) A square real-rational strictly proper stable transfer function matrix $G(s)$ is negative imaginary if and only if $sG(s)$ is positive real.

(b) A square real-rational proper stable transfer function matrix $G(s)$ is negative imaginary if and only if $M_0 = M_0^T$ and $s(G(s) - M_0)$ is positive real.

The properties above can be related to (5) as the derivative in the output generates an additional factor s when taking Laplace transforms. Next, we show an important property of the function $\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right)$.

Lemma 4. *Let a real $G \in \mathcal{RH}_\infty^{m \times m}$ be given. Then $\Lambda\left(\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right)\right) = \Lambda\left(-\mathrm{i}\left(G(-\mathrm{i}\omega) - G^H(-\mathrm{i}\omega)\right)\right)$ for all $\omega \in \mathbb{R}$.*

Proof. First note that $\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right) = -\mathrm{i}\left(G^H(\mathrm{i}\omega) - G(\mathrm{i}\omega)\right)$ which means that $\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right)$ is Hermitian and thus has a purely real spectrum for all real values of ω. Thus we can conclude that

$$\begin{aligned}\Lambda\left(\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right)\right) &= \Lambda\left(\left(\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right)\right)^T\right) \\ &= \Lambda\left(\mathrm{i}\left(G^H(-\mathrm{i}\omega) - G(-\mathrm{i}\omega)\right)\right) \\ &= \Lambda\left(-\mathrm{i}\left(G(-\mathrm{i}\omega) - G^H(-\mathrm{i}\omega)\right)\right).\end{aligned}$$

$\square$

Following from Lemma 4, the eigenvalue curves of the matrix-valued function $\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right)$ are symmetric with respect to the origin.

3. Spectral Characterizations for Negative Imaginariness

In this section we derive algebraic characterizations for negative imaginariness of transfer functions in terms of spectral conditions of certain structured matrix pencils. We formulate these conditions by using the given descriptor system realization $(\lambda E - A, B, C, D)$ without additively decomposing the transfer function as in (4). This has some advantages for computational considerations as computing the decomposition (4) might be an ill-conditioned problem and thus should be avoided if possible. For this purpose we introduce the matrix and pencil structures that we will need in the following [5, 35]. Consider a matrix pencil $\lambda\mathcal{N} - \mathcal{M} \in \mathbb{C}[\lambda]^{n\times n}$. Such a matrix pencil is called even if $\mathcal{N}$ is skew-Hermitian and $\mathcal{M}$ is Hermitian. It is called odd if $\mathcal{N}$ is Hermitian and $\mathcal{M}$ is skew-Hermitian. Now assume that n is an even number, i.e., $n = 2m$. Define the skew-symmetric matrix $\mathcal{J} := \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}$. The matrix pencil $\lambda\mathcal{N} - \mathcal{M}$ is called skew-Hamiltonian/Hamiltonian if $\mathcal{N}$ is skew-Hamiltonian (i.e., $(\mathcal{N}\mathcal{J})^H = -\mathcal{N}\mathcal{J}$) and $\mathcal{M}$ is Hamiltonian (i.e., $(\mathcal{M}\mathcal{J})^H = \mathcal{M}\mathcal{J}$). Similarly, it is called Hamiltonian/skew-Hamiltonian if $\mathcal{N}$ is Hamiltonian and $\mathcal{M}$ is skew-Hamiltonian. Pencils of this structure have many interesting properties. Maybe the most important one is that all these pencils have a spectrum with Hamiltonian eigensymmetry, that is, if λ is an eigenvalue, so is also $-\bar{\lambda}$.

By using the matrix pencils with the above structures we can first formulate and prove the following result.

Theorem 1. *Let $G \in \mathcal{RH}_\infty^{m\times m}$. Then, $\mathrm{i}\left(G(\mathrm{i}\omega_0) - G^H(\mathrm{i}\omega_0)\right)$ is singular if and only if the even matrix pencil*

$$\lambda\mathcal{N} - \mathcal{M} := \lambda \begin{bmatrix} 0 & \mathrm{i}E & 0 \\ \mathrm{i}E^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \mathrm{i}A & \mathrm{i}B \\ -\mathrm{i}A^T & 0 & -\mathrm{i}C^T \\ -\mathrm{i}B^T & \mathrm{i}C & \mathrm{i}(D - D^T) \end{bmatrix} \tag{7}$$

has the eigenvalue $\mathrm{i}\omega_0$.

Proof. Let $\mathrm{i}\left(G(\mathrm{i}\omega_0) - G^H(\mathrm{i}\omega_0)\right)$ be a singular matrix. Then,

$$\begin{aligned}
&\mathrm{i}\left(G(\mathrm{i}\omega_0) - G^H(\mathrm{i}\omega_0)\right) \\
&= \mathrm{i}C\left(\mathrm{i}\omega_0(\mathrm{i}E) - \mathrm{i}A\right)^{-1}\mathrm{i}B + \mathrm{i}D - \mathrm{i}B^T\left(\mathrm{i}\omega_0\left(\mathrm{i}E^T\right) - \left(-\mathrm{i}A^T\right)\right)^{-1}\left(-\mathrm{i}C^T\right) - \mathrm{i}D^T \\
&= \begin{bmatrix} \mathrm{i}C & -\mathrm{i}B^T \end{bmatrix} \begin{bmatrix} \mathrm{i}\omega_0(\mathrm{i}E) - \mathrm{i}A & 0 \\ 0 & \mathrm{i}\omega_0\left(\mathrm{i}E^T\right) - \left(-\mathrm{i}A^T\right) \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{i}B \\ -\mathrm{i}C^T \end{bmatrix} + \mathrm{i}\left(D - D^T\right) \\
&=: \mathrm{i}\omega_0\hat{\mathcal{N}} - \hat{\mathcal{M}}
\end{aligned} \tag{8}$$

is singular. In other words, the matrix pencil $\lambda\hat{\mathcal{N}} - \hat{\mathcal{M}}$ has the eigenvalue $\mathrm{i}\omega_0$. Now we analyze $\lambda\hat{\mathcal{N}} - \hat{\mathcal{M}}$ in more detail. We can exploit the Schur complement structure of $\lambda\hat{\mathcal{N}} - \hat{\mathcal{M}}$ and extend this matrix pencil to

$$\lambda\tilde{\mathcal{N}} - \tilde{\mathcal{M}} := \begin{bmatrix} \lambda(\mathrm{i}E) - \mathrm{i}A & 0 & \mathrm{i}B \\ 0 & \lambda\left(\mathrm{i}E^T\right) - (-\mathrm{i}A^T) & -\mathrm{i}C^T \\ -\mathrm{i}C & \mathrm{i}B^T & \mathrm{i}(D - D^T) \end{bmatrix},$$

which has the same *finite* eigenvalues as $\lambda\hat{\mathcal{N}} - \hat{\mathcal{M}}$ [43]. By performing some simple equivalence transformations we obtain the matrix pencil $\lambda\mathcal{N} - \mathcal{M}$ as in (7). The converse direction can be proven easily. Assume $\lambda\mathcal{N} - \mathcal{M}$ has the eigenvalue $\mathrm{i}\omega_0$. Then $\mathrm{i}\omega_0\hat{\mathcal{N}} - \hat{\mathcal{M}}$ is singular. Then, by (8) also $\mathrm{i}\left(G(\mathrm{i}\omega_0) - G^H(\mathrm{i}\omega_0)\right)$ is singular. $\square$

From Theorem 1 we can easily conclude that $G \in \mathcal{RH}_\infty^{m\times m}$ is strictly negative imaginary if and only if $M_0 = M_0^T$, there exists an $\omega_0 > 0$ such that $\mathrm{i}\left(G(\mathrm{i}\omega_0) - G^H(\mathrm{i}\omega_0)\right) \succ 0$, and the corresponding even matrix pencil $\lambda\mathcal{N} - \mathcal{M}$ has no nonzero, finite, purely imaginary eigenvalues. Graphically, this means that the eigenvalue curves of $\mathrm{i}\left(G(\mathrm{i}\omega_0) - G^H(\mathrm{i}\omega_0)\right)$ lie all above the zero level in the positive frequency range.

However, there is the boundary case of eigenvalue curves that touch the zero level (and hence generate purely imaginary eigenvalues in $\lambda\mathcal{N} - \mathcal{M}$) but do not cross it. A graphical interpretation of different situation is given in Figure 1. It can be seen that there are many different cases that have to be considered. We will show later how we can treat all these in a uniform way.

To analyze this in more detail we need more sophisticated tools from linear algebra which are briefly summarized in the following. To formulate our results we need some canonical forms of matrix pencils. The *Kronecker canonical form* is a generalization of the Weierstraß canonical form (3) to singular or nonsquare matrix pencils. By $A \oplus B = \operatorname{diag}(A, B)$ we denote the direct sum of two matrices.

Proposition 1. *[25, 32] For every matrix pencil $\lambda E - A \in \mathbb{C}[\lambda]^{n\times m}$ there exist nonsingular matrices $P \in \mathbb{C}^{n\times n}$ and $Q \in \mathbb{C}^{m\times m}$ such that*

$$P(\lambda E - A)Q = \operatorname{diag}(C_1(\lambda), C_2(\lambda), C_3(\lambda), C_4(\lambda)),$$

where

$$C_1(\lambda) = \bigoplus_{j=1}^{k_1} \left(\lambda \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{\rho_j\times\rho_j} - \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}_{\rho_j\times\rho_j} \right),$$

$$C_2(\lambda) = \bigoplus_{j=1}^{k_2} \left(\lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}_{\sigma_j\times\sigma_j} - \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{\sigma_j\times\sigma_j} \right),$$

$$C_3(\lambda) = \bigoplus_{j=1}^{k_3} \left(\lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}_{\varepsilon_j\times(\varepsilon_j+1)} - \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}_{\varepsilon_j\times(\varepsilon_j+1)} \right),$$

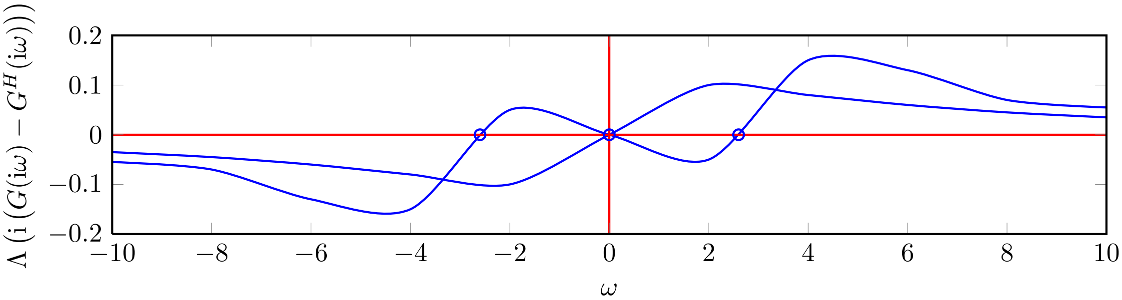
$$C_4(\lambda) = \bigoplus_{j=1}^{k_4} \left(\lambda \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}_{(\delta_j+1)\times\delta_j} - \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix}_{(\delta_j+1)\times\delta_j} \right).$$

This decomposition is unique up to permutations of the blocks.

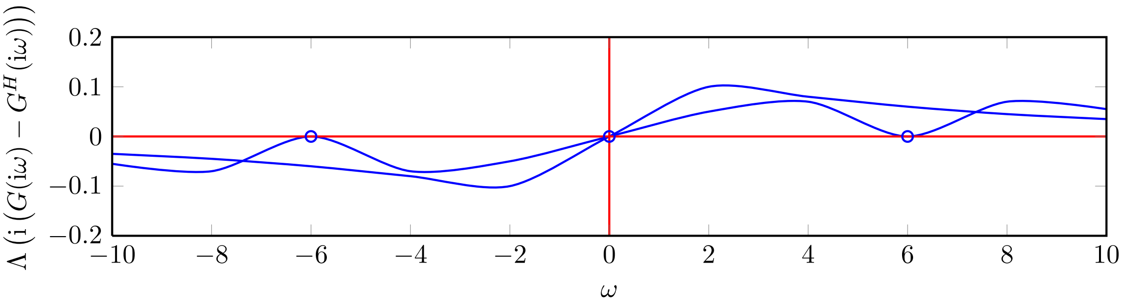
For square matrices. the blocks $C_1(\lambda)$ and $C_2(\lambda)$ correspond to the finite and infinite eigenvalues, respectively. Both form the regular structure of the pencil. The blocks $C_3(\lambda)$ and $C_4(\lambda)$ correspond to the singular structure. However, in case of a structured matrix pencil, the transformation to Kronecker canonical form generally does not preserve the structure. Fortunately, for even matrix pencils, there exists a structured Kronecker-like canonical form which we call *even Kronecker canonical form*, see the following proposition.

Proposition 2. *[32, 41] For every even matrix pencil $\lambda N - M \in \mathbb{C}[\lambda]^{n\times n}$ there exists a nonsingular matrix $U \in \mathbb{C}^{n\times n}$ such that*

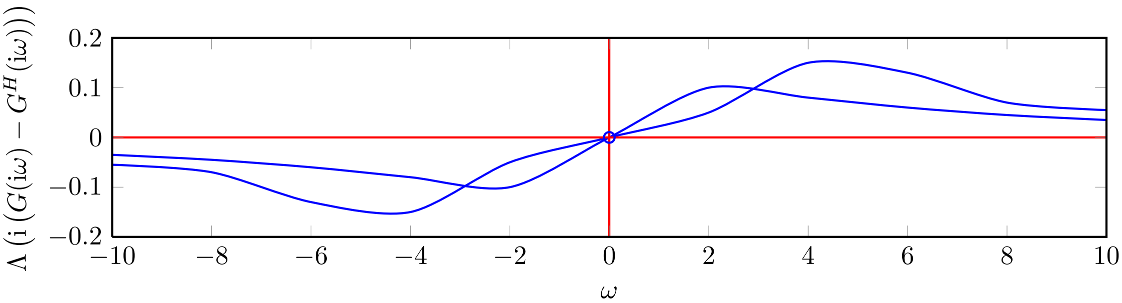
$$U^H(\lambda N - M)U = \operatorname{diag}(D_1(\lambda), D_2(\lambda), D_3(\lambda), D_4(\lambda)),$$



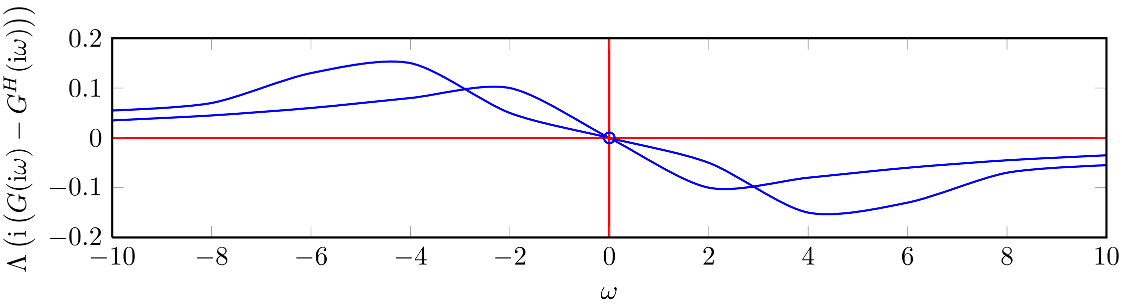
(a) Eigenvalue curve is crossing the zero level at nonzero frequency points — $\lambda\mathcal{N} - \mathcal{M}$ has two nonzero, purely imaginary eigenvalues — $G(s)$ is not negative imaginary.



(b) Eigenvalue curve is touching the zero level from above in the positive frequency range — $\lambda\mathcal{N} - \mathcal{M}$ has two *double* nonzero, purely imaginary eigenvalues — $G(s)$ is negative imaginary.



(c) Eigenvalue curves lie all above the zero level in the positive frequency range and are not touching it — $\lambda\mathcal{N} - \mathcal{M}$ has no finite, nonzero, purely imaginary eigenvalues — $G(s)$ is negative imaginary.



(d) Eigenvalue curves lie all below the zero level in the positive frequency range and are not touching it — $\lambda\mathcal{N} - \mathcal{M}$ has no finite, nonzero, purely imaginary eigenvalues — $G(s)$ is not negative imaginary.

Figure 1: Graphical interpretation of some possible situations

where

$$
D_1(\lambda) = \bigoplus_{j=1}^{k_1} \left(\left[\begin{array}{cccc|cccc}
 & & & & -\lambda+\mu & -1 & & \\
 & & & & & \ddots & \ddots & \\
 & & & & & & \ddots & -1 \\
 & & & & & & & -\lambda+\mu \\ \hline
\lambda+\bar{\mu} & & & & & & & \\
-1 & \ddots & & & & & & \\
 & \ddots & \ddots & & & & & \\
 & & -1 & \lambda+\bar{\mu} & & & &
\end{array} \right]_{2\rho_j \times 2\rho_j} \right), \quad \mu \in \mathbb{C}^+,
$$

$$
D_2(\lambda) = \bigoplus_{j=1}^{k_2} \left(\lambda s_j \begin{bmatrix} & & -\mathrm{i} \\ & \iddots & \\ -\mathrm{i} & & \end{bmatrix}_{\sigma_j \times \sigma_j} - s_j \begin{bmatrix} & & -1 & \mu \\ & \iddots & \iddots & \\ -1 & \iddots & & \\ \mu & & & \end{bmatrix}_{\sigma_j \times \sigma_j} \right), \quad \mu \in \mathbb{R},
$$

$$
D_3(\lambda) = \bigoplus_{j=1}^{k_3} \left(\lambda t_j \begin{bmatrix} & & \mathrm{i} & 0 \\ & \iddots & \iddots & \\ \mathrm{i} & \iddots & & \\ 0 & & & \end{bmatrix}_{\varepsilon_j \times \varepsilon_j} - t_j \begin{bmatrix} & & -1 \\ & \iddots & \\ -1 & & \end{bmatrix}_{\varepsilon_j \times \varepsilon_j} \right),
$$

$$
D_4(\lambda) = \bigoplus_{j=1}^{k_4} \left(\left[\begin{array}{ccc|cccc}
 & & & 1 & -\lambda & & \\
 & & & & \ddots & \ddots & \\
 & & & & & 1 & -\lambda \\ \hline
1 & & & & & & \\
\lambda & \ddots & & & & & \\
 & \ddots & 1 & & & & \\
 & & \lambda & & & &
\end{array} \right]_{(2\delta_j+1)\times(2\delta_j+1)} \right),
$$

and s_j, $t_j \in \{-1, 1\}$ are called the signatures of the corresponding blocks. This decomposition is unique up to permutation of the blocks.

The blocks in $D_1(\lambda)$ correspond to pairs $(\mu, -\bar{\mu})$ of eigenvalues where $\mu \notin \mathrm{i}\mathbb{R}$. The blocks in $D_2(\lambda)$ and $D_3(\lambda)$ correspond to the finite, purely imaginary eigenvalues and the infinite eigenvalues, respectively. The blocks in $D_4(\lambda)$ reflect the singular structure of $\lambda N - M$. In the following we present some statements about the inertia of the blocks in the even Kronecker canonical form. Recall, that the inertia of a Hermitian matrix A is denoted by $\mathrm{In}(A) = (\pi_+, \pi_0, \pi_-)$, where π_+, π_0 and π_- are the numbers of positive, zero and negative eigenvalues, respectively.

Proposition 3. *[10, 11, 32] Let an even matrix pencil $\lambda N - M$ be given in even Kronecker canonical form as in Lemma 2 and let $\mathcal{D}_j(\lambda)$ be the jth blocks from either $D_1(\lambda)$, $D_2(\lambda)$, $D_3(\lambda)$, or $D_4(\lambda)$. Then the following is satisfied.*

(1) If $\mathcal{D}_j(\lambda)$ is from $D_1(\lambda)$, then

$$
\mathrm{In}(\mathcal{D}_j(\mathrm{i}\omega)) = (\rho_j, 0, \rho_j) \quad \textit{for all} \quad \omega \in \mathbb{R}.
$$

(2) If $\mathcal{D}_j(\lambda)$ is from $D_2(\lambda)$ and σ_j is even, then

$$\mathrm{In}(\mathcal{D}_j(\mathrm{i}\omega)) = \begin{cases} (\sigma_j/2, 0, \sigma_j/2), & \text{if} \quad \mu \neq \omega, \\ (\sigma_j/2-1, 1, \sigma_j/2-1) + \mathrm{In}(s_j), & \text{if} \quad \mu = \omega. \end{cases}$$

(3) If $\mathcal{D}_j(\lambda)$ is from $D_2(\lambda)$ and σ_j is odd, then

$$\mathrm{In}(\mathcal{D}_j(\mathrm{i}\omega)) = \begin{cases} ((\sigma_j-1)/2, 0, (\sigma_j-1)/2) + \mathrm{In}(s_j(\omega-\mu)), & \text{if} \quad \mu \neq \omega, \\ ((\sigma_j-1)/2, 1, (\sigma_j-1)/2), & \text{if} \quad \mu = \omega. \end{cases}$$

(4) If $\mathcal{D}_j(\lambda)$ is from $D_3(\lambda)$ and ε_j is even, then

$$\mathrm{In}(\mathcal{D}_j(\mathrm{i}\omega)) = (\varepsilon_j/2, 0, \varepsilon_j/2) \quad \text{for all} \quad \omega \in \mathbb{R}.$$

(5) If $\mathcal{D}_j(\lambda)$ is from $D_3(\lambda)$ and ε_j is odd, then

$$\mathrm{In}(\mathcal{D}_j(\mathrm{i}\omega)) = ((\varepsilon_j-1)/2, 0, (\varepsilon_j-1)/2) + \mathrm{In}(t_j) \quad \text{for all} \quad \omega \in \mathbb{R}.$$

(6) If $\mathcal{D}_j(\lambda)$ is from $D_4(\lambda)$, then

$$\mathrm{In}(\mathcal{D}_j(\mathrm{i}\omega)) = (\delta_j, 1, \delta_j) \quad \text{for all} \quad \omega \in \mathbb{R}.$$

Furthermore we need some technical definitions and lemmas.

For a rational matrix-valued function $H : \mathbb{C}\backslash D \longrightarrow \mathbb{C}^{n\times m}$, where $D \subset \mathbb{C}$ is the finite set of poles, we define the normal rank of H by $\mathrm{normalrank}(H) = \max_{s\in\mathbb{C}\backslash D} \operatorname{rank} H(s)$ [32].

Proposition 4. *[32] Let $\lambda\mathcal{N} - \mathcal{M}$ is in (7). Then there exists a congruence transformation $U(\mathrm{i}\omega)$ for all $\mathrm{i}\omega \notin \Lambda(E, A)$ such that*

$$U^H(\mathrm{i}\omega)(\mathrm{i}\omega\mathcal{N} - \mathcal{M})U(\mathrm{i}\omega) = \begin{bmatrix} 0 & \mathrm{i}\omega(\mathrm{i}E) - \mathrm{i}A & 0 \\ \mathrm{i}\omega\left(\mathrm{i}E^T\right) - \left(-\mathrm{i}A^T\right) & 0 & 0 \\ 0 & 0 & -\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right) \end{bmatrix}$$

where

$$U(\mathrm{i}\omega) = \begin{bmatrix} I_n & 0 & -\left(\mathrm{i}\omega\left(\mathrm{i}E^T\right) - \left(-\mathrm{i}A^T\right)\right)^{-1}\mathrm{i}C^T \\ 0 & I_n & \left(\mathrm{i}\omega\left(\mathrm{i}E\right) - \mathrm{i}A\right)^{-1}\mathrm{i}B \\ 0 & 0 & I_m \end{bmatrix}.$$

With these tools we can now prove the following theorem.

Theorem 2. *Let $G \in \mathcal{RH}_\infty^{m\times m}$ and let $d = \mathrm{normalrank}\left(\mathrm{i}\left(G(s) - G^H(s)\right)\right)$. Then the following statements are equivalent.*

(1) $G(s)$ is negative imaginary.

(2) The even Kronecker canonical form of $\lambda\mathcal{N} - \mathcal{M}$ consists only of the following blocks:

- *(i) Whenever there exists an even block of type $D_2(\lambda)$ associated to a $\mu = \omega_0 > 0$, it has positive signature and there exists an equally sized block of type $D_2(\lambda)$ associated to $\mu = -\omega_0$ with negative signature.*
- *(ii) There exist exactly d odd blocks of type $D_2(\lambda)$ corresponding to $\mu = 0$ with negative signature.*

(iii) Blocks of type $D_3(\lambda)$ are either of even size or the number of odd blocks of type $D_3(\lambda)$ with positive and negative signature is equal.

(iv) There exist exactly $m-d$ blocks of type $D_4(\lambda)$.

Proof. First we show (1)⇒(2). From the negative imaginariness and stability of $G(s)$ it follows that

$$\begin{aligned} &\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right) \succcurlyeq 0 \text{ for all } \omega \geq 0, \text{ and} \\ &\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right) \preccurlyeq 0 \text{ for all } \omega \leq 0, \end{aligned}$$

following from Lemma 4. Then there exists a function $a : \mathbb{R} \longrightarrow \mathbb{N}$ which is zero except for a finite set of values of ω such that

- $\mathrm{In}(\mathrm{i}\omega\mathcal{N} - \mathcal{M}) = (n, m-d+a(\omega), n+d-a(\omega))$ for $\omega > 0$,
- $\mathrm{In}(-\mathcal{M}) = (n, m, n) = (n, m-d+a(0), n)$,
- $\mathrm{In}(\mathrm{i}\omega\mathcal{N} - \mathcal{M}) = (n+d-a(\omega), m-d+a(\omega), n)$ for $\omega < 0$.

Roughly speaking, the function $a(\omega)$ describes the change of inertia in the case, that eigenvalue curves touch the zero level at ω. Now we have to analyze which block structures in the even Kronecker canonical form of $\lambda\mathcal{N} - \mathcal{M}$ can produce the inertia pattern above. First of all, $\lambda\mathcal{N} - \mathcal{M}$ has at least $m-d$ zero eigenvalues for all values of ω. Hence, according to the even Kronecker canonical form we have $m-d$ blocks of type $D_4(\lambda)$. We consider now the subpencil $\lambda\mathcal{N}_1 - \mathcal{M}_1$ of $\lambda\mathcal{N} - \mathcal{M}$ without these blocks which has the inertia

- $\mathrm{In}(\mathrm{i}\omega\mathcal{N}_1 - \mathcal{M}_1) = (n_1, a(\omega), n_1+d-a(\omega))$ for $\omega > 0$,
- $\mathrm{In}(-\mathcal{M}_1) = (n_1, d, n_1) = (n_1, a(0), n_1)$,
- $\mathrm{In}(\mathrm{i}\omega\mathcal{N}_1 - \mathcal{M}_1) = (n_1+d-a(\omega), a(\omega), n_1)$ for $\omega < 0$,

where $n_1 = n - \sum\limits_{j=1}^{k_4} \delta_j$. From this structure, we can deduce that there exist d odd blocks of type $D_2(\lambda)$ corresponding to $\mu = 0$ with negative signature. By again removing these from $\lambda\mathcal{N}_1 - \mathcal{M}_1$ we obtain the subpencil $\lambda\mathcal{N}_2 - \mathcal{M}_2$ with

- $\mathrm{In}(\mathrm{i}\omega\mathcal{N}_2 - \mathcal{M}_2) = (n_2, a(\omega), n_2 - a(\omega))$ for $\omega > 0$,
- $\mathrm{In}(-\mathcal{M}_2) = (n_2, 0, n_2)$,
- $\mathrm{In}(\mathrm{i}\omega\mathcal{N}_2 - \mathcal{M}_2) = (n_2 - a(\omega), a(\omega), n_2)$ for $\omega < 0$,

where $n_2 = n_1 - \sum\limits_{\substack{j=1 \\ \sigma_j \text{ odd}}}^{k_2} \sigma_j$. Now, we see that the remaining blocks of type $D_2(\lambda)$ are of even size. Whenever there exist such a block associated to a $\mu = \omega_0 > 0$, it has positive signature and there exists an equally sized block of type $D_2(\lambda)$ associated to $\mu = -\omega_0$ with negative signature. When removing these blocks as well, there remains a subpencil $\lambda\mathcal{N}_3 - \mathcal{M}_3$ of $\lambda\mathcal{N}_2 - \mathcal{M}_2$ with

- $\mathrm{In}(\mathrm{i}\omega\mathcal{N}_3 - \mathcal{M}_3) = (n_3, 0, n_3)$ for $\omega > 0$,
- $\mathrm{In}(-\mathcal{M}_3) = (n_3, 0, n_3)$,
- $\mathrm{In}(\mathrm{i}\omega\mathcal{N}_3 - \mathcal{M}_3) = (n_3, 0, n_3)$ for $\omega < 0$,

with $n_3 = n_2 - \sum_{\substack{j=1 \\ \sigma_j \text{ even}}}^{k_2} \sigma_j$. This shows that all blocks of type $D_3(\lambda)$ are either of even size or the number of odd blocks of type $D_3(\lambda)$ with positive and negative signature is equal. This shows (1)⇒(2).

To prove (2)⇒(1) one has to use the same argumentation backwards. By constructing a matrix pencil with the given blocks, one can show the properties of the inertia of the matrix pencil $\lambda\mathcal{N} - \mathcal{M}$ as given here hold. From that, one can conclude that $G(s)$ is negative imaginary by employing Lemma 4. □

For solving numerical problems we transform the imaginary even matrix pencil $\lambda\mathcal{N} - \mathcal{M}$ from (7) into a real odd matrix pencil by dividing both matrices by i. We obtain

$$\lambda\mathcal{H} - \mathcal{S} := \lambda \begin{bmatrix} 0 & E & 0 \\ E^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ -A^T & 0 & -C^T \\ -B^T & C & D - D^T \end{bmatrix}. \tag{9}$$

This allows us to use real instead of complex arithmetic.

4. Enforcement of Negative Imaginariness

Often, the systems that we consider are only approximations to the real system dynamics. This happens, if we, e.g., apply model order reduction [6] to a large-scale system or if we approximate the system by rational interpolation via frequency response data (like vector fitting [3], or interpolation via Löwner matrix pencils [26]). In this way it can easily happen that the negative imaginariness of the system is lost due to the modeling or approximation error. It is important to keep this property since otherwise this could lead to instabilities during the simulation of the model. Therefore, one is interested in a post-processing procedure to restore negative imaginariness without introducing a too large perturbations to the dynamical system. The method we will use here, is an adaption of the concepts presented for passivity enforcement in [18, 19, 36]. From Theorems 1 and 2 it follows that (strict) negative imaginariness is connected to the spectrum of a related imaginary even (or as shown above real odd) matrix pencil. Thus our method is based on the computation of a perturbed descriptor system with realization $(\lambda\tilde{E} - \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ and transfer function $\tilde{G} \in \mathcal{RH}_\infty^{m\times m}$ which is negative imaginary and the error $\|\tilde{G} - G\|$ is small in some system norm. The computation is performed by perturbing the nonzero, finite, purely imaginary eigenvalues of the related matrix pencils off the imaginary axis. In our considerations, we keep the matrix pencil $\lambda E - A$ to preserve the poles of the system. So, there is no risk to loose stability. Following from the decomposition (4), we have to perturb B_1 or C_1 if there is violation of negative imaginariness in the dynamic part. We will discuss in detail which matrix is the best choice for that. Furthermore, we have to modify the matrices D, B_2 or C_2 if the matrix M_0 is not symmetric.

First we present some technical results that we will make use of for the derivation of the algorithm.

4.1. Some Useful Results

First, we need a basic spectral perturbation result for general matrix pencils, see [37].

Proposition 5. *Let $\lambda B - A \in \mathbb{R}[\lambda]^{n\times n}$ be a given matrix pencil and let v, $w \in \mathbb{C}^n$ right and left eigenvectors corresponding to a simple eigenvalue $\lambda = (\alpha, \beta) = \left(w^H A v, w^H B v\right)$. Let $\lambda(B+\Delta B)-(A+\Delta A)$ be a perturbed matrix pencil with eigenvalues $\tilde{\lambda} = \left(\tilde{\alpha}, \tilde{\beta}\right)$. Then it holds*

$$\left(\tilde{\alpha}, \tilde{\beta}\right) = (\alpha, \beta) + \left(w^H \Delta A v, w^H \Delta B v\right) + \mathcal{O}\left(\varepsilon^2\right), \tag{10}$$

where $\varepsilon = \left\| \begin{bmatrix} \Delta A & \Delta B \end{bmatrix} \right\|_2$.

Next, we want to apply this lemma to the special case of an odd matrix pencil. Let v be a right eigenvector of the odd matrix pencil $\lambda\mathcal{H} - \mathcal{S}$ corresponding to the eigenvalue λ. Then we obtain

$$0 = \lambda\mathcal{H}v - \mathcal{S}v.$$

Now, by taking the conjugate transpose of the above equation and using $\mathcal{H}^T = \mathcal{H}$ and $\mathcal{S}^T = -\mathcal{S}$, we obtain

$$0 = \bar{\lambda}v^H\mathcal{H}^T - v^H\mathcal{S}^T = \bar{\lambda}v^H\mathcal{H} + v^H\mathcal{S}.$$

So, when λ is purely imaginary and hence $\lambda = -\bar{\lambda}$, we get that if v is an associated right eigenvector, v is also a corresponding left eigenvector. Let $\lambda = (\alpha, \beta)$ be a simple, purely imaginary eigenvalue of an odd matrix pencil $\lambda\mathcal{H} - \mathcal{S}$. For a perturbed matrix pencil of the form $\lambda(\mathcal{H} + \varepsilon\mathcal{H}') - (\mathcal{S} + \varepsilon\mathcal{S}')$, formula (10) can be written as

$$\left(\tilde{\alpha}, \tilde{\beta}\right) = (\alpha, \beta) + \left(\varepsilon v^H\mathcal{S}'v, \varepsilon v^H\mathcal{H}'v\right) + \mathcal{O}\left(\varepsilon^2\right), \tag{11}$$

Theorem 3. *Consider a transfer function $G \in \mathcal{RH}_\infty^{m\times m}$. Let furthermore v be a right eigenvector of $\lambda\mathcal{H} - \mathcal{S}$ as in* (9) *corresponding to a nonzero, simple, finite, purely imaginary eigenvalue $\mathrm{i}\omega_0$ and let $\nu(\omega)$ be an eigenvalue curve of $H(\mathrm{i}\omega) = \mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right)$ that crosses the level zero at ω_0, i.e., $\nu(\omega_0) = 0$. Then the slope of $\nu(\omega)$ is positive (negative) at ω_0 if $v^H\mathcal{H}v > 0$ ($v^H\mathcal{H}v < 0$).*

Proof. We want to motivate the technique of the proof by the following idea. To decide whether the curve increases or decreases at the point ω_0, we could compute the point $\omega_0 + \delta$, where the curve crosses the level ε with $\varepsilon > 0$ and then check whether δ is positive or negative. Thus, the actual proof starts by analyzing the eigenvalues of the perturbed matrix pencil

$$\lambda\mathcal{H} - \mathcal{S}_\varepsilon := \lambda\mathcal{H} - (\mathcal{S} + \varepsilon\mathcal{S}') + \mathcal{O}\left(\varepsilon^2\right),$$

where

$$\mathcal{S}' = \left.\frac{\mathrm{d}\mathcal{S}_\varepsilon}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}\varepsilon I_m \end{bmatrix}\right|_{\varepsilon=0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}I_m \end{bmatrix}.$$

The matrix $\mathcal{S}_\varepsilon$ is obtained by analyzing at which frequencies the eigenvalue curves of $H(\mathrm{i}\omega)$ cross the level ε, or equivalently at which frequencies the eigenvalue curves of $H(\mathrm{i}\omega) - \varepsilon I_m$ cross the zero level, see (8). Note that we do not consider a perturbation of the matrix $\mathcal{H}$, since

$$\mathcal{H}' = \left.\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = 0.$$

Furthermore, the matrix $\mathcal{S}_\varepsilon'$ is skew-Hermitian. Let $\mathrm{i}\omega_0$ be a finite eigenvalue of $\lambda\mathcal{H} - \mathcal{S}$ and let $\mathrm{i}\omega_\varepsilon$ be the corresponding perturbed eigenvalue of $\lambda\mathcal{H} - \mathcal{S}_\varepsilon$. Then, by (11) it follows that

$$\begin{aligned} \mathrm{i}\omega_\varepsilon &= \frac{v^H\mathcal{S}v + \varepsilon v^H\mathcal{S}'v}{v^H\mathcal{H}v} + \mathcal{O}\left(\varepsilon^2\right) \\ &= \mathrm{i}\omega_0 + \varepsilon\frac{v^H\mathcal{S}'v}{v^H\mathcal{H}v} + \mathcal{O}\left(\varepsilon^2\right). \end{aligned} \tag{12}$$

In other words, we have

$$\left.\frac{\mathrm{d}\omega_\varepsilon}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \frac{v^H\mathcal{S}'v}{\mathrm{i}v^H\mathcal{H}v}.$$

Since ν and ε are interchangeable, it follows that the slope of the eigenvalue curve crossing the level zero at ω_0 can be written as

$$\xi := \left.\frac{\mathrm{d}\nu}{\mathrm{d}\omega_\varepsilon}\right|_{\omega_\varepsilon=\omega_0} = \frac{1}{\left.\frac{\mathrm{d}\omega_\varepsilon}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}} = \frac{\mathrm{i}v^H\mathcal{H}v}{v^H\mathcal{S}'v}.$$

Now, we conclude the assertion as $\mathcal{S}' = \mathrm{i}\hat{\mathcal{S}}$ with a positive semidefinite matrix $\hat{\mathcal{S}}$. $\square$

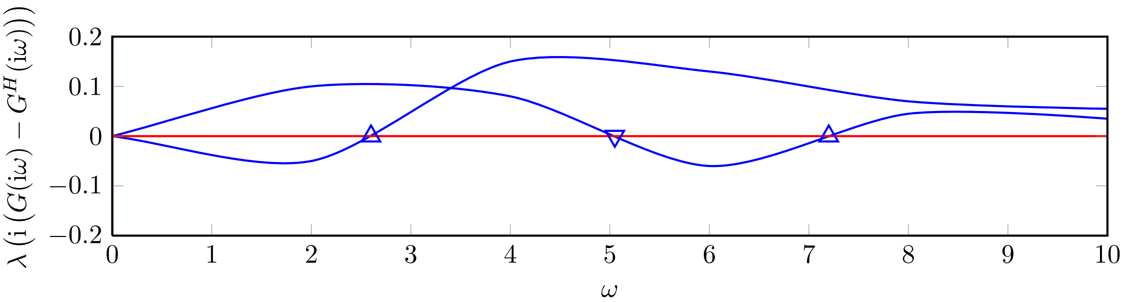


Figure 2: Characterization of negative imaginariness violation points and corresponding slopes

In Figure 2, a non-negative imaginary transfer function is depicted with intersection points of the eigenvalue curves with the zero level and corresponding slopes denoted by triangles. With this characterization we can now think about moving the nonzero, finite, purely imaginary eigenvalues of $\lambda\mathcal{H}-\mathcal{S}$ off the imaginary axis in order to enforce negative imaginariness. Therefore we need the finite, positive imaginary eigenvalues of $\lambda\mathcal{H}-\mathcal{S}$ and the corresponding eigenvectors. To compute these, we reformulate the odd eigenvalue problem into a Hamiltonian/skew-Hamiltonian one and use a structure-preserving algorithm [5] to solve it. Then, we can also use a structure-exploiting technique to obtain the corresponding eigenvectors. We will describe this in detail in Subsection 4.7.

4.2. Choice of the New Frequencies

From Figure 2 we can see that it is reasonable to assume that the "size" of the violation of negative imaginariness decreases if we move the nonzero, finite, purely imaginary eigenvalues of $\lambda\mathcal{H}-\mathcal{S}$ with negative slope to the right and those with positive slope to the left. Let the frequencies ω_i, where the eigenvalue curves cross the level zero be ordered in increasing order, i.e. $0 < \omega_1 < \omega_2 < \ldots < \omega_k$. Choosing the displacement proportionally to the distance between ω_{i+1} and ω_i, we obtain the following equations

$$\tilde{\omega}_i = \begin{cases} \omega_i + \alpha(\omega_{i+1}-\omega_i), & v_i^H\mathcal{H}v_i < 0,\ i \neq k \neq 1, \\ (1+2\alpha)\omega_i & v_i^H\mathcal{H}v_i < 0,\ i = k, \\ \omega_i - \alpha(\omega_i-\omega_{i-1}), & v_i^H\mathcal{H}v_i > 0,\ i \neq 1 \neq k, \\ (1-2\alpha)\omega_i, & v_i^H\mathcal{H}v_i > 0,\ i = 1, \end{cases} \tag{13}$$

Here, $\alpha \in (0, 0.5]$ is a tuning parameter. It is tempting to use $\alpha = 0.5$ since then the transfer function would be negative imaginary. But this corresponds to a rather large perturbation. This is dangerous because it might take us out of the region, where the first order perturbation theory holds. Therefore we suggest to use smaller values of α (depending on the problem) and to apply the whole method iteratively, until the negative imaginariness is enforced. Other choices of $\tilde{\omega}_i$ are also possible, see [18]. We remark, that when $v_k^H\mathcal{H}v_k < 0$, the system violates the negative imaginary property at infinity. To restore this we have to move the eigenvalue $\mathrm{i}\omega_k$ to infinity. It is not possible to do this numerically. Hence we define a threshold η and declare all eigenvalues whose magnitudes are larger than η as numerically infinite.

There are particular situations where the rule above does not lead to the correct result. This has also not been yet covered by the available literature. Consider for example the situation depicted in Figure 3. Here, there are two *intersected* intervals which negative imaginariness is violated in. This is characterized by two successing intersection points of the eigenvalue curves with the zero level which have negative slope followed by two intersection points with positive slope. When successively applying formula (13) the second and third frequency point would form a double intersection point (assuming that we are able to exactly perturb these frequency points which is not the case). This means that the corresponding matrix pencil $\lambda\mathcal{H}-\mathcal{S}$ has a double nonzero, finite, purely imaginary eigenvalue. However, in this case we also have two

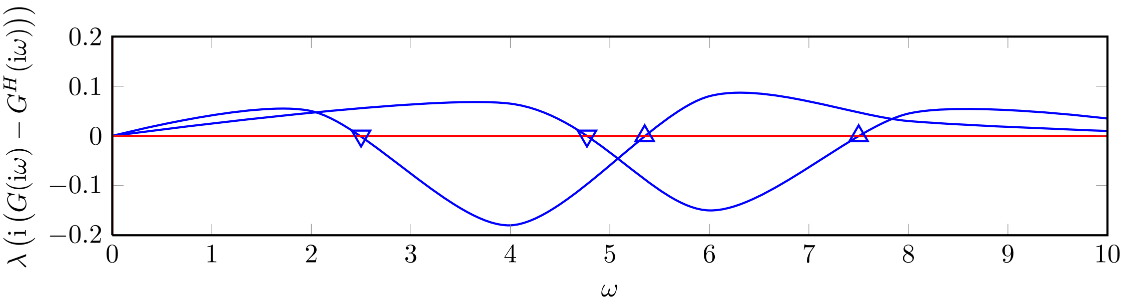


Figure 3: Non-negative imaginary transfer function with intersected frequency intervals with negative imaginariness violations

linearly independent eigenvectors which means that this eigenvalue does not generate nontrivial blocks in the Kronecker canonical form. On the other hand, in the case that the frequency intervals are not intersected (like in Figure 2), the converged eigenvalues would form an associated block of size two in the Kronecker canonical form as there exists only one linearly independent eigenvector. So we add the following rule to the update formula (13):

$$\tilde{\omega}_i = \begin{cases} \omega_i + \alpha(\omega_{i+2} - \omega_i), & v_i^H \mathcal{H} v_i < 0,\ i \neq k-1,\, k, \\ \omega_i - \alpha(\omega_i - \omega_{i-2}), & v_i^H \mathcal{H} v_i > 0,\ i \neq 1,\, 2, \end{cases}$$

if

$$\begin{aligned} &|\omega_{i+1} - \omega_i| < \delta \quad \text{and} \quad \left\{ \left| \frac{\mathrm{i} v_i^H \mathcal{H} v_i}{v_i^H \mathcal{S}' v_i} \right| > \varepsilon \quad \text{or} \quad \left| \frac{\mathrm{i} v_{i+1}^H \mathcal{H} v_{i+1}}{v_{i+1}^H \mathcal{S}' v_{i+1}} \right| > \varepsilon \right\}, \\ &|\omega_i - \omega_{i-1}| < \delta \quad \text{and} \quad \left\{ \left| \frac{\mathrm{i} v_{i-1}^H \mathcal{H} v_{i-1}}{v_{i-1}^H \mathcal{S}' v_{i-1}} \right| > \varepsilon \quad \text{or} \quad \left| \frac{\mathrm{i} v_i^H \mathcal{H} v_i}{v_i^H \mathcal{S}' v_i} \right| > \varepsilon \right\}, \end{aligned} \tag{14}$$

respectively, where δ and ε are predefined tolerances.

4.3. Choice of the System Norm

As proposed in [19], we compute the perturbation that minimizes the $\mathcal{H}_2$-norm of the error $\mathcal{E}(s) := \tilde{G}(s) - G(s)$ given by

$$\|\mathcal{E}\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathcal{E}(\mathrm{i}\omega)\|_F^2 \,\mathrm{d}\omega \right)^{\frac{1}{2}}. \tag{15}$$

Using the decomposition (4), we have $\mathcal{E}(s) = \mathcal{E}_{\mathrm{sp}}(s) + \underbrace{M_0 + \mathcal{E}_{\mathrm{i}}(s)}_{=:\mathcal{P}(s)}$, so we can also write (15) as [39]

$$\|\mathcal{E}\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathcal{E}_{\mathrm{sp}}(\mathrm{i}\omega)\|_F^2 \,\mathrm{d}\omega + \frac{1}{2\pi} \int_0^{2\pi} \left\|\mathcal{P}(e^{\mathrm{i}\omega})\right\|_F^2 \mathrm{d}\omega \right)^{\frac{1}{2}}. \tag{16}$$

Since we only want to perturb B_1 or C_1, we can drop the second term of the right-hand side of (16) and get

$$\|\mathcal{E}\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathcal{E}_{\mathrm{sp}}(\mathrm{i}\omega)\|_F^2 \,\mathrm{d}\omega \right)^{\frac{1}{2}} = \|\mathcal{E}_{\mathrm{sp}}\|_{\mathcal{H}_2}.$$

Assume that the descriptor system (1) is given in the decoupled form (4) and that $M_0 = M_0^T$. Consider the observability Gramian $\mathcal{G}_{\text{fo}}$ of the slow subsystem [13] $(\lambda I_{n_{\text{f}}} - J, B_1, C_1, 0)$ which is defined as the unique, positive semidefinite solution of the Lyapunov equation [38]

$$\mathcal{G}_{\text{fo}} J + J^T \mathcal{G}_{\text{fo}} = -C_1^T C_1. \tag{17}$$

Since $\mathcal{G}_{\text{fo}}$ can be represented as

$$\mathcal{G}_{\text{fo}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\omega I_{n_{\text{f}}} - J)^{-T} C_1^T C_1 (\mathrm{i}\omega I_{n_{\text{f}}} - J)^{-1} \,\mathrm{d}\omega,$$

we have $\|\mathcal{E}_{\text{sp}}\|_{\mathcal{H}_2} = \|L\Delta\|_F$ where L is a lower triangular Cholesky factor of $\mathcal{G}_{\text{fo}}$, i.e., $\mathcal{G}_{\text{fo}} = L^T L$, and Δ is a perturbation of B_1, i.e., $\Delta = \tilde{B}_1 - B_1$ with $\tilde{B}_1$ corresponding to a negative imaginary system.

We remark that it is not necessary to compute the fully decoupled realization (4) to solve the Lyapunov equation (17) to obtain L. This is also not reasonable since the computation of the Weierstraß canonical form might be arbitrarily ill-conditioned and thus should be avoided. There are algorithms which compute a slightly generalized condensed form of the matrix pencil $\lambda E - A$, that is

$$W(\lambda E - A)T = \lambda \begin{bmatrix} E_{11} & 0 \\ 0 & E_{22} \end{bmatrix} - \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

where W, $T \in \mathbb{R}^{n\times n}$ are nonsingular and $\lambda E_{11} - A_{11}$ and $\lambda E_{22} - A_{22}$ are the subpencils of $\lambda E - A$ that correspond to its finite and infinite eigenvalues, respectively. These algorithm basically work in two steps. In Step 1 an upper triangular form with eigenvalue separation of the pencil $\lambda E - A$ is computed, i.e.,

$$\mathcal{P}(\lambda E - A)\mathcal{Q} = \lambda \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{18}$$

This can be done by the QZ algorithm with subsequent eigenvalue reordering [17], the GUPTRI algorithm [14, 15], or the disk function method [4, 40]. By setting $B := \mathcal{P}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $C := C\mathcal{Q} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$, we obtain a corresponding restricted equivalence transformation of the descriptor system. Now, Step 2 consists of block-diagonalizing the pencil (18). This can be done by solving the generalized Sylvester equation

$$A_{11}Y + ZA_{22} + A_{12} = 0, \quad E_{11}Y + ZE_{22} + E_{12} = 0, \tag{19}$$

see, e.g. [21, 22, 23, 24]. Then, we define $\mathcal{Z} := \begin{bmatrix} I_{n_f} & Z \\ 0 & I_{n_\infty} \end{bmatrix}$, $\mathcal{Y} := \begin{bmatrix} I_{n_f} & Y \\ 0 & I_{n_\infty} \end{bmatrix}$, and get

$$\mathcal{Z}\left(\lambda \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\right)\mathcal{Y} = \lambda \begin{bmatrix} E_{11} & 0 \\ 0 & E_{22} \end{bmatrix} - \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}. \tag{20}$$

By updating B and C, we obtain $B := \mathcal{Z}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B_1 + ZB_2 \\ B_2 \end{bmatrix}$ and $C := \begin{bmatrix} C_1 & C_2 \end{bmatrix}\mathcal{Y} = \begin{bmatrix} C_1 & C_1Y + C_2 \end{bmatrix}$. To compute $\|\mathcal{E}_{\text{sp}}\|_{\mathcal{H}_2}$, we can now solve the generalized Lyapunov equation

$$E_{11}^T \mathcal{G}_{\text{fo}} A_{11} + A_{11}^T \mathcal{G}_{\text{fo}} E_{11} = -C_1^T C_1. \tag{21}$$

instead of (17).

Note, that it is sufficient to perform only Step 1 since E_{11}, A_{11}, and C_1 are not changed while performing Step 2. However, this is only possible when we only change the matrix B_1 during the enforcement procedure. This would no longer hold, if we would also change C_1. This is the reason why we only apply perturbations to B_1 in this paper. Furthermore, note that we can compute L directly without explicitly computing $\mathcal{G}_{\text{fo}}$ beforehand [7, 20].

4.4. Enforcement Procedure

Now, as we know how to move nonzero, purely imaginary eigenvalues of odd matrix pencils $\lambda\mathcal{H} - \mathcal{S}$ and which system norm we use to compute the optimal perturbation, we are now going to actually compute this perturbation, similarly as in [18, 19, 36]. We consider the matrix pencil (9), where $\lambda E - A$ is now given in the form (18) and B and C are properly updated, i.e.,

$$\lambda\mathcal{H} - \mathcal{S} = \lambda \begin{bmatrix} 0 & 0 & E_{11} & E_{12} & 0 \\ 0 & 0 & 0 & E_{22} & 0 \\ E_{11}^T & 0 & 0 & 0 & 0 \\ E_{12}^T & E_{22}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & A_{11} & A_{12} & B_1 \\ 0 & 0 & 0 & A_{22} & B_2 \\ -A_{11}^T & 0 & 0 & 0 & -C_1^T \\ -A_{12}^T & -A_{22}^T & 0 & 0 & -C_2^T \\ -B_1^T & -B_2^T & C_1 & C_2 & D - D^T \end{bmatrix} \tag{22}$$

We perturb the matrix pencil (22) by replacing B_1 by $B_1 + \Delta$. The perturbed matrix pencil $\lambda\mathcal{H} - \tilde{\mathcal{S}}$ can then be written as $\lambda\mathcal{H} - \tilde{\mathcal{S}} = \lambda\mathcal{H} - \left(\mathcal{S} + \hat{\mathcal{S}}\right) + \mathcal{O}\left(\|\Delta\|^2\right)$ with

$$\hat{\mathcal{S}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \Delta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\Delta^T & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{23}$$

Let v_i be a right eigenvector of $\lambda\mathcal{H} - \mathcal{S}$ corresponding to an eigenvalue $\mathrm{i}\omega_i$. Then, by (12) the imaginary eigenvalues $\mathrm{i}\tilde{\omega}_i$ of $\lambda\mathcal{H} - \tilde{\mathcal{S}}$ and those of $\lambda\mathcal{H} - \mathcal{S}$ are related up to first order perturbations via

$$\tilde{\omega}_i - \omega_i = \frac{v_i^H \hat{\mathcal{S}} v_i}{\mathrm{i} v_i^H \mathcal{H} v_i}. \tag{24}$$

The numerator of the right-hand side of (24) can be expressed as

$$\begin{aligned} v_i^H \hat{\mathcal{S}} v_i &= v_{i1}^H \Delta v_{i5} - v_{i5}^H \Delta^T v_{i1} \\ &= 2\mathrm{i}\,\mathrm{Im}\left(v_{i1}^H \Delta v_{i5}\right), \end{aligned} \tag{25}$$

where $v_i = \begin{bmatrix} v_{i1}^T & \dots & v_{i5}^T \end{bmatrix}^T \in \mathbb{C}^{2n+m}$ is partitioned according to the block structure of (22). Applying the vectorization operator to (25) yields

$$v_i^H \hat{\mathcal{S}} v_i = 2\mathrm{i}\,\mathrm{Im}\left(v_{i5}^T \otimes v_{i1}^H\right) \mathrm{vec}\,(\Delta).$$

Inserting this into (24) gives

$$\frac{2}{v_i^H \mathcal{H} v_i} \mathrm{Im}\left(v_{i5}^T \otimes v_{i1}^H\right) \mathrm{vec}\,(\Delta) = \tilde{\omega}_i - \omega_i. \tag{26}$$

This a linear relation between $\mathrm{vec}\,(\Delta)$ and $\tilde{\omega}_i - \omega_i$. Collecting this information for every nonzero, finite, purely imaginary eigenvalue, we obtain the linear system

$$Z\,\mathrm{vec}\,(\Delta) = \tilde{\omega} - \omega, \tag{27}$$

where $\tilde{\omega} = \begin{bmatrix} \tilde{\omega}_1 & \dots & \tilde{\omega}_k \end{bmatrix}$, $\omega = \begin{bmatrix} \omega_1 & \dots & \omega_k \end{bmatrix}$, and the i-th row of $Z \in \mathbb{R}^{k \times n_{\mathrm{f}} m}$ has the form

$$e_i^T Z = \frac{2}{v_i^H \mathcal{H} v_i} \mathrm{Im}\left(v_{i5}^T \otimes v_{i1}^H\right).$$

To compute the perturbation Δ that satisfies (27) and minimizes $\|\mathcal{E}_{\mathrm{sp}}\|_{\mathcal{H}_2}$, we have to solve the following minimization problem:

$$\min_{\Delta \in \mathbb{R}^{n_{\mathrm{f}} \times m}} \|L\Delta\|_F \quad \text{subject to} \quad Z\,\mathrm{vec}(\Delta) = \tilde{\omega} - \omega.$$

If the slow subsystem of (1) is R-observable [13], its observability Gramian $\mathcal{G}_{\mathrm{fo}}$ is positive definite and hence its Cholesky factor L is nonsingular [38]. By changing the basis $\Delta_L := L\Delta$, we obtain the equivalent minimization problem

$$\min_{\Delta_L \in \mathbb{R}^{n_\mathrm{f} \times m}} \|\Delta_L\|_F \quad \text{subject to} \quad Z_L \operatorname{vec}(\Delta_L) = \tilde{\omega} - \omega, \tag{28}$$

where $Z_L = Z\left(I \otimes L^{-1}\right)$. Note, that we do not have to build the matrix $I \otimes L^{-1}$ explicitly, since the i-th row of Z_L can be computed as

$$e_i^T Z_L = e_i^T Z \left(I \otimes L^{-1}\right) = \frac{2}{v_i^H \mathcal{H} v_i} \operatorname{Im}\left(v_{i5}^T \otimes v_{i1}^H L^{-1}\right). \tag{29}$$

In this case the minimization problem (28) reduces to the standard least squares problem

$$\min_{\Delta_L \in \mathbb{R}^{n_\mathrm{f} \times m}} \|\operatorname{vec}(\Delta_L)\|_2 \quad \text{subject to} \quad Z_L \operatorname{vec}(\Delta_L) = \tilde{\omega} - \omega.$$

The solution of this problem is given by

$$\operatorname{vec}(\Delta_L) = Z_L^\dagger (\tilde{\omega} - \omega),$$

where $Z_L^\dagger$ denotes the Moore-Penrose inverse of Z_L. The computation of $Z_L^\dagger$ requires a singular value decomposition of the $k \times n_\mathrm{f} m$ matrix Z_L which costs $\mathcal{O}\left(n_\mathrm{f} m k^2\right)$ floating point operations. The required perturbation is then computed as

$$\Delta = L^{-1} \Delta_L.$$

4.5. Enforcement of $M_0 = M_0^T$

As shown by Lemma 4, a negative imaginary transfer function $G \in \mathcal{RH}_\infty^{m \times m}$ satisfies $M_0 = M_0^T$, where $M_0 = G(\mathrm{i}\infty)$. It might happen, that this property is also lost during the modeling process. In this section we will briefly describe how to restore the symmetry of M_0. First, we actually have to compute this matrix. This can be done by decoupling the system (1) into its slow and fast subsystems, respectively. This is achieved by decoupling the matrix pencil $\lambda E - A$ into its subpencils corresponding to finite and infinite eigenvalues, respectively, as done in (20). Note, that this computation might be ill-conditioned, as solving a generalized Sylvester equation for the decoupling might be. Hence, this operation should avoided when it is clear, that M_0 is symmetric. Now we can write the transfer function $G(s)$ as

$$G(s) = C_1 \left(sE_{11} - A_{11}\right)^{-1} B_1 + C_2 \left(sE_{22} - A_{22}\right)^{-1} B_2 + D.$$

Then it holds

$$\begin{aligned} M_0 &= \lim_{\omega \to \infty} G(\mathrm{i}\omega) \\ &= \lim_{\omega \to \infty} \left(C_1 \left(\mathrm{i}\omega E_{11} - A_{11}\right)^{-1} B_1\right) + \lim_{\omega \to \infty} \left(C_2 \left(\mathrm{i}\omega E_{22} - A_{22}\right)^{-1} B_2 + D\right) \\ &= D - C_2 A_{22}^{-1} B_2. \end{aligned}$$

Now assume that M_0 is not symmetric. Then

$$M_0 - M_0^T = \mathcal{F} = \mathcal{T} - \mathcal{T}^T,$$

where $\mathcal{T}$ is defined as the strictly upper triangular part of the skew-symmetric error matrix $\mathcal{F}$. In this way we perturb the matrix D as

$$\tilde{D} := D - \mathcal{T}.$$

The error caused by this perturbation in the $\mathcal{H}_2$-norm of the system is given by

$$\|\mathcal{E}\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_0^{2\pi} \left\|\mathcal{P}(e^{\mathrm{i}\omega})\right\|_F^2 \mathrm{d}\omega\right)^{\frac{1}{2}} = \|\mathcal{T}\|_F,$$

see (16).

Algorithm 1 Algorithm for Enforcing Symmetry of M_0

Input: Asymptotically stable descriptor system $G = (\lambda E - A, B, C, D)$.

Output: A descriptor system $\tilde{G} = (\lambda E - A, B, C, \tilde{D})$ satisfying $M_0 = M_0^T$.

1: Triangularize the matrix pencil $\lambda E - A$, i.e., compute orthogonal $\mathcal{P}$ and $\mathcal{Q}$ such that

$$\mathcal{P}(\lambda E - A)\mathcal{Q} = \lambda \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

.

2: Set $B := \mathcal{P}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ and $C := C\mathcal{Q} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$.

3: Solve the generalized Sylvester equation

$$A_{11}Y + ZA_{22} + A_{12} = 0, \quad E_{11}Y + ZE_{22} + E_{12} = 0.$$

4: Update $C_2 := C_1 Y + C_2$.

5: Compute $M_0 := D - C_2 A_{22}^{-1} B_2$.

6: Compute the strictly upper triangular part of $M_0 - M_0^T$, denoted by $\mathcal{T}$.

7: Set $\tilde{D} := D - \mathcal{T}$.

4.6. The Overall Process

From the considerations above we can now state the procedures for enforcing the symmetry of M_0 in Algorithm 1 and negative imaginariness in Algorithm 2. Note, that when Algorithm 1 has been performed, the triangularization of $\lambda E - A$ has already been done, so this step can be omitted in Algorithm 2.

We briefly summarize how to solve some specific subproblems with available software tools. In particular we mention routines implemented in MATLAB® and Fortran (within the software packages LAPACK[1] and SLICOT[2]). Algorithms which have only been implemented in Fortran can be called by MATLAB® by using its `mex` functionality. See Table 1 for an overview. Note that the SLICOT routine `MB04BD` is actually designed to compute the eigenvalues of a skew-Hamiltonian/Hamiltonian matrix pencil. However, as pointed out in the next subsection, there is a close connection between skew-Hamiltonian/Hamiltonian and odd matrix pencils.

4.7. Reformulation of the Odd Eigenvalue Problem

This subsection provides some details about the solution of the odd eigenvalue problem. First, we transform the matrix pencil $\lambda\mathcal{H} - \mathcal{S}$ to a related Hamiltonian/skew-Hamiltonian pencil $\lambda\mathbf{H} - \mathbf{S}$. Then, we apply the real-case version of the structure-preserving method presented in [5] to compute the eigenvalues

[1] `http://www.netlib.org/lapack/`
[2] `http://www.slicot.org/`
[3] `http://www8.cs.umu.se/~guptri/`

Table 1: Survey of available software

Operation	MATLAB	LAPACK/SLICOT
Block triangularizing $\lambda E - A$ as in (18)	`qz`, `ordqz`	`DGGES`
	`guptri`[3]	`GUPTRI`[3]
Solving generalized Sylvester equations as in (19)	—	`SB04OD`
Solving generalized Lyapunov equations as in (21)	`lyapchol`	`SG03BD`
Computing imaginary eigenvalues of $\lambda\mathcal{H} - \mathcal{S}$	—	`MB04BD`

Algorithm 2 Negative Imaginariness Enforcement Algorithm

Input: Asymptotically stable and R-observable descriptor system $G = (\lambda E - A, B, C, D)$ such that $\lim\limits_{\omega\to\infty} \left(\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right)\right) = 0$, control parameters $0 < \alpha \le 0.5$, $\delta > 0$, $\varepsilon > 0$.

Output: A negative imaginary descriptor system $\tilde{G} = (\lambda E - A, \tilde{B}, C, D)$.

Triangularize the matrix pencil $\lambda E - A$, i.e., compute orthogonal $\mathcal{P}$ and $\mathcal{Q}$ such that

$$\mathcal{P}(\lambda E - A)\mathcal{Q} = \lambda \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

.

Set $B := \mathcal{P}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ and $C := C\mathcal{Q} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$.

Compute the Cholesky factor L of the proper observability Gramian $\mathcal{G}_{\mathrm{fo}} = L^T L$ by solving the generalized Lyapunov equation (21).

Compute the purely imaginary eigenvalues of the odd matrix pencil $\lambda\mathcal{H} - \mathcal{S}$ from (22) with positive imaginary part.

while $\lambda\mathcal{H} - \mathcal{S}$ has nonzero, finite, purely imaginary eigenvalues **do**

  Choose new eigenvalues as in (13) and (14).

  Solve $\min\limits_{\Delta_L \in \mathbb{R}^{n_{\mathrm{f}} \times m}} \|\mathrm{vec}\,(\Delta_L)\|_2$ subject to $Z_L\,\mathrm{vec}(\Delta_L) = \tilde{\omega} - \omega$ with Z_L as in (29).

  Update $B_1 := B_1 + L^{-1}\Delta_L$ and update $\mathcal{S}$ accordingly.

  Compute the positive imaginary eigenvalues and the corresponding eigenvectors of $\lambda\mathcal{H} - \mathcal{S}$.

end while

Set $\tilde{B} := \mathcal{P}^T B$.

---

of $\lambda\mathbf{S} - \mathbf{H}$. The related eigenvectors can also be computed in a structure-exploiting manner by using the condensed form which is computed to get the eigenvalues. This has been presented in [8].

Consider now the odd matrix pencil $\lambda\mathcal{H} - \mathcal{S} \in \mathbb{R}^{2n+m\times 2n+m}$. Recall that every Hamiltonian/skew-Hamiltonian matrix pencil has an even dimension. So, if m is an odd number we first increase the dimension of $\lambda\mathcal{H} - \mathcal{S}$ by one. We define the numbers

$$r := m \bmod 2, \quad k := \frac{1}{2}(m + r), \quad q := n - k.$$

We repartition

$$E = \begin{array}{c} \begin{array}{cc} k & q \end{array} \\ \begin{bmatrix} E_1 & E_2 \end{bmatrix} \end{array} \begin{array}{c} \\ n \end{array}, \quad A = \begin{array}{c} \begin{array}{cc} k & q \end{array} \\ \begin{bmatrix} A_1 & A_2 \end{bmatrix} \end{array} \begin{array}{c} \\ n \end{array}, \quad C = \begin{array}{c} \begin{array}{cc} k & q \end{array} \\ \begin{bmatrix} C_1 & C_2 \end{bmatrix} \end{array} \begin{array}{c} \\ p \end{array}, \tag{30}$$

and set

$$\lambda\tilde{\mathbf{H}} - \tilde{\mathbf{S}} := \lambda \begin{array}{c} \begin{array}{cc} n & r \end{array} \\ \begin{bmatrix} \mathcal{H} & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ n \\ r \end{array} - \begin{array}{c} \begin{array}{cc} n & r \end{array} \\ \begin{bmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ n \\ r \end{array}.$$

In this way we preserve $\lambda\mathcal{H} - \mathcal{S}$ if m is even, and increase the dimension by 1 if m is odd. In this case an additional infinite eigenvalue is introduced. Then, by using (30) we obtain

$$\lambda\tilde{\mathbf{H}} - \tilde{\mathbf{S}} := \lambda \begin{array}{c} \begin{array}{ccccc} n & k & q & m & r \end{array} \\ \left[\begin{array}{cc|ccc} 0 & E_1 & E_2 & 0 & 0 \\ E_1^T & 0 & 0 & 0 & 0 \\ \hline E_2^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ n \\ k \\ q \\ m \\ r \end{array} - \begin{array}{c} \begin{array}{ccccc} n & k & q & m & r \end{array} \\ \left[\begin{array}{cc|ccc} 0 & A_1 & A_2 & B & 0 \\ -A_1^T & 0 & 0 & -C_1^T & 0 \\ \hline -A_2^T & 0 & 0 & -C_2^T & 0 \\ -B^T & C_1 & C_2 & D - D^T & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ n \\ k \\ q \\ m \\ r \end{array}.$$

Table 2: Numerical results for example system 1

α	0.5	0.4	0.3	0.2	0.1	0.05	0.02	0.01
rel. error	0.28648	0.22947	0.17189	0.11500	0.08544	0.07776	0.07548	0.07543
# iter.	1	2	1	4	14	25	64	120

Table 3: Numerical results for example system 2

α	0.5	0.4	0.3	0.2	0.1	0.05	0.02	0.01
rel. error	–	–	–	1.46320	0.58618	0.47051	0.43616	0.42906
# iter.	–	–	–	3	11	94	271	499

By multiplying this pencil by $\mathcal{J}$ from the left we obtain the desired Hamiltonian/skew-Hamiltonian matrix pencil

$$\lambda\mathbf{H}-\mathbf{S}:=\lambda\begin{array}{c@{}c} \begin{array}{ccccc} n & k & q & m & r \end{array} & \\ \left[\begin{array}{cc|ccc} E_2^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & -E_1 & -E_2 & 0 & 0 \\ -E_1^T & 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{c} q \\ m \\ r \\ n \\ k \end{array} \end{array} - \begin{array}{c@{}c} \begin{array}{ccccc} n & k & q & m & r \end{array} & \\ \left[\begin{array}{cc|ccc} -A_2^T & 0 & 0 & -C_2^T & 0 \\ -B^T & C_1 & C_2 & D-D^T & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -A_1 & -A_2 & -B & 0 \\ A_1^T & 0 & 0 & C_1^T & 0 \end{array}\right] & \begin{array}{c} q \\ m \\ r \\ n \\ k \end{array} \end{array}.$$

Now, we can compute the purely imaginary eigenvalues of $\lambda\mathbf{S}-\mathbf{H}$ and the associated eigenvectors in a structure-preserving way. Due to the symmetry of $\mathrm{i}\left(G(\mathrm{i}\omega)-G^H(\mathrm{i}\omega)\right)$ we only need the eigenvalues with positive imaginary part. Let $\mathrm{i}\omega$ with $\omega>0$ be an eigenvalue of $\lambda\mathbf{S}-\mathbf{H}$ with eigenvector v. Then it holds:

$$\begin{aligned} &\mathrm{i}\omega\mathbf{S}v-\mathbf{H}v=0 \\ \Longleftrightarrow&\mathbf{S}v-\frac{1}{\mathrm{i}\omega}\mathbf{H}v=0 \\ \Longleftrightarrow&\mathbf{S}v+\mathrm{i}\frac{1}{\omega}\mathbf{H}v=0 \\ \Longleftrightarrow&\mathbf{S}\bar{v}-\mathrm{i}\frac{1}{\omega}\mathbf{H}\bar{v}=0. \end{aligned}$$

In other words, v is an eigenvector of $\lambda\mathbf{S}-\mathbf{H}$ to $\mathrm{i}\omega$ if and only if $\bar{v}$ is an eigenvector of $\lambda\mathbf{H}-\mathbf{S}$ to $\mathrm{i}\frac{1}{\omega}$. Now, the theory presented for odd matrix pencils in this section directly applies to Hamiltonian/skew-Hamiltonian pencils as well. One only has to take care that now instead of v, $\mathcal{J}v$ is a left eigenvector associated to a purely imaginary eigenvalue.

4.8. Illustrative Example

In this section we present some numerical results of the algorithm for enforcing negative imaginariness. As an example we use a constrained damped mass-spring system described in [30]. In [44], it has been shown that such systems fulfill the negative imaginary property. The system we consider has $n=11$ descriptor variables and $m=1$ input and output. To obtain non-negative imaginary test examples we perturb the state matrix A by a matrix $\hat{A}$ with small norm. Here, we have a look on two such example systems which have been created by a relatively large perturbation of the matrix A.

First, we analyze the results for different values of the tuning parameter α and present the relative error measured in the $\mathcal{H}_2$-norm and the number of iterations needed to make the systems negative imaginary. For the first example, the perturbation of A is still that small that the algorithm gives reasonable results for all values of α. However, for Example 2 we increased the size of the perturbation and then the violation

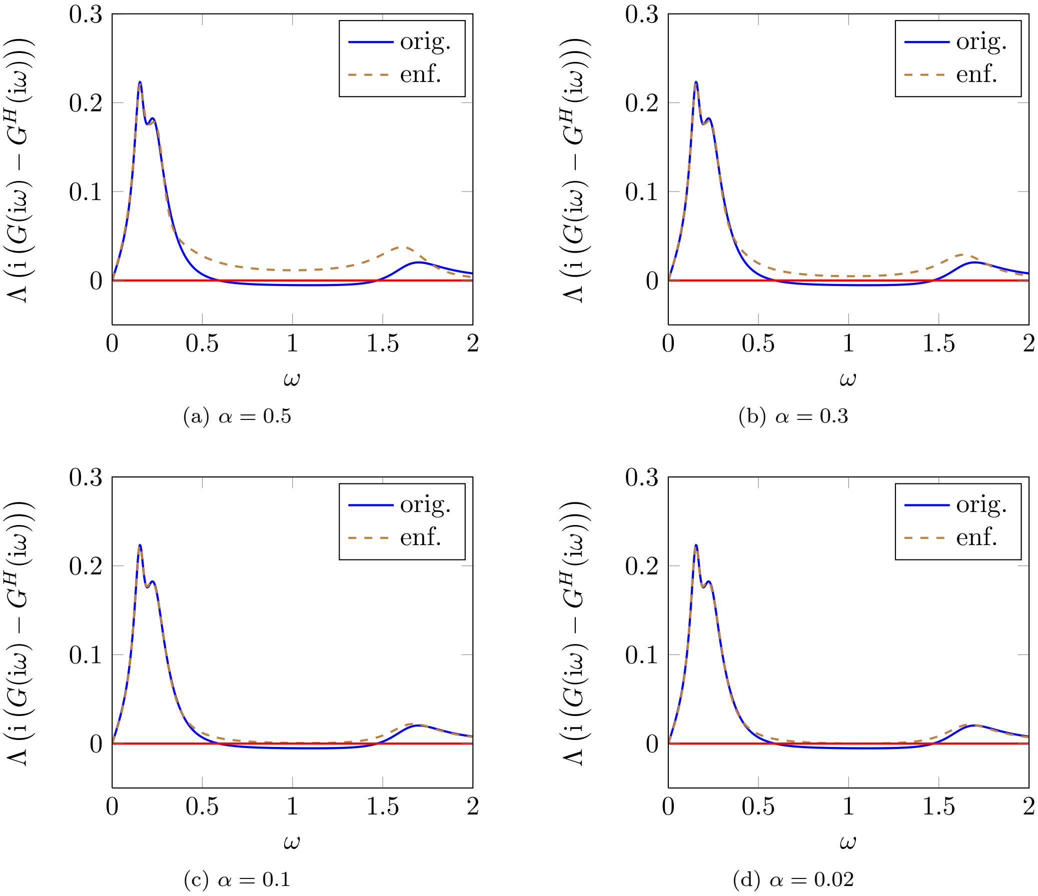


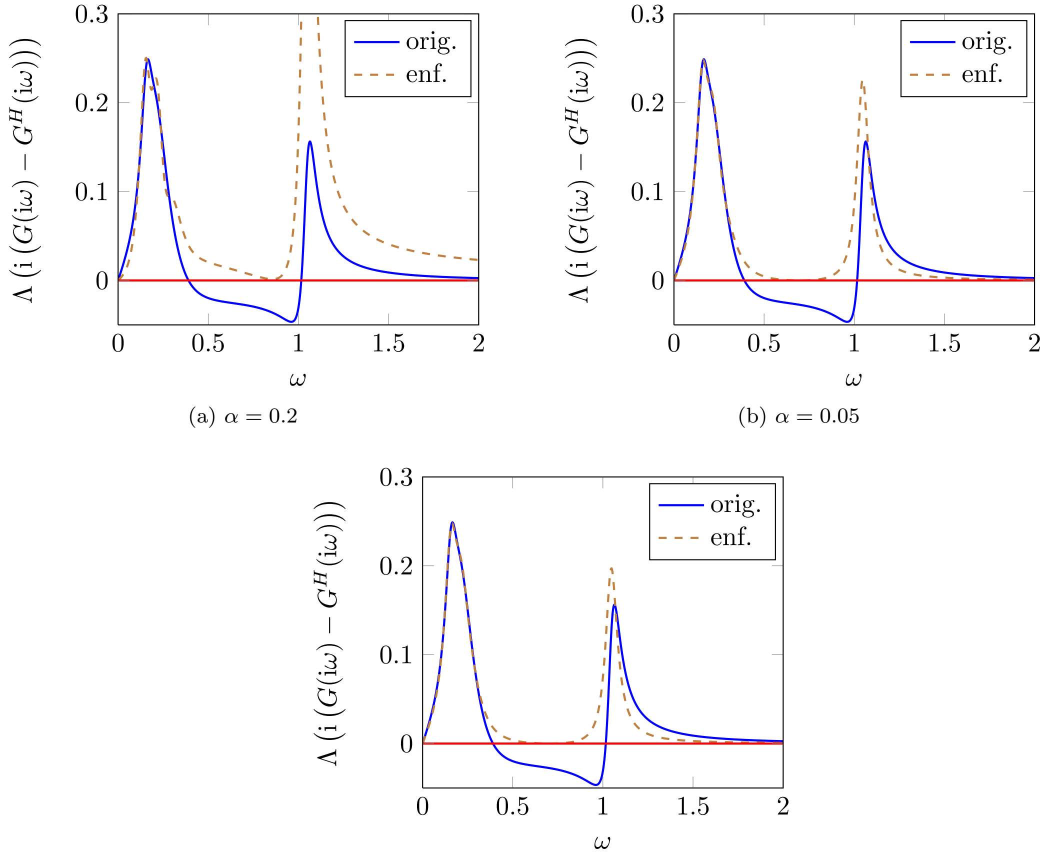
Figure 4: Results for enforcing negative imaginariness of example 1

of negative imaginariness is so large that the enforcement algorithm fails, if α is too large. We only get reasonable results if we further decrease α and make the perturbation of the system in each step sufficiently small. We observed that when we run the algorithm, it often happens that there occur additional negative imaginariness violations for larger frequencies. This is due to the fact that for these frequencies, the eigenvalues of $\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right)$ are already very close to zero and thus can be easily perturbed to negative values. Therefore, the algorithm has to enforce negative imaginariness (repeatedly) for these frequencies which drastically increases the iteration numbers for smaller α.

The numerical results are also depicted in Figures 4 and 5. For Example 1 we see that for larger values of α, we perform a slightly too large perturbation as the eigenvalue curves have some distance from the zero level. However, for smaller values of α this distance gets smaller and the approximation gets better. For Example 2 one can see that for $\alpha = 0.2$ we have a large error around $\omega = 1.1$ as there is a very high peak for the negative imaginary system. But again, as α decreases, also the size of the peak gets closer to the one of the original system and the approximation gets better.

5. Conclusions and Outlook

In this paper we have introduced the negative imaginary property for transfer functions related to descriptor systems. We have shown equivalent conditions for negative imaginariness in terms of the spectrum of a certain even matrix pencil. Therefore we have analyzed the corresponding even Kronecker canonical



(a) $\alpha = 0.2$

(b) $\alpha = 0.05$

(c) $\alpha = 0.01$

Figure 5: Results for enforcing negative imaginariness of Example 2

form of this pencil and showed that it has to fulfill a certain block structure. In the second part of the paper, we have introduced a numerical method for restoring negative imaginariness in the case that it has been lost when applying a system approximation algorithm such as done in model order reduction. This numerical method relies on the structure-preserving computation of the purely imaginary eigenvalues and associated eigenvectors of related skew-Hamiltonian/Hamiltonian matrix pencils. Finally, we have presented some numerical results and we have discussed the behavior of the enforcement algorithm. A future research topic might be the analysis of an negative imaginariness enforcement procedure which also allows the perturbation of other matrices than B_1 to obtain more accurate results.

References

[1] B. D. O. Anderson and S. Vongpanitlerd. *Network Analysis and Synthesis – A Modern Systems Theory Approach.* Prentice-Hall, Englewood Cliffs, N. J., 1973.

[2] D. Angeli. Systems with counterclockwise input-output dynamics. *IEEE Trans. Automat. Control*, 51(7):1130–1143, July 2006.

[3] A. C. Antoulas. *Approximation of Large-Scale Dynamical Systems.* SIAM Publications, Philadelphia, PA, 2005.

[4] P. Benner and R. Byers. Disk functions and their relationship to the matrix sign function. In *Proceedings of the European Control Conference 97*, Waterloo, Belgium, 1997. BELWARE Information Technology. Paper 936.

[5] P. Benner, R. Byers, V. Mehrmann, and H. Xu. Numerical Computation of Deflating Subspaces of Embedded Hamiltonian Pencils. Technical report, Chemnitz University of Technology, Faculty of Mathematics, Germany, June 1999. SFB393-Preprint 99-15.

[6] P. Benner, V. Mehrmann, and D. Sorensen, editors. *Dimension Reduction of Large-Scale Systems*, volume 45 of *Lecture Notes in Computational Science and Engineering*. Springer-Verlag, Berlin/Heidelberg, Germany, 2005.

[7] P. Benner and E. S. Quintana-Ortí. Solving stable generalized Lyapunov equations with the matrix sign function. *Numer. Algorithms*, 20(1):75–100, 1999.

[8] P. Benner and M. Voigt. On the computation of particular eigenvectors of Hamiltonian matrix pencils. *Proc. Appl. Math. Mech.*, 11(1):753–754, December 2011.

[9] C. Cai and G. Hagen. Coupling of stable subsystems with counterclockwise input-output dynamics. In *Proceedings of the 2010 American Control Conference*, pages 3458–3463, Baltimore, MD, 2010.

[10] D. Clements. A state-space approach to indefinite spectral factorization. *SIAM J. Matrix. Anal. Appl.*, 21(3):743–767, 2000.

[11] D. Clements and K. Glover. Spectral factorization via Hermitian pencils. 122–124:797–846, 1989.

[12] J. D. Cobb. Controllability, observability and duality in singular systems. *IEEE Trans. Automat. Control*, AC-29(12):1076–1082, December 1984.

[13] L. Dai. *Singular Control Systems*, volume 118 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Heidelberg, 1989.

[14] J. Demmel and B. Kågström. The generalized Schur decomposition of an arbitrary pencil $A - zB$: robust software with error bounds and applications. Part I: theory and algorithms. *ACM Trans. Math. Softw.*, 19(2):160–174, 1993.

[15] J. Demmel and B. Kågström. The generalized Schur decomposition of an arbitrary pencil $A - zB$: robust software with error bounds and applications. Part II: software and applications. *ACM Trans. Math. Softw.*, 19(2):175–201, 1993.

[16] R. W. Freund and F. Jarre. An extension of the positive real lemma to descriptor systems. *Optim. Methods Softw.*, 19(1):69–87, 2004.

[17] G. H. Golub and C. F. Van Loan. *Matrix Computations*. The John Hopkins University Press, Baltimore/London, third edition, 1996.

[18] S. Grivet-Talocia. Passivity enforcement via perturbation of Hamiltonian matrices. *IEEE Trans. Circuits and Systems*, 51(9):1755–1769, September 2004.

[19] S. Grivet-Talocia and A. Ubolli. On the generation of large passive macromodels for complex interconnect structures. *IEEE Trans. Adv. Pack.*, 29(1):39–54, February 2006.

[20] S. J. Hammarling. Numerical solution of the stable, non-negative definite Lyapunov equation. *IMA J. Numer. Anal.*, 2:303–323, 1982.

[21] B. Kågström and P. Van Dooren. Additive decomposition of a transfer function with respect to a specified region. In *Proceedings of MTNS-89*, volume 3, pages 469–477, Boston, 1990. Birkhäuser Boston Inc.

[22] B. Kågström and P. Van Dooren. A generalized state-space approach for the additive decomposition of a transfer matrix. Technical report, Institute of Information Processing, University of Umeå, Sweden, April 1991. Report UMINF-91.12.

[23] B. Kågström and L. Westin. GSYLV - Fortran routines for the generalized Schur method with Dif estimators for solving the generalized Sylvester equation. Technical report, Institute of Information Processing, University of Umeå, Sweden, July 1987. Report UMINF-132.86.

[24] B. Kågström and L. Westin. Generalized Schur methods with condition estimators for solving the generalized Sylvester equation. *IEEE Trans. Automat. Control*, 34:745–751, 1989.

[25] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations – Analysis and Numerical Solution*. Textbooks in Mathematics. European Mathematical Society, 2006.

[26] S. Lefteriu and A. C. Antoulas. A new approach to modeling multiport systems from frequency-domain data. *IEEE Trans. Comput. Aided Des. Integr. Circuits Syst.*, 29(1):14–27, January 2010.

[27] A. Loria and H. Nijmeijer. Passivity based control. *Control Systems, Robotics and Automation*, XIII.

[28] M. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon. Enforcing a system model to be negative imaginary via perturbation of Hamiltonian matrices. In *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference*, pages 3748–3752, Orlando, FL, USA, December 2011.

[29] M. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon. Spectral conditions for the negative imaginary property of transfer function matrices. In *Proceedings of the 18th IFAC World Congress*, pages 1302–1306, Milan, Italy, September 2011.

[30] V. Mehrmann and T. Stykel. Balanced truncation model reduction for large-scale systems in descriptor form. In Benner et al. [6], chapter 3, pages 89–116.

[31] I. R. Petersen and A. Lanzon. Feedback control of negative-imaginary systems. *IEEE Control Systems Magazine*, 30(5):54–72, October 2010. Keynote Survey Article.

[32] T. Reis. Lur'e equations and even matrix pencils. *Lin. Alg. Appl.*, 434:152–173, 2011.

[33] T. Reis and T. Stykel. Positive real and bounded real balancing for model reduction of descriptor systems. *Int. J. Control*, 83(1):74–88, 2010.

[34] H. H. Rosenbrock. Structural properties of linear dynamical systems. *Int. J. Control*, 20:191–202, 1974.

[35] C. Schröder. *Palindromic and Even Eigenvalue Problems — Analysis and Numerical Methods*. PhD thesis, Institut für Mathematik, Technische Universität Berlin, Germany, May 2008.

[36] C. Schröder and T. Stykel. Passivation of LTI systems. Technical report, DFG Research Center MATHEON, 2007. Technical Report 368-2007.

[37] G. W. Stewart and J.-G. Sun. *Matrix Perturbation Theory*. Academic Press, New York, 1990.

[38] T. Stykel. Gramian-based model reduction for descriptor systems. *Math. Control Signals Systems*, 16(4):297–319, 2004.

[39] T. Stykel. On some norms for descriptor systems. *IEEE Trans. Automat. Control*, 51(5):842–847, May 2006.

[40] X. Sun and E.S. Quintana-Ortí. Spectral division methods for block generalized Schur decompositions. *Mathematics of*

Computation, 73:1827–1847, 2004.
[41] R. Thompson. The characterisitic polynomial of a principal subpencil of a Hermitian matrix pencil. *Lin. Alg. Appl.*, 14:135–177, 1976.
[42] G. C. Verghese, B. C. Levy, and T. Kailath. A generalized state-space for singular systems. *IEEE Trans. Automat. Control*, AC-26(4):811–831, 1981.
[43] M. Voigt. $\mathcal{L}_\infty$-norm computation for descriptor systems, July 2010.
[44] J. Xiong, I. R. Petersen, and A. Lanzon. A negative imaginary lemma and the stability of interconnections of linear negative imaginary systems. *IEEE Trans. Automat. Control*, 55(10):2342–2347, October 2010.