Peter Benner Tobias Breiten

Two-sided moment matching methods for nonlinear model reduction

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Abstract

In this paper, we discuss a recently introduced approach for nonlinear model order reduction. The new method is motivated by the concept of moment matching known from model reduction techniques for linear systems and can be generalized by means of generalized transfer functions arising for a large class of smooth nonlinear control affine dynamical systems. We will extend the existing concepts by making use of some basic tools known from tensor theory. This will allow a more efficient computation of the reduced-order model as well as the possibility of constructing two-sided projection methods which are theoretically shown to yield more accurate reduced-order models. Moreover, we will test both, one-sided and two-sided projection methods, on several semi-discretized nonlinear partial differential equations which already have been used as test examples in the context of nonlinear model reduction and compare them with the common nonlinear reduction technique proper orthogonal decomposition. We will further point out the main advantages and drawbacks of our new method.

Keywords: nonlinear model reduction, quadratic-bilinear control systems, moment matching, Krylov subspaces, tensor calculus

Author's addresses:

Peter Benner
Computational Methods in Systems and Control Theory
Max Planck Institute for Dynamics of Complex Technical Systems
Sandtorstr. 1
39106 Magdeburg
Germany
(benner@mpi-magdeburg.mpg.de)

Tobias Breiten
Computational Methods in Systems and Control Theory
Max Planck Institute for Dynamics of Complex Technical Systems
Sandtorstr. 1
39106 Magdeburg
Germany
(breiten@mpi-magdeburg.mpg.de)

1 Introduction

One of the most important challenges in the field of numerical analysis certainly is the study and analysis of complex dynamical processes described by ordinary differential equations (ODEs) and/or partial differential equations (PDEs), respectively. Although computational power is increasing at vast rates, the fast simulation of complex dynamical systems still often is to resource-intensive for the fine granularity of models necessary for an understanding of real-life applications in full detail. In particular, in order to solve a certain PDE numerically, one often starts out by a spatial discretization which leads to a large-scale system of ODEs. However, the number of state variables of such a system easily might exceed dimensions up to $\mathcal{O}(10^5)$, making a fast and reliable simulation hardly possible. Particularly, in a many query context, e.g. a design study, it is necessary to simulate the system for varying forcing terms. Here, model order reduction (MOR) can be used to significantly accelerate the repeated simulation. Although far from being a trivial task, theory as well as numerical methods for linear systems are quite well-established and recently more and more interest is dedicated to nonlinear control systems of the form

$$\Sigma_{NL}: \quad \begin{cases} \dot{x}(t) = f(x(t)) + bu(t), \\ y(t) = c^T x(t), \quad x_0 = 0, \end{cases} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear state evolution function and $b,\, c \in \mathbb{R}^n$ denote the input and output vector, respectively. Moreover, $x(t),\, u(t),\, y(t) \in \mathbb{R}^n$ are called the state, input and output of the system, respectively. The term $bu(t)$ often is obtained after spatial discretization of a PDE from a source term of the form $S(x,t)$ by separation of variables $S(x,t) = b(x)u(t)$. In general, the initial state of the system x_0 does not have to be zero. However, since all the concepts rely on this fact, throughout the paper, we will assume that $x_0 = 0$. Nevertheless, if this is not the case, one can always transform the above system by introducing a reference state variable $\tilde{x} = x - x_0$ which fulfills this condition such that this is no restriction for more general systems. As already mentioned above, if the state dimension n becomes too large, one usually is interested in a reduced order model of the same structure

$$\Sigma_{NLR}: \quad \begin{cases} \dot{\hat{x}}(t) = \hat{f}(\hat{x}(t)) + \hat{b}u(t), \\ \hat{y}(t) = \hat{c}^T \hat{x}(t), \quad \hat{x}_0 = 0, \end{cases} \tag{2}$$

with $\hat{f} : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{n}}$, $\hat{b},\, \hat{c} \in \mathbb{R}^{\hat{n}}$ and $\hat{n} \ll n$. In contrast to linear systems, one of the main difficulties clearly is the construction of a reduced evolution function $\hat{f}$. Trajectory-based methods like proper orthogonal decomposition (POD), see e.g. [2, 6, 8, 15, 16], rely on a Galerkin projection $\mathcal{P} = \mathcal{V}\mathcal{V}^T$

and compute $\hat{f} = \mathcal{V}^T f(\mathcal{V}\hat{x})$. While this definitely preserves the nonlinear structure of the original system, it also displays a major bottleneck of the classic POD approach. To be more precise, note that the function f still has to be evaluated on the original state space $\mathbb{R}^n$, making the simulation of the reduced-order system too expensive. However, there exist several ways to circumvent this problem, e.g., the empirical interpolation method (EIM), missing point estimation (MPE), best points interpolation method (BPIM) and the discrete empirical interpolation method (DEIM). For those methods, we refer to, e.g., [2, 4, 8, 9, 18]. We refer to e.g. [6, 8, 15, 16], for a detailed discussion on POD. Motivated by the same idea, the reduced basis method is a further popular and successful approach in the context of nonlinear model order reduction, see e.g. [4, 9].

Another way is to replace the nonlinearity by a weighted combination of linear systems which then can be efficiently treated by well-known linear reduction methods like balanced truncation or interpolation (see [1]). For a more detailed insight into the resulting trajectory piecewise linear (TPWL) method, the reader is referred to [21], where more information can be found.

So far, the above mentioned methods all share the common drawback of input dependency, i.e., in order to construct a reduced-order model one at first needs some snapshots of a given or computed solution trajectory of the original model. If this has been done, one indeed can get very accurate approximations of the system. However, as soon as the input function is varied, which is common in control, optimization and design problems, no rigorous assertions on the error for the new dynamics can be specified. In this paper, we will pick up a method which extends the concept of interpolation or moment matching, respectively, discussed for linear systems in, e.g., [10]. The main idea was introduced in [11], where the author shows how to transform a specific class of nonlinear control systems into a system of so-called quadratic-bilinear differential algebraic equations (QBDAEs). For those, in [11] an approximation procedure based on generalized moment matching about the interpolation point 0 was discussed and evaluated by means of some typical numerical test examples in the context of nonlinear model reduction. Basically, the method can be seen as a suitable extension of ideas which have been discussed for systems with a similar structure in, e.g., [3, 5, 20, 19]. The main advantage of the approach is that it tries to construct a reduced-order model that aims at capturing the input-output behavior of the underlying system, making it input independent and thus allowing to use the reduced-order model for varying controls.

The structure of the paper now is as follows. In the next section, we will state the main properties of quadratic-bilinear differential algebraic equations. This will include a brief review on the concept of variational analysis which allows to replace the nonlinear system by a nested sequence of pseudo-linear subsystems and subsequently opens up the possibility to derive generalized transfer functions. In Section 3, we then recall some tools from tensor theory. This will be helpful in order to improve the computation and the accuracy of the reduced-order model. The main result then is proven in Section 4, where we will see how to construct appropriate two-sided projection methods for quadratic-bilinear differential algebraic equations. Finally, we will carefully implement and test some numerical examples in Section 5 and underline advantages and difficulties of the new approach. We conclude with a summary and an outlook for topics of further research.

2 Quadratic-bilinear DAEs

In this section, we will review the basic properties of systems of quadratic-bilinear differential algebraic equations (QBDAEs). These systems are of the form

$$\Sigma_{QB}: \quad \begin{cases} E\dot{x}(t) = Ax(t) + Hx(t)\otimes x(t) + Nx(t)u(t) + bu(t), \\ \quad y(t) = c^T x(t), \quad x_0 = 0, \end{cases} \tag{3}$$

where $E, A, N \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{n\times n^2}$, and $b, c \in \mathbb{R}^n$. Analog to more general nonlinear systems of the form (1), here $u(t), y(t) \in \mathbb{R}$ are input and output variables, respectively. At this point, note the special structure of the matrix H which denotes the Hessian of the right hand side. Due to commutativity of the variables in $x(t)\otimes x(t)$, it is always possible to arrange the entries of H such that the commutativity is handed over to the matrix itself. To be more precise, for two arbitrary vectors $u, v \in \mathbb{R}^n$, we have $H\,(u\otimes v) = H\,(v\otimes u)$. Since this concept will be important later on, we will study a simple example which underscores the main idea.

Example 2.1. *Let us consider a two-dimensional purely quadratic system of the form*

$$\dot{x}(t) = Hx(t)\otimes x(t), \qquad \textit{with } H = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}.$$

Writing down the dynamics explicitly, we obtain

$$\begin{aligned} \dot{x}_1(t) &= ax_1(t)^2 + bx_1(t)x_2(t) + cx_2(t)x_1(t) + dx_2(t)^2, \\ \dot{x}_2(t) &= ex_1(t)^2 + fx_1(t)x_2(t) + gx_2(t)x_1(t) + hx_2(t)^2. \end{aligned}$$

Using $j = \frac{b+c}{2}$ and $k = \frac{f+g}{2}$, the above system is equivalent to

$$\begin{aligned} \dot{x}_1(t) &= ax_1(t)^2 + jx_1(t)x_2(t) + jx_2(t)x_1(t) + dx_2(t)^2, \\ \dot{x}_2(t) &= ex_1(t)^2 + kx_1(t)x_2(t) + kx_2(t)x_1(t) + hx_2(t)^2. \end{aligned}$$

Hence, we can replace H *by* $\tilde{H} = \begin{bmatrix} a & j & j & d \\ e & k & k & h \end{bmatrix}$. *However, one now easily observes that for arbitrary* $u, v \in \mathbb{R}^2$, *it holds*

$$\tilde{H}\,(u \otimes v) = \tilde{H}\,(v \otimes u) = \begin{bmatrix} au_1v_1 + ju_1v_2 + ju_2v_1 + ku_2v_2 \\ eu_1v_1 + ku_1v_2 + ku_2v_1 + fu_2v_2 \end{bmatrix}.$$

Obviously, the above also holds true for arbitrary $n > 2$.

As has already been shown in [11], QBDAEs are very useful in the context of nonlinear model order reduction. In particular, a large class of smooth nonlinear control affine systems can be transformed into a system of QBDAEs. This is done via introducing new state variables for the occurring nonlinearities of the underlying control system. The new dynamics then can be derived by symbolic differentiation or adding algebraic constraints. For a more detailed discussion on this topic, we refer to [11]. However, it should be mentioned that this transformation concept has already been known as McCormick-relaxation for several years, see [17]. The fact that the idea has not been used for model reduction purposes might be surprising. On the other hand, at a first glance it seems counterintuitive to first increase the state dimension of a control system which actually should be reduced.

Before we proceed with the concepts of variational analysis for these systems, we will mention some differences to the theory discussed in [11]. There the author includes a further term of the form

$$Lx(t) \otimes x(t)u(t), \quad L \in \mathbb{R}^{n \times n^2}.$$

However, although it might further increase the state dimension of a transformed system, it should be emphasized that by introducing a new state variable $z(t) := x(t) \otimes x(t)$, the nonlinearity becomes purely bilinear, i.e. $Lz(t)u(t)$. Since this simplifies the structure of the transfer functions that will be introduced in the following, we will always assume that the system under consideration does not contain multiplicative couplings of quadratic and bilinear variables. Moreover, in [11], the systems are denoted as quadratic-linear since the state variable $x(t)$ appears quadratically while the input variable appears linearly. On the other hand, one can interpret system (3) as a combination of a purely quadratic system and a bilinear control system, justifying the notation QBDAE.

Let us now turn our attention to the analysis of QBDAEs. Even for more general nonlinear systems, it is well-known (see e.g. [22]) that instead of solving (3), one can iteratively look for solutions of a sequence of pseudo linear systems of the form

$$
\begin{aligned}
E\dot{x}_1(t) &= Ax_1(t) + bu(t),\\
E\dot{x}_2(t) &= Ax_2(t) + A_2x_1(t)\otimes x_1(t) + Nx_1(t)u(t),\\
E\dot{x}_3(t) &= Ax_3(t) + A_2\left(x_1(t)\otimes x_2(t) + x_2(t)\otimes x_1(t)\right) + Nx_2(t)u(t),\\
&\vdots
\end{aligned}
$$

where, e.g., in the 2nd system, $x_1(t)$ is used as additional input for the linear ODE defining x_2. The solution $x(t)$ of (3) then can be derived as $x(t) = \sum_{i=1}^{\infty} x_i$. For the previous approach, one assumes that the nonlinear system under consideration consists of a series of homogeneous subsystems, meaning that the transient response to an input of the form $\alpha u(t)$ is given as

$$
x(t) = \alpha x_1(t) + \alpha^2 x_2(t)^2 + \alpha^3 x_3(t)^3 + \ldots,
$$

where the x_i are given as the solution of the above sequence.

A similar technique allows an input-output characterization in the frequency domain. According to [22], if one is interested in the first two transfer functions of (3), one can consider an input $u(t) = e^{s_1t} + e^{s_2t}$ which is supposed to yield a transient response

$$
x(t) = H_{10}e^{s_1t} + H_{01}e^{s_2t} + H_{20}e^{2s_1t} + H_{02}e^{2s_2t} + H_{11}e^{(s_1+s_2)t}.
$$

Inserting this expression into the state equation (3) and comparing the coefficients then leads to the first two generalized *symmetric* transfer functions

$$
\begin{aligned}
G_1(s_1) &= c^T \underbrace{(s_1E - A)^{-1}}_{F(s_1)} b,\\
G_2(s_1,s_2) &= \frac{1}{2}c^T\left((s_1+s_2)E - A\right)^{-1} H\left(F(s_1)\otimes F(s_2) + F(s_2)\otimes F(s_1)\right)\\
&\quad + \frac{1}{2}c^T\left((s_1+s_2)E - A\right)^{-1} N\left(F(s_1) + F(s_2)\right)\\
&= c^T\left((s_1+s_2)E - A\right)^{-1} H\left(F(s_1)\otimes F(s_2)\right)\\
&\quad + \frac{1}{2}c^T\left((s_1+s_2)E - A\right)^{-1} N\left(F(s_1) + F(s_2)\right).
\end{aligned}
$$

Similarly, one can derive higher order transfer functions, see e.g. [11, 22].

3 Tensors and matricizations

In this section, we now want to briefly review some concepts known from tensor theory, see e.g. [14]. This will be helpful in understanding the structure of H and will yield some properties that are beneficial for model reduction purposes. Although the following ideas exist for arbitrary order, here it will suffice to stick to the three-dimensional tensor case. Recall that in quadratic-bilinear systems of the form (3), the matrix $H \in \mathbb{R}^{n \times n^2}$ corresponds to the terms of second order and thus denotes the Hessian tensor of the right hand side. Moreover, we can interpret H as the so-called *matricization* of a tensor $\mathcal{H} \in \mathbb{R}^{n^3}$. To be more specific, a tensor $\mathcal{H}$ is a vector indexed by a product index set

$$\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3, \quad \#\mathcal{I}_j = n.$$

For such a tensor $\mathcal{H}$, the t-matricization $\mathcal{H}^{(t)}$ is defined as

$$\mathcal{H}^{(t)} \in \mathbb{R}^{\mathcal{I}_t \times \mathcal{I}_{t'}}, \;\; \mathcal{H}^{(t)}_{(i_\mu)\mu\in t,\,(i_\mu)\mu\in t'} := \mathcal{H}_{(i_1,i_2,i_3)}, \quad t' := \{1,2,3\}\backslash t.$$

Example 3.1. *For a given 3-tensor* $\mathcal{H}_{(i_1,i_2,i_3)}$ *with* $i_1, i_2, i_3 \in \{1, 2\}$, *we have the following matricizations:*

$$\mathcal{H}^{(1)} = \begin{bmatrix} \mathcal{H}_{(1,1,1)} & \mathcal{H}_{(1,2,1)} & \mathcal{H}_{(1,1,2)} & \mathcal{H}_{(1,2,2)} \\ \mathcal{H}_{(2,1,1)} & \mathcal{H}_{(2,2,1)} & \mathcal{H}_{(2,1,2)} & \mathcal{H}_{(2,2,2)} \end{bmatrix},$$
$$\mathcal{H}^{(2)} = \begin{bmatrix} \mathcal{H}_{(1,1,1)} & \mathcal{H}_{(2,1,1)} & \mathcal{H}_{(1,1,2)} & \mathcal{H}_{(2,1,2)} \\ \mathcal{H}_{(1,2,1)} & \mathcal{H}_{(2,2,1)} & \mathcal{H}_{(1,2,2)} & \mathcal{H}_{(2,2,2)} \end{bmatrix},$$
$$\mathcal{H}^{(3)} = \begin{bmatrix} \mathcal{H}_{(1,1,1)} & \mathcal{H}_{(2,1,1)} & \mathcal{H}_{(1,2,1)} & \mathcal{H}_{(2,2,1)} \\ \mathcal{H}_{(1,1,2)} & \mathcal{H}_{(2,1,2)} & \mathcal{H}_{(1,2,2)} & \mathcal{H}_{(2,2,2)} \end{bmatrix}.$$

In context of so-called multimoment matching of the transfer functions of a quadratic-bilinear system, in the next section, we will be faced with terms of the form $w^T H(u \otimes v)$, with $u, v, w \in \mathbb{R}^n$. At this point recall the symmetric structure of the Hessian tensor discussed in Section 2. There we have seen that, due to the structure of the terms in $x(t) \otimes x(t)$, we can always rearrange the entries of H in such a way that it holds

$$H(u \otimes v) = H(v \otimes u), \tag{4}$$

for arbitrary u and v. Moreover, in terms of the above notation, if we associate H with the 1-matricization of $\mathcal{H} \in \mathbb{R}^{n^3}$, it follows that the two remaining matricizations of the underlying tensor coincide, i.e $\mathcal{H}^{(2)} = \mathcal{H}^{(3)}$ and, thus,

$$w^T H(u \otimes v) = u^T \mathcal{H}^{(2)}(v \otimes w) = u^T \mathcal{H}^{(3)}(v \otimes w).$$

The above identity will be the crucial tool in constructing two-sided projection methods for reducing a quadratic-bilinear control system of the form (3). Furthermore, interpretation of H as a matricization of an underlying tensor $\mathcal{H}$ has the additional advantage of computing the reduced-order system in a beneficial way. Let us consider two matrices $\mathcal{V}, \mathcal{W} \in \mathbb{R}^{n \times \hat{n}}$, with orthonormal columns. The reduced-order model then is given as

$$\hat{E} = \mathcal{W}^T E \mathcal{V},\; \hat{A} = \mathcal{W}^T A \mathcal{V},\; \hat{H} = \mathcal{W}^T H(\mathcal{V} \otimes \mathcal{V}),\; \hat{b} = \mathcal{W}^T b,\; \hat{c} = \mathcal{V}^T c.$$

However, since $\mathcal{V}$ usually is a dense matrix, building $\mathcal{V}\otimes\mathcal{V}$ can easily become computationally infeasible. Note that storing $\mathcal{V}\otimes\mathcal{V}$ requires $\mathcal{O}(n^2\cdot\hat{n}^2)$ memory. One way out obviously is given by the splitting

$$\mathcal{V}\otimes\mathcal{V}=(\mathcal{V}\otimes I_n)(I_{\hat{n}}\otimes\mathcal{V}),$$

which will reduce the necessary storage complexity to $\mathcal{O}(n^2\cdot\hat{n}+n\cdot\hat{n}^2)$. However, this still might be too expensive. On the other hand, if we use the matricization idea, we can start with computing $H_{\mathcal{W}}=\mathcal{W}^T H\in\mathbb{R}^{\hat{n}\times n^2}$ and then proceed with reshaping $H_{\mathcal{W}}$ into one of the two remaining matricizations. This will lead to a matrix $\tilde{H}_{\mathcal{W}}\in\mathbb{R}^{n\times\hat{n}n}$ that we might multiply with $\mathcal{V}^T$ from the left in order to obtain $\bar{H}_{\mathcal{V}\mathcal{W}}\in\mathbb{R}^{\hat{n}\times\hat{n}n}$. Similarly, we can now repeat this process and reshape $\bar{H}_{\mathcal{V}\mathcal{W}}$ into the last matricization, followed by a multiplication with $\mathcal{V}^T$. Finally, if we reshape the result into the first matricization again, we end up with the same matrix we would have obtained with $\mathcal{W}^T H(\mathcal{V}\otimes\mathcal{V})$. Hence, we can compute the reduced system Hessian without ever explicitly forming the matrix $\mathcal{V}\otimes\mathcal{V}$, leading to a storage complexity of only $\mathcal{O}(n\cdot\hat{n})$.

4 Two-sided multimoment matching

In order to simplify the notation in this section, we will introduce the following two definitions concerning rational Krylov subspaces.

Definition 4.1. *Let $E,A\in\mathbb{R}^{n\times n}$, $b\in\mathbb{R}^n$, $q\in\mathbb{N}$ and $\sigma\in\mathbb{C}$. Then we define the associated* rational Krylov subspace *as*

$$\mathcal{K}_q\left(E,A,b,\sigma\right):=\mathcal{K}_q\left((\sigma E-A)^{-1}E,(\sigma E-A)^{-1}b\right).$$

Definition 4.2. *Let $E,A\in\mathbb{R}^{n\times n}, j\in\mathbb{N}$ and $\sigma\in\mathbb{C}$. Then we define*

$$\mathcal{A}^j_{E,\sigma}:=\left((\sigma E-A)^{-1}E\right)^j(\sigma E-A)^{-1}$$

and

$$\mathcal{A}^{T,j}_{E,\sigma}:=\left((\sigma E^T-A^T)^{-1}E^T\right)^j(\sigma E^T-A^T)^{-1}.$$

Let us now come back to the actual topic of model order reduction. Recall that for a given set of nonlinear dynamical equations of the form (1), we want to construct an approximation which fulfills $\hat{y}(t)\approx y(t)$. As we have seen in Section 2, instead of the general nonlinear system Σ_{NL}, for a large class of systems, we might consider a transformed equation of quadratic-bilinear structure (3). Following the previous discussion, we know that the input-output behavior in frequency domain can be characterized via an infinite series of nested transfer functions

$$G_1(s_1),\ G_2(s_1,s_2),\ G_3(s_1,s_2,s_3),\ldots$$

Consequently, if we can ensure that, up to a given prespecified order q, it holds

$$G_i(s_1,\dots,s_i) \approx \hat{G}_i(s_1,\dots,s_i), \quad i = 1,\dots,q,$$

we can certainly expect the outputs of the original and the reduced-order model to be similar. Hence, let us have a closer look at the structure of $G_i(s_1,\dots,s_i)$. Similar to [11], here we will restrict ourselves to the first two transfer functions G_1 and G_2. Recall from the linear case that for a given point σ, we can locally expand G_1 in a Taylor series, see e.g. [1, 10]. In more detail, we have

$$G_1(s_1) = \sum_{i=0}^{\infty} m_i (s_1 - \sigma)^i \tag{5}$$

with so-called moments $m_i = (-1)^i \cdot c^T \mathcal{A}_{E,\sigma}^i b$. Thus, if we construct a reduced-order system such that some of its moments coincide with the original system, i.e. $m_i = \hat{m}_i,\ i = 1,\dots,q$, the transfer functions of both systems locally should be equal. To be more specific, since the moments m_i are the derivatives of G_1 evaluated at σ, we have

$$\frac{\partial^i G_1}{\partial s_1^i}(\sigma) = \frac{\partial^i \hat{G}_1}{\partial s_1^i}(\sigma), \quad i = 1,\dots,q. \tag{6}$$

Similarly, we can expand the second transfer function of a quadratic-bilinear system. Although there exists a lot of freedom in choosing a pair (σ_1,σ_2) of interpolations points, here we stick to the case were both points coincide, i.e. $\sigma_1 = \sigma_2 = \sigma$. Since the physical meaning of the frequency variables s_1 and s_2 is ambiguous anyway, this is not a too severe restriction. Moreover, in the procedure described in Theorem 4.1, this assumption will allow to recycle vectors for certain Krylov subspaces and thus reduce the required complexity of the resulting algorithm. Accordingly, we then obtain the following multivariate Taylor expansion of the second transfer function

$$\begin{aligned} G_2(s_1,s_2) = &\sum_{i,j,k} m_{i,j,k} (s_1+s_2-2\sigma)^i (s_1-\sigma)^j (s_2-\sigma)^k \\ &+ \sum_{i,\ell_1,\ell_2} m_{i,\ell_1,\ell_2} (s_1+s_2-2\sigma)^i \left((s_1-\sigma)^l + (s_2-\sigma)^m \right), \end{aligned}$$

with multimoments given as

$$\begin{aligned} m_{i,j,k} &= (-1)^{i+j+k+1} \cdot \frac{1}{2} c^T \mathcal{A}_{E,2\sigma}^i \; H \left(\mathcal{A}_{E,\sigma}^j b \otimes \mathcal{A}_{E,\sigma}^k b + \mathcal{A}_{E,\sigma}^k b \otimes \mathcal{A}_{E,\sigma}^j b \right), \\ m_{i,\ell_1,\ell_2} &= (-1)^{i+\ell_1} \cdot \frac{1}{2} c^T \mathcal{A}_{E,2\sigma}^i \; N \mathcal{A}_{E,\sigma}^{\ell_1} b + (-1)^{i+\ell_2} \frac{1}{2} \cdot c^T \mathcal{A}_{E,2\sigma}^i \; N \mathcal{A}_{E,\sigma}^{\ell_2} b. \end{aligned}$$

Analog to the transfer function G_1 of the linear subsystem, it is easily seen that $m_{i,j,k}$ and m_{i,ℓ_1,ℓ_2} basically determine the derivatives of the second transfer function G_2. Hence, it seems reasonable to construct a reduced-order system in such a way that for a given pair of interpolation points (σ, σ), the derivatives of $\hat{G}_2$ coincide with those of the original transfer function up to a certain order q. The following result now states how to choose an appropriate sequence of nested Krylov subspaces that extends the known results for one-sided projections specified in [11].

Theorem 4.1. *Let* $\Sigma = (E, A, H, N, b, c)$ *denote a system of quadratic-bilinear differential algebraic equations of dimension* n. *Let* $q_1, q_2 \in \mathbb{N}$ *with* $q_2 \leq q_1$. *Assume that a reduced QBDAE system is constructed by a Petrov-Galerkin type projection*

$$\hat{E} = \mathcal{W}^T E \mathcal{V}, \quad \hat{A} = \mathcal{W}^T A \mathcal{V}, \quad \hat{H} = \mathcal{W}^T H \mathcal{V} \otimes \mathcal{V},$$
$$\hat{N} = \mathcal{W}^T N \mathcal{V}, \quad \hat{b} = \mathcal{W}^T b, \quad \hat{c} = \mathcal{V}^T c,$$

where $\operatorname{span}(\mathcal{V})$ *and* $\operatorname{span}(\mathcal{W})$ *are orthonormal bases for the union of the following column spaces*

$$
\begin{aligned}
&V_1 = \mathcal{K}_{q_1}(E, A, b, \sigma), \quad W_1 = \mathcal{K}_{q_1}\left(E^T, A^T, c, 2\sigma\right)\\
&\textbf{for } i = 1 : q_2\\
&\quad V_2^i = \mathcal{K}_{q_2-i+1}(E, A, NV_1(:,i), 2\sigma)\\
&\quad W_2^i = \mathcal{K}_{q_2-i+1}\left(E^T, A^T, N^T W_1(:,i), \sigma\right)\\
&\quad\quad \textbf{for } j = 1 : \min(q_2 - i + 1, i)\\
&\quad\quad\quad V_3^{i,j} = \mathcal{K}_{q_2-i-j+2}(E, A, HV_1(:,i) \otimes V_1(:,j), 2\sigma)\\
&\quad\quad\quad W_3^{i,j} = \mathcal{K}_{q_2-i-j+2}\left(E^T, A^T, \mathcal{H}^{(2)} V_1(:,i) \otimes W_1(:,j), \sigma\right),
\end{aligned}
$$

i.e.,

$$\operatorname{span}(\mathcal{V}) = \operatorname{span}(V_1) \cup \bigcup_i \operatorname{span}\left(V_2^i\right) \cup \bigcup_{i,j} \operatorname{span}\left(V_3^{i,j}\right),$$
$$\operatorname{span}(\mathcal{W}) = \operatorname{span}(W_1) \cup \bigcup_i \operatorname{span}\left(W_2^i\right) \cup \bigcup_{i,j} \operatorname{span}\left(W_3^{i,j}\right).$$

Then it holds:

$$\frac{\partial^i G_1}{\partial s_1^i}(\sigma) = \frac{\partial^i \hat{G}_1}{\partial s_1^i}(\sigma), \quad \frac{\partial^i G_1}{\partial s_1^i}(2\sigma) = \frac{\partial^i \hat{G}_1}{\partial s_1^i}(2\sigma), \quad i = 0, \ldots, q_1 - 1,$$
$$\frac{\partial^{i+j}}{\partial s_1^i s_2^j} G_2(\sigma) = \frac{\partial^{i+j}}{\partial s_1^i s_2^j} \hat{G}_2(\sigma), \qquad i + j \leq 2q_2 - 1.$$

Proof. The assertion for the first transfer function immediately follows from known moment matching results for linear systems, see e.g. [1]. Hence, we only have to consider the second transfer function. Here, it suffices to focus on the contributions of the quadratic part of the system. For the bilinear contributions, we refer to e.g. [5], where two-sided multimoment matching for these systems is studied. Using that

$$\frac{\partial}{\partial y}\left(A(y)^{-1}\right) = -A(y)^{-1}\frac{\partial A(y)}{\partial y}A(y)^{-1},$$

aside from constant factors, we thus have to concentrate on terms of the form

$$c^T \mathcal{A}_{E,2\sigma}^{j} H\left(\mathcal{A}_{E,\sigma}^{k}\, b \otimes \mathcal{A}_{E,\sigma}^{\ell} b\right),$$

with $j+k+\ell \leq 2q_2 - 1$ and, w.l.o.g., $k \geq \ell$. From the results for the first transfer function, we know that

$$\mathcal{V}\hat{\mathcal{A}}_{\hat{E},\sigma}^{i}\hat{b} = \mathcal{A}_{E,\sigma}^{i} b, \quad \mathcal{W}\hat{\mathcal{A}}_{\hat{E}^T,\sigma}^{T,i}\hat{c} = \mathcal{A}_{E,\sigma}^{T,i} c, \tag{7}$$

for $i = 1,\ldots,q_1 - 1$. This yields the statement for $j,k,\ell \leq q_2 - 1$. Let us now assume that $j = 2q_2 - 1, k = \ell = 0$. Note that we have

$$\mathcal{V}\mathcal{V}^T \mathcal{A}_{E,2\sigma}^{0} H\left(\mathcal{A}_{E,\sigma}^{0} b \otimes \mathcal{A}_{E,\sigma}^{0} b\right) = \mathcal{A}_{E,2\sigma}^{0} H\left(\mathcal{A}_{E,\sigma}^{0} b \otimes \mathcal{A}_{E,\sigma}^{0} b\right). \tag{8}$$

This follows from the construction of span $(\mathcal{V})$ and the property of $\mathcal{V}$ being orthonormal. Next, it holds

$$\begin{aligned}
&\mathcal{V}\hat{\mathcal{A}}_{\hat{E},2\sigma}^{0}\hat{H}\left(\hat{\mathcal{A}}_{\hat{E},\sigma}^{0}\hat{b} \otimes \hat{\mathcal{A}}_{\hat{E},\sigma}^{0}\hat{b}\right)\\
&\quad = \mathcal{V}\hat{\mathcal{A}}_{\hat{E},2\sigma}^{0}\mathcal{W}^T H\left(\mathcal{V}\hat{\mathcal{A}}_{\hat{E},\sigma}^{0}\hat{b} \otimes \mathcal{V}\hat{\mathcal{A}}_{\hat{E},\sigma}^{0}\hat{b}\right)\\
&\quad = \mathcal{V}\hat{\mathcal{A}}_{\hat{E},2\sigma}^{0}\mathcal{W}^T H\left(\mathcal{A}_{E,\sigma}^{0} b \otimes \mathcal{A}_{E,\sigma}^{0} b\right)\\
&\quad = \mathcal{V}\hat{\mathcal{A}}_{\hat{E},2\sigma}^{0}\mathcal{W}^T \left(\mathcal{A}_{E,2\sigma}^{0}\right)^{-1}\mathcal{A}_{E,2\sigma}^{0} H\left(\mathcal{A}_{E,\sigma}^{0} b \otimes \mathcal{A}_{E,\sigma}^{0} b\right)\\
&\quad = \mathcal{V}\hat{\mathcal{A}}_{\hat{E},2\sigma}^{0}\mathcal{W}^T \left(\mathcal{A}_{E,2\sigma}^{0}\right)^{-1}\mathcal{V}\mathcal{V}^T\mathcal{A}_{E,2\sigma}^{0} H\left(\mathcal{A}_{E,\sigma}^{0} b \otimes \mathcal{A}_{E,\sigma}^{0} b\right)\\
&\quad = \mathcal{V}\mathcal{V}^T\mathcal{A}_{E,2\sigma}^{0} H\left(\mathcal{A}_{E,\sigma}^{0} b \otimes \mathcal{A}_{E,\sigma}^{0} b\right)\\
&\quad = \mathcal{A}_{E,2\sigma}^{0} H\left(\mathcal{A}_{E,\sigma}^{0} b \otimes \mathcal{A}_{E,\sigma}^{0} b\right).
\end{aligned}$$

With the same arguments, one can iteratively show that

$$\mathcal{V}\hat{\mathcal{A}}_{\hat{E},2\sigma}^{i}\hat{H}\left(\hat{\mathcal{A}}_{\hat{E},\sigma}^{0}\hat{b} \otimes \hat{\mathcal{A}}_{\hat{E},\sigma}^{0}\hat{b}\right) = \mathcal{A}_{E,2\sigma}^{i} H\left(\mathcal{A}_{E,\sigma}^{0} b \otimes \mathcal{A}_{E,\sigma}^{0} b\right), \tag{9}$$

for $i = 0,\ldots,q_2 - 1$. Hence, let us consider

$$\hat{c}^T\hat{\mathcal{A}}_{\hat{E},2\sigma}^{2q_2-1}\hat{H}\left(\hat{\mathcal{A}}_{\hat{E},\sigma}^{0}\hat{b} \otimes \hat{\mathcal{A}}_{\hat{E},\sigma}^{0}\hat{b}\right).$$

By Definition 4.2, we have

$$\begin{aligned}\hat{\mathcal{A}}_{\hat{E},2\sigma}^{2q_2-1} &= \left((2\sigma\hat{E}-\hat{A})^{-1}\hat{E}\right)^{q_2-1}\left((2\sigma\hat{E}-\hat{A})^{-1}\hat{E}\right)\left((2\sigma\hat{E}-\hat{A})^{-1}\hat{E}\right)^{q_2-1} \\ &= \hat{\mathcal{A}}_{\hat{E},2\sigma}^{q_2-1}\mathcal{W}^T E\mathcal{V}\hat{\mathcal{A}}_{\hat{E},2\sigma}^{q_2-1}.\end{aligned}$$

Thus, it follows

$$\hat{c}^T\hat{\mathcal{A}}_{\hat{E},2\sigma}^{2q_2-1}\hat{H}\left(\hat{\mathcal{A}}_{\hat{E},\sigma}^{0}\hat{b}\otimes\hat{\mathcal{A}}_{\hat{E},\sigma}^{0}\hat{b}\right) = \hat{c}^T\hat{\mathcal{A}}_{\hat{E},2\sigma}^{q_2-1}\mathcal{W}^T E\mathcal{V}\hat{\mathcal{A}}_{\hat{E},2\sigma}^{q_2-1}\hat{H}\left(\hat{\mathcal{A}}_{\hat{E},\sigma}^{0}\hat{b}\otimes\hat{\mathcal{A}}_{\hat{E},\sigma}^{0}\hat{b}\right)$$

From (7) and (9), we can conclude that this is equal to

$$c^T\mathcal{A}_{E,2\sigma}^{q_2-1}E\mathcal{A}_{E,2\sigma}^{q_2-1}H\left(\mathcal{A}_{E,\sigma}^{0}b\otimes\mathcal{A}_{E,\sigma}^{0}b\right).$$

However, this is the same as

$$c^T\mathcal{A}_{E,2\sigma}^{2q_2-1}H\left(\mathcal{A}_{E,\sigma}^{0}b\otimes\mathcal{A}_{E,\sigma}^{0}b\right).$$

In the following, we will now assume that $k = 2q_2-1, j = \ell = 0$. Analog to (8), one easily obtains

$$\mathcal{W}\mathcal{W}^T\mathcal{A}_{E,\sigma}^{T,0}\mathcal{H}^{(2)}\left(\mathcal{A}_{E,\sigma}^{0}b\otimes\mathcal{A}_{E,\sigma}^{T,0}c\right) = \mathcal{A}_{E,\sigma}^{T,0}\mathcal{H}^{(2)}\left(\mathcal{A}_{E,\sigma}^{0}b\otimes\mathcal{A}_{E,\sigma}^{T,0}c\right).$$

Again, this is true since

$$\mathcal{A}_{E,\sigma}^{T,0}\mathcal{H}^{(2)}\left(\mathcal{A}_{E,\sigma}^{0}b\otimes\mathcal{A}_{E,\sigma}^{T,0}c\right)\in\operatorname{span}(\mathcal{W})$$

and $\mathcal{W}^{\mathcal{T}}\mathcal{W} = \mathcal{I}$. With this in mind, we subsequently observe

$$\begin{aligned}&\mathcal{W}\hat{\mathcal{A}}_{\hat{E},\sigma}^{T,0}\mathcal{V}^T\mathcal{H}^{(2)}\left(\mathcal{V}\hat{\mathcal{A}}_{\hat{E},\sigma}^{0}\hat{b}\otimes\mathcal{W}\hat{\mathcal{A}}_{\hat{E},2\sigma}^{T,0}\hat{c}\right) \\ &\quad= \mathcal{W}\hat{\mathcal{A}}_{\hat{E},\sigma}^{T,0}\mathcal{V}^T\mathcal{H}^{(2)}\left(\mathcal{A}_{E,\sigma}^{0}b\otimes\mathcal{A}_{E,2\sigma}^{T,0}c\right) \\ &\quad= \mathcal{W}\hat{\mathcal{A}}_{\hat{E},\sigma}^{T,0}\mathcal{V}^T\left(\mathcal{A}_{E,\sigma}^{T,0}\right)^{-1}\mathcal{A}_{E,\sigma}^{T,0}\mathcal{H}^{(2)}\left(\mathcal{A}_{E,\sigma}^{0}b\otimes\mathcal{A}_{E,2\sigma}^{T,0}c\right) \\ &\quad= \mathcal{W}\hat{\mathcal{A}}_{\hat{E},\sigma}^{T,0}\mathcal{V}^T\left(\mathcal{A}_{E,\sigma}^{T,0}\right)^{-1}\mathcal{W}\mathcal{W}^T\mathcal{A}_{E,\sigma}^{T,0}\mathcal{H}^{(2)}\left(\mathcal{A}_{E,\sigma}^{0}b\otimes\mathcal{A}_{E,2\sigma}^{T,0}c\right) \\ &\quad= \mathcal{W}\mathcal{W}^T\mathcal{A}_{E,\sigma}^{T,0}\mathcal{H}^{(2)}\left(\mathcal{A}_{E,\sigma}^{0}b\otimes\mathcal{A}_{E,2\sigma}^{T,0}c\right) \\ &\quad= \mathcal{A}_{E,\sigma}^{T,0}\mathcal{H}^{(2)}\left(\mathcal{A}_{E,\sigma}^{0}b\otimes\mathcal{A}_{E,2\sigma}^{T,0}c\right).\end{aligned}$$

Iteratively using the above arguments, we finally get

$$\mathcal{W}\hat{\mathcal{A}}_{\hat{E},\sigma}^{T,i}\mathcal{V}^T\mathcal{H}^{(2)}\left(\mathcal{V}\hat{\mathcal{A}}_{\hat{E},\sigma}^{0}\hat{b}\otimes\mathcal{W}\hat{\mathcal{A}}_{\hat{E},2\sigma}^{T,0}\hat{c}\right) = \mathcal{A}_{E,\sigma}^{T,i}\mathcal{H}^{(2)}\left(\mathcal{A}_{E,\sigma}^{0}b\otimes\mathcal{A}_{E,2\sigma}^{T,0}c\right), \tag{10}$$

for $i = 0, \ldots, q_2 - 1$. What we have to consider for $k = 2q_2 - 1, j = \ell = 0$ is

$$\hat{c}^T \hat{\mathcal{A}}^0_{\hat{E},2\sigma} \hat{H} \left(\hat{\mathcal{A}}^{2q_2-1}_{\hat{E},\sigma} \hat{b} \otimes \hat{\mathcal{A}}^0_{\hat{E},\sigma} \hat{b} \right).$$

According to Definition 4.2 and (7) and (9), this term is rewritten as

$$\begin{aligned}
&\hat{c}^T \hat{\mathcal{A}}^0_{\hat{E},2\sigma} \hat{H} \left(\hat{\mathcal{A}}^{q_2-1}_{\hat{E},\sigma} \mathcal{W}^T E \mathcal{V} \hat{\mathcal{A}}^{q_2-1}_{\hat{E},\sigma} \hat{b} \otimes \hat{\mathcal{A}}^0_{\hat{E},\sigma} \hat{b} \right) \\
&\quad = \hat{c}^T \hat{\mathcal{A}}^0_{\hat{E},2\sigma} \hat{H} \left(\hat{\mathcal{A}}^{q_2-1}_{\hat{E},\sigma} \mathcal{W}^T E \mathcal{A}^{q_2-1}_{E,\sigma} b \otimes \hat{\mathcal{A}}^0_{\hat{E},\sigma} \hat{b} \right) \\
&\quad = \hat{c}^T \hat{\mathcal{A}}^0_{\hat{E},2\sigma} \mathcal{W}^T H \left(\mathcal{V} \hat{\mathcal{A}}^{q_2-1}_{\hat{E},\sigma} \mathcal{W}^T E \mathcal{A}^{q_2-1}_{E,\sigma} b \otimes \mathcal{V} \hat{\mathcal{A}}^0_{\hat{E},\sigma} \hat{b} \right) \\
&\quad = b^T \mathcal{A}^{T,q_2-1}_{E,\sigma} E^T \mathcal{W} \hat{\mathcal{A}}^{T,q_2-1}_{\hat{E},\sigma} \mathcal{V}^T \mathcal{H}^{(2)} \left(\mathcal{V} \hat{\mathcal{A}}^0_{\hat{E},\sigma} \hat{b} \otimes \mathcal{W} \hat{\mathcal{A}}^{T,0}_{\hat{E},2\sigma} \hat{c} \right) \\
&\quad = b^T \mathcal{A}^{T,q_2-1}_{E,\sigma} E^T \mathcal{A}^{T,q_2-1}_{E,\sigma} \mathcal{H}^{(2)} \left(\mathcal{A}^0_{E,\sigma} b \otimes \mathcal{A}^{T,0}_{E,2\sigma} c \right) \\
&\quad = c^T \mathcal{A}^0_{E,2\sigma} H \left(\mathcal{A}^{q_2-1}_{E,\sigma} E \mathcal{A}^{q_2-1}_{E,\sigma} b \otimes \mathcal{A}^0_{E,\sigma} b \right) \\
&\quad = c^T \mathcal{A}^0_{E,2\sigma} H \left(\mathcal{A}^{2q_2-1}_{E,\sigma} b \otimes \mathcal{A}^0_{E,\sigma} b \right).
\end{aligned}$$

Since the previous extremal cases contain the essential ideas, we omit a detailed derivation for the remaining combinations j, k, ℓ with $j + k + \ell \leq 2q_2 - 1$. □

To sum up, we have seen that we indeed can construct two-sided projection methods for systems of QBDAEs. At least theoretically, making use of such a projection essentially doubles the number of interpolated derivatives of the first two transfer functions and thus should lead to better approximations by the reduced-order model. However, as has already been indicated in [5], in context of nonlinear model reduction, the benefit of matching more multimoments might come along with a loss of numerical stability and thus has to be treated with care.

5 Numerical examples

In this section, we now want to study the procedure specified in Theorem 4.1 by means of some numerical examples. Besides a very common and well-known model reduction benchmark arising in circuit theory, we investigate different large-scale ODEs resulting from the semi-discretization of several nonlinear partial differential equations. Here, we refrain from sophisticated finite element discretizations and instead use a simple finite difference scheme for all test cases.

In general, moment matching type methods only allow to make an assertion on the approximation of the input-output behavior of a dynamical system. However, we will see that one can often reconstruct the full state

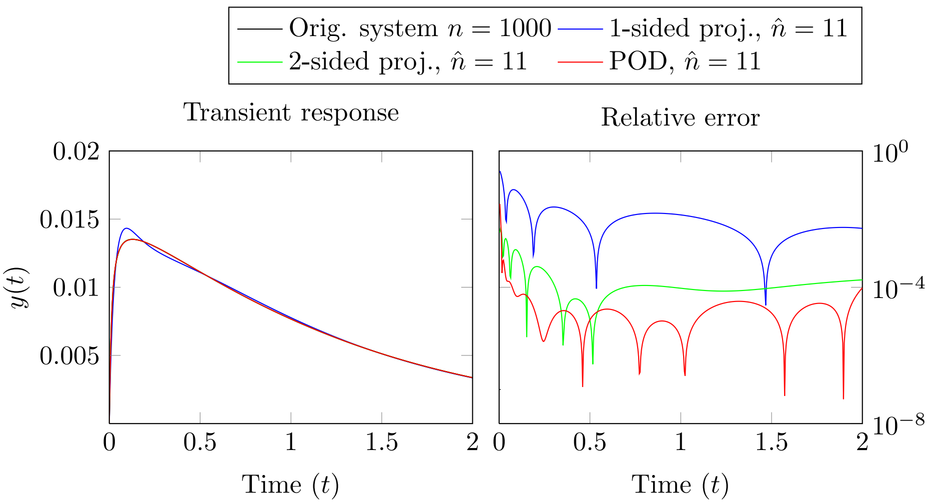


Figure 1: **A nonlinear RC circuit.** Comparison of moment matching methods and POD subject to boundary control $u(t) = e^{-t}$.

vector $x \approx \mathcal{V}\hat{x}$ by a prolongation with the projection matrix $\mathcal{V}$. Moreover, for some problems one might only be interested in the steady state behavior without controlling the process. In this context, we investigate the approximation quality for two uncontrolled systems with nonzero initial condition.

All simulations were generated on an Intel® Dual-Core CPU E5400, 2 MB cache, 3 GB RAM, Ubuntu Linux 10.04 (i686), MATLAB® Version 7.11.0 (R2010b) 32-bit (glnx86).

5.1 A nonlinear RC circuit

The first example we want to study is a scalable nonlinear transmission line circuit which is one of the standard test examples in the context of moment matching based reduction techniques, see e.g. [3, 5, 11, 19, 20]. Since the applications have been studied and discussed in the given references, here we will refrain from a more detailed analysis. However, we want to point out that the nonlinearities result from the diode I-V characteristic $i_D = e^{40v_D} - 1$. As has been discussed in [11], by a suitable change of state variables, the dynamics can be described by a quadratic-bilinear control systems of dimension $2n$, where n denotes the number of capacitors and resistors of the circuit, respectively.

In Figure 1, we see a comparison between the new method discussed here and the classical one-sided method discussed in [11]. Moreover, we compute a POD-based approximation by taking 100 snapshots of the original solution for the input excitation $u(t) = e^{-t}$. Obviously, the POD reduced-order model performs the best. However, the two-sided method exhibits a comparable approximation quality while the one-sided approach performs

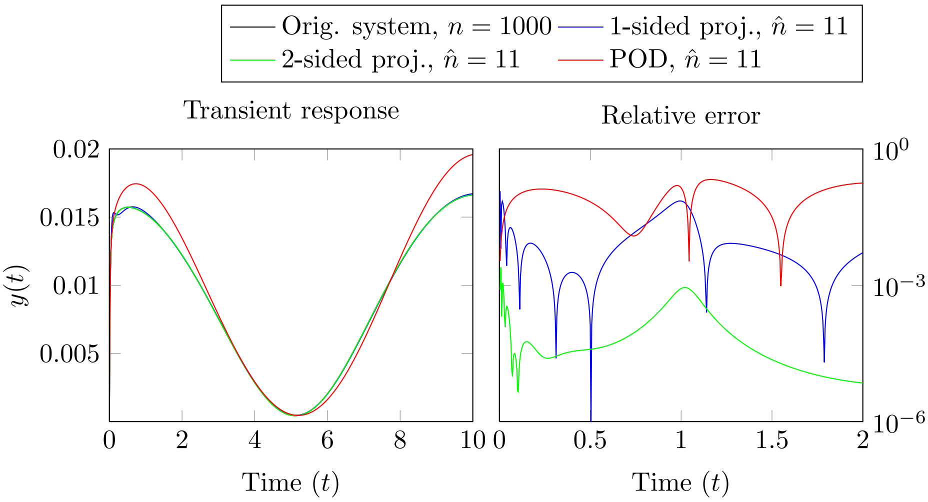


Figure 2: **A nonlinear RC circuit**. Comparison of moment matching methods and POD subject to boundary control $u(t) = (\cos(2\pi \frac{t}{10}) + 1)/2$.

the worst. All reduced-order models are of dimension $\hat{n} = 11$. The moment matching based techniques are generated according to Theorem 4.1 with values $\sigma = 1,\ q_1 = 5, q_2 = 2$.

In order to test our method with respect to input variations, in Figure 2, we show the approximations for the input signal $u(t) = (\cos(2\pi \frac{t}{10}) + 1)/2$. Clearly, the POD approximation shows a significant deviation from the original output. On the other side, the two-sided method still reflects the dynamics very accurately and also outperforms the one-sided technique as well.

5.2 Burgers' equation

Let us now consider the one-dimensional Burgers' equation on $\Omega = (0,1) \times (0,T)$, leading to the following set of equations

$$v_t + v \cdot v_x = \nu \cdot v_{xx}, \quad \text{in } (0,1) \times (0,T), \tag{11}$$

$$\alpha v(0,\cdot) + \beta v_x(0,\cdot) = u(t), \quad \text{in } (0,T), \tag{12}$$

$$v_x(1,\cdot) = 0, \quad \text{in } (0,T), \tag{13}$$

$$v(x,0) = v_0(x), \ \text{in } (0,1), \tag{14}$$

where ν is the viscosity parameter and $v_0(x)$ denotes the initial condition of the system. This equation can be seen as a standard numerical test example for nonlinear model reduction and optimal control, respectively, and has already been extensively studied in e.g. [15, 16]. In the context of this paper, the above PDE is of particular interest since a semi-discretization automatically leads to a quadratic-bilinear control system of the form (3).

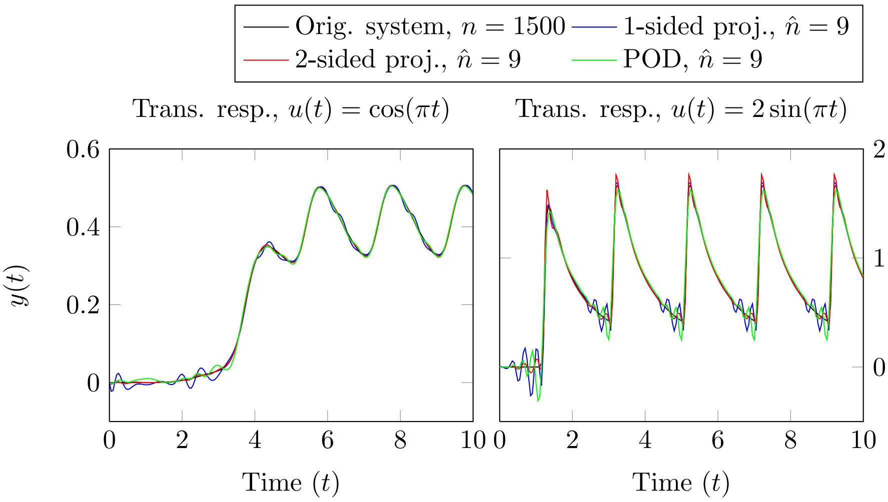


Figure 3: **Burgers' equation**. Comparison of moment matching methods and POD subject to boundary control ($\nu = 0.02$).

5.2.1 Boundary control

Let us assume that the equation is subject to a boundary control on the left side of the interval, i.e. $\alpha = 1$ and $\beta = 0$. Furthermore, we assume the initial state of the system to be zero, i.e. $v_0(x) = 0$. For the viscosity parameter ν we start by choosing the value 0.02. However, while for larger values of ν, the accuracy of the reduced-order models often become better, decreasing ν makes the model more difficult to reduce.

In Figure 3, we show the results for the reduction of an original system which was spatially discretized using $n = 1000$ points and $T = (0, 10)$. The reduced-order models are of dimension $\hat{n} = 9$ and are generated by the procedure specified in Theorem 4.1 with $\sigma = 0.0288, q_1 = 4$ and $q_2 = 2$. The specific interpolation point σ is chosen to minimize the $\mathcal{H}_2$-optimal model reduction problem for the linearized system, see e.g. [12]. This minimization is done by the iterative rational Krylov algorithm from [12]. For the one-sided projection method, we simply set $\mathcal{W} = \mathcal{V}$. The measured output of the system is assumed to be the value at the right boundary, leading to an output vector $c = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^T$. Besides a comparison between one-sided and two-sided projection, we compute a POD approximation by making use of the SVD of the solution matrix of the original problem over the whole interval range. The 100 snapshots are chosen uniformly within this interval. As can be seen in Figure 3, for the control $u(t) = \cos(\pi t)$, all approaches faithfully reproduce the dynamics of the original system although the one-sided approach exhibits some smaller oscillations. In order to investigate the methods with regard to robustness to input variations, we slightly change the control to $u(t) = 2\sin(\pi t)$. Increasing the amplitude of $u(t)$ seems to

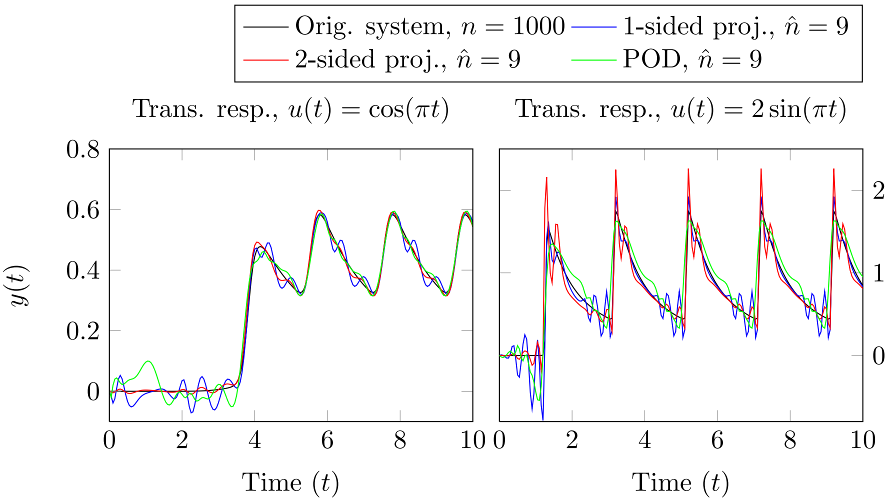


Figure 4: **Burgers' equation**. Comparison of moment matching methods and POD subject to boundary control ($\nu = 0.01$).

make the reduction process more difficult. For the POD approximation, we use the projection subspace derived by the first input signal. As expected, we see that this results in a less accurate reduced-order model indicating the input dependency of POD. On the other hand, for the two-sided projection we observe overshoots at the sharper fronts of the curve. Nevertheless, altogether for this parameter configuration of σ, q_1, q_2, we can conclude that the new method performs well and seems to outperform the one-sided projection. Though, it has to be mentioned that for the two-sided approach many of the parameter constellations lead to instable reduced-order models. A similar observation already was discussed in [5]. Hence, a reasonable choice of the interpolation points together with the order of the matched derivatives seems to be an important aspect of further research.

5.2.2 The uncontrolled case

In order to test the efficiency of the reduction method, we also want to investigate the performance when the system under consideration exhibits a non-zero initial condition. In view of the above mentioned setting, we use $\alpha = 0, \beta = 1$ and $v_0(x) = 1 + \sin((2x+1)\pi)$. After a semi-discretization with $n = 1000$, the system is rewritten to a system with zero initial condition, leading to a single-input and single-output (SISO) QBDAE system with constant input vector $u(t)$. Again, the viscosity parameter is $\nu = 0.01$ while we choose $T = (0, 2)$. In contrast to the previous example, we now consider the entire state x. Since we want to compare the results for a two-sided reduction method, we artificially have to choose a certain output matrix c such that we can run the procedure from Theorem 4.1. Here, we use

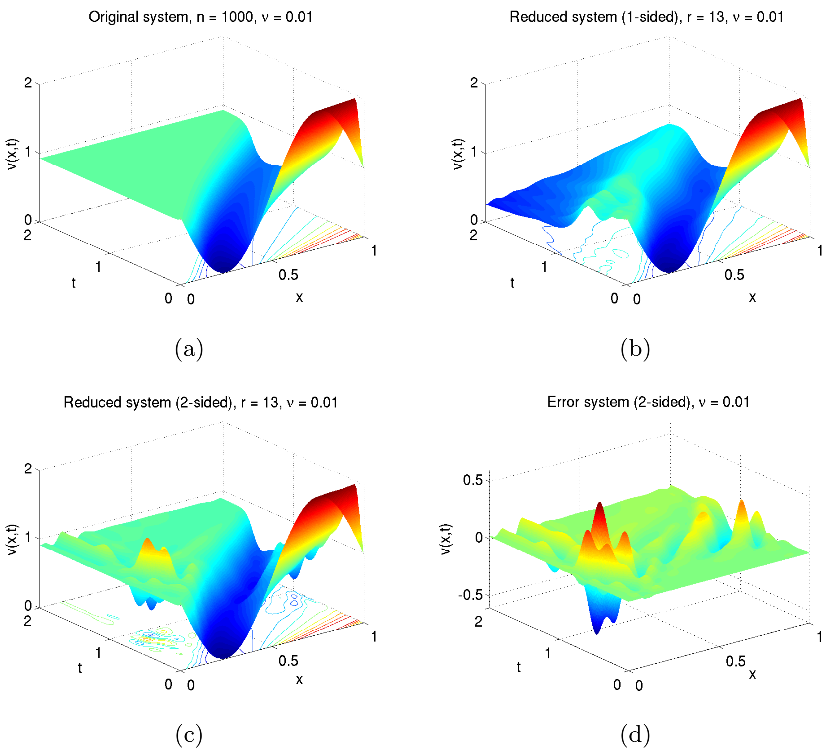


Figure 5: Comparison of uncontrolled solutions for the Burgers equation.

$c = \frac{1}{k}\begin{bmatrix}1 & \dots & 1\end{bmatrix}^T$, i.e. the average value of $v(x,t)$ on the interval $(0,1)$.

In Figure 5, we show the different steady state solutions for the original system (Figure 5(a)), the reduced-order system obtained by an orthogonal projection (Figure 5(b)) and the reduced-order system resulting from an oblique projection (Figure 5(c)). For the reduction process we choose $\sigma = 5, q_1 = 10$ and $q_2 = 2$, leading to reduced-order models of dimension $\hat{n} = 13$. Here, the interpolation point now is chosen as the one performing the best among several random choices. Obviously, the one-sided approach deviates significantly from the original solution, while the two-sided method produces some undesired peaks. However, one still has to keep in mind that we cannot make a theoretical assertion on the reconstruction of a state vector but only on the input-output behavior of the system. If we keep this in mind, the approximations still might be appropriate for the analysis of the uncontrolled dynamics. Note that we do not compare the results with POD at this point since we do have a specific constant input which does not vary. Hence, it is clear that POD will outperform the moment matching approaches due to its intrinsic properties. To be more precise, recall that for a given input which is not subject to variation, the approximation given by POD is optimal due the properties of the singular value decomposition.

5.3 Chafee-Infante equation

Next, we consider the one-dimensional Chafee-Infante equation. For more details on this nonlinear PDE, we refer to [7, 13]. The equation exhibits a cubic nonlinearity and is subject to similar initial and boundary conditions as the Burgers' equation, namely

$$
\begin{aligned}
v_t + v^3 &= v_{xx} + v && \text{in } (0,1)\times(0,T), && (15)\\
\alpha v(0,\cdot) + \beta v_x(0,\cdot) &= u(t), && \text{in } (0,T), && (16)\\
v_x(1,\cdot) &= 0, && \text{in } (0,T), && (17)\\
v(x,0) &= v_0(x), && \text{in } (0,1). && (18)
\end{aligned}
$$

Following the discussion in [13], we once more use a finite difference scheme for the spatial discretization. The resulting system of nonlinear ODEs then has to be transformed to quadratic-bilinear structure. This is done by introducing a new state variable $w_i = v_i^2$. Computing the derivative of w_i leads to $\dot{w}_i = 2v_i\dot{v}_i$ which can be rewritten in the desired QBDAE form (3).

5.3.1 Boundary control

Completely analog to Section 5.2, we start with the boundary controlled equation on $T = (0, 10)$ and a zero initial condition $v_0(x) = 0$. We further use the same output, i.e. the value at the right boundary, leading to an output vector $c = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^T$. The discretization was done with $n = 750$ points. Hence, after transformation to QBDAE form, the system consists of $2 \cdot 750$ states.

The reductions are generated with $\sigma = 1$, $q_1 = 4$ and $q_2 = 3$, yielding systems of dimension $\hat{n} = 9$. Similar to the Burgers' equation, we run IRKA in order to get an $\mathcal{H}_2$-optimal interpolation point for the linearized system, leading to the specific choice $\sigma = 1$. Again, in Figure 6, we visualize the approximations of our new method and compare them with a one-sided projection as well as POD. For the input $u(t) = (1 + \cos(\pi t))/2$, we see that the new approach clearly outperforms the one-sided projection. On the other hand, it cannot compete with POD.

Now we slightly change the input signal to $u(t) = 25 \cdot (1 + \sin(\pi t))/2$. The corresponding results are given in Figure 7. Though a bit surprising, we observe that the reduced order model for the one-sided approach completely fails in reproducing the original dynamics. Once more, we do not vary the projection subspace of POD but simply use the one for the first test signal specified above. Here, we now see that POD indeed also has problems in the approximation of the maxima of the transient response which is not the case for the two-sided approach.

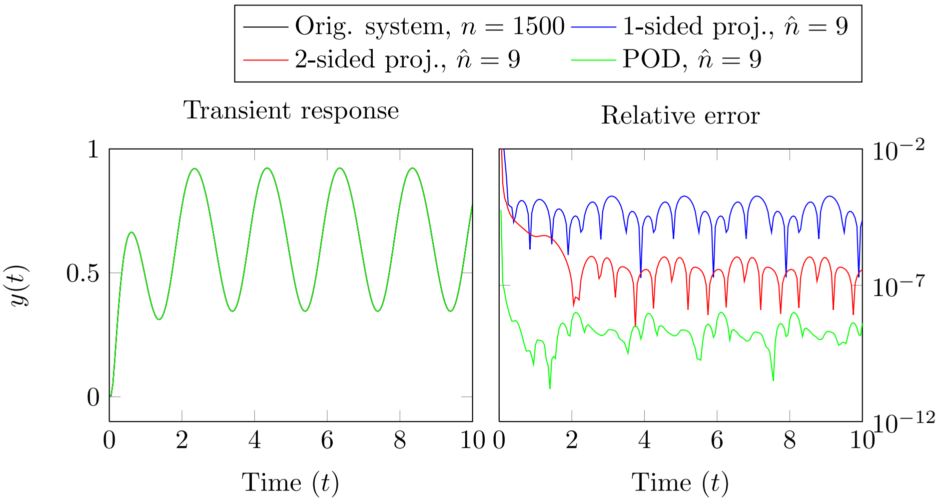


Figure 6: **Chafee-Infante equation**. Comparison of moment matching methods and POD subject to boundary control $u(t) = (1 + cos(\pi t))/2$.

5.3.2 The uncontrolled case

For the uncontrolled case, we set $\alpha = 0$, $\beta = 1$ and implement a non-zero initial condition which was already discussed in [13]. To be more precise, we have $v_0(x) = \frac{1}{10} + \frac{7}{10} \cdot \sin^2((2 \cdot x + 1)\pi)$. In Figure 8, we compare the full state vector for the time interval $T = (0, 0.02)$ for a semi-discretization with $n = 750$. The reduced-order systems are of dimension $\hat{n} = 10$ and result from the model reduction parameters $\sigma = 3$, $q_1 = 3$, $q_2 = 3$, which basically are chosen at random. As we can see, both approaches yield very accurate reconstructions. However, due to several parameter studies, it seems that the one-sided projection method performs more robust with respect to stability issues of the reduced-order model.

5.4 FitzHugh-Nagumo system

Finally, as a last example we study the FitzHugh-Nagumo system modeling activation and deactivation dynamics of a spiking neuron which has been under consideration in the context of POD-based model reduction in [8]. Formally, the model is described by the following system of coupled nonlinear PDEs

$$\epsilon v_t(x,t) = \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g, \quad (19)$$

$$w_t(x,t) = hv(x,t) - \gamma w(x,t) + g, \quad (20)$$

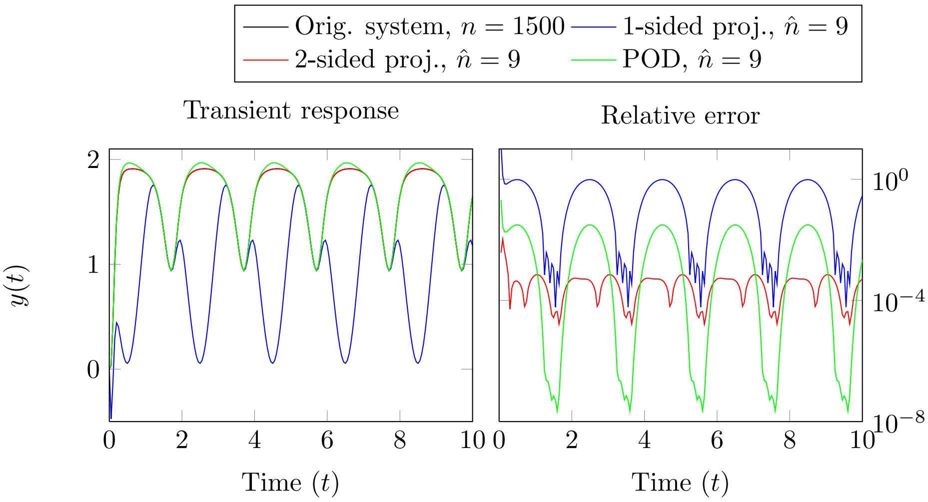


Figure 7: **Chafee-Infante equation**. Comparison of moment matching methods and POD subject to boundary control $u(t) = 25 \cdot (1 + \sin(\pi t))/2$.

with $f(v) = v(v - 0.1)(1 - v)$ and initial and boundary conditions

$$v(x,0) = 0, \qquad w(x,0) = 0, \qquad x \in [0,1], \tag{21}$$
$$v_x(0,t) = -i_0(t), \qquad v_x(1,t) = 0, \qquad t \geq 0, \tag{22}$$

where $\epsilon = 0.015$, $h = 0.5$, $\gamma = 2$, $g = 0.05$, $i_0(t) = 5 \cdot 10^4 t^3 \exp(-15t)$. Again, one can easily use a finite difference scheme, resulting in a system of nonlinear (cubic) ODEs. Similar to the Chafee-Infante equation, introducing an additional dynamical variable $z_i = v_i^2$ allows to reformulate the dynamics as a system of QBDAEs of dimension $3 \cdot n$, where n is the number of degrees of freedom used in the finite difference scheme. However, in contrast to the first two examples, the system no longer is of SISO type since the constant parameter g as well as the stimulus $i_0(t)$ have to be incorporated within the modeling process. In order to apply the previously discussed reduction techniques, we run the corresponding algorithm once for each column of the input vector.

Here, we follow the setting in [8] and use a discretization with $n = 1000$ points. In Figure 9, we show the reduction results measured in terms of the limit cycle behavior which is a typical phenomenon when modeling neuronal dynamics. For the comparison between one-sided and two-sided projections, we assume the output matrix $C \in \mathbb{R}^{2\times 3n}$ to sort out the values $v(0,t)$ and $w(0,t)$, i.e. the limit cycle at the left boundary. The results shown in Figure 9(a) are constructed with parameter values $\sigma = 100$, $q_1 = 2$, $q_2 = 2$ and the reduced-order models both are of dimension $\hat{n} = 14$. Although the approximation of the two-sided reduced model performs better, based on this specific example we cannot recommend using the new approach. This

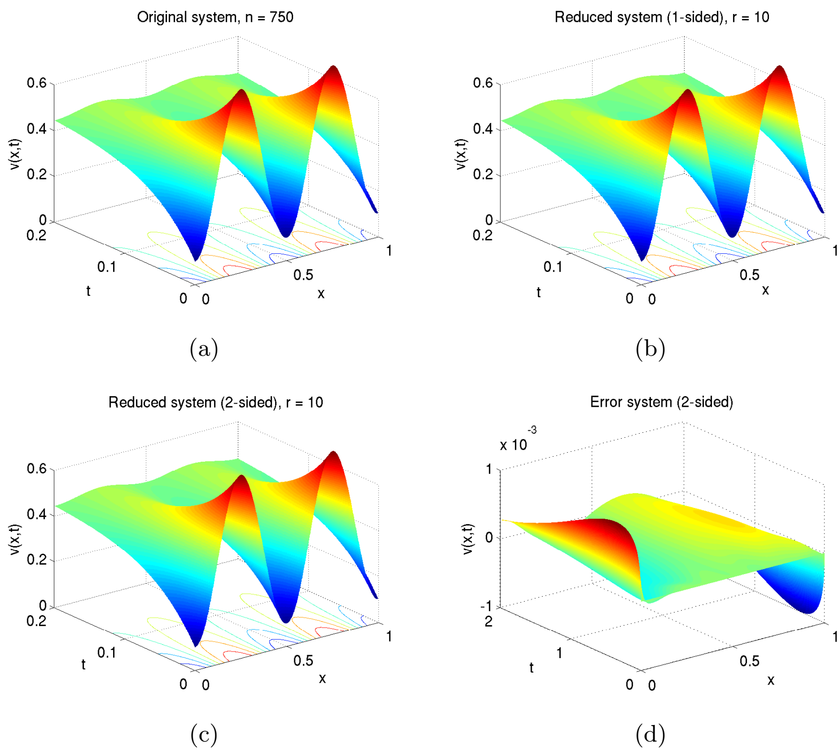
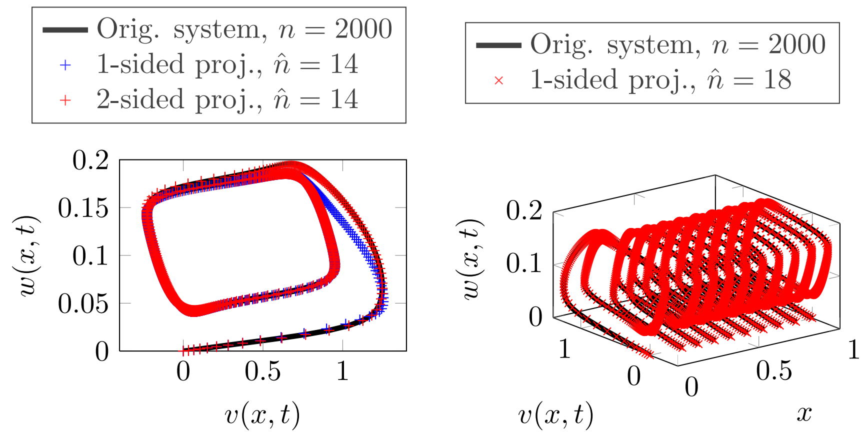


Figure 8: Steady state solution for the Chafee-Infante equation

is simply due to the fact that nearly all generated reduced-order models become instable and it does not seem to be obvious how to circumvent this significant drawback. On the other hand, most reductions obtained by using random interpolation points yield accurate approximations for the one-sided technique. For example, in Figure 9(b) we plot the limit cycle behavior similar to the one studied in [8] for a discretization of $n = 1000$ and the parameter setting $\sigma = 14$, $q_1 = 5$, $q_2 = 2$ and a reduced-order system of dimension $\hat{n} = 18$. Although the results are not as accurate as in [8], where a sufficient reduction to a system of dimension $\hat{n} = 10$ is reported, we are certainly able to construct an appropriate reduced-order model.

6 Conclusions

In this paper, we have studied a recently introduced new approach for model order reduction of nonlinear control systems. In contrast to other methods in this field of research, the technique relies on generalized moment matching and thus is input independent, i.e., no training trajectories are needed. Besides a slight extension of existing results for the case of $\sigma = 0$, we have shown how the sequence of nested Krylov subspaces has to be chosen in or-



(a) (b)

Figure 9: **FitzHugh-Nagumo system.** Limit cycle behavior for original and reduced-order systems.

der to interpolate at arbitrary interpolation points $\sigma \neq 0$. Moreover, we used some basic tools and properties known from tensor theory in order to show how one can improve the efficiency of the necessary projection step leading to the reduced-order system. In particular, we have seen that one can avoid building up the matrix $\mathcal{V} \otimes \mathcal{V}$ which easily might exceed the given memory capacity. The main contribution of this paper then is the construction of an appropriate two-sided projection method which theoretically allows to double the number of interpolated derivatives of the first two transfer functions. However, here one has to be careful in applying the new method since the gain of accuracy sometimes destroys the stability of the underlying system making a reduction unreliable. Nevertheless, by means of a standard nonlinear model reduction test example and several nonlinear partial differential equations, we have proven that the moment matching approach indeed seems to have potential and even allows to reconstruct typical dynamics observed in fluid mechanics and neuron modeling, respectively. Moreover, for three examples we could show that the new method can compete with proper orthogonal decomposition and, in some cases, might be advantageous if the input signal is known to exhibit larger variations. Hence, it might be an interesting field of further research. In particular, the study of suitable or somehow optimal interpolation points seems to be an important issue. Similarly, investigating possible structure preserving methods which prevent from constructing unstable reduced-order models should be one of the major challenges in order to improve the applicability of the new method.

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