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Adaptive discontinuous Galerkin methods for state constrained optimal control problems governed by convection diffusion equations

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MAGDEBURG

MPIMD/13-16 **September 19, 2013**

Abstract

We study a posteriori error estimates for the numerical approximations of state constrained optimal control problems governed by convection diffusion equations, regularized by Moreau-Yosida and Lavrentiev-based techniques. The upwind Symmetric Interior Penalty Galerkin (SIPG) method is used as a discontinuous Galerkin (DG) discretization method. We derive different residual-based error indicators for each regularization technique due to the regularity issues. An adaptive mesh refinement indicated by a posteriori error estimates is applied. Numerical examples are presented to illustrate the effectiveness of the adaptivity for both regularization techniques.

Impressum:

Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg

Publisher:
Max Planck Institute for
Dynamics of Complex Technical Systems

Address:
Max Planck Institute for
Dynamics of Complex Technical Systems
Sandtorstr. 1
39106 Magdeburg

www.mpi-magdeburg.mpg.de/preprints

1 Introduction

In this paper, we consider optimal control problems governed by linear partial differential equations (PDEs) and subject to inequality constraints on the state variable, so called state constraints, formulated as follows:

$$\text{minimize } J(y,u) \text{ s.t. } A(y) = f, \; y^a(x) \le y(x) \le y^b(x),$$

where J denotes the cost functional to be minimized, A represents the linear differential operator, and the inequality constraints on the state variable y are formulated pointwise on the computational domain using lower and upper bounds $y^a, y^b \in C(\bar{\Omega})$. Such problems are known to be difficult from the theoretical as well as the numerical point of view, due to the fact that Lagrange multipliers corresponding to state constraints exhibit low function space regularity [5, 7]. To overcome this difficulty several techniques in the literature have been proposed. A natural idea is to relax the state constraints by either substituting them by a mixed control-state constraint, i.e., Lavrentiev regularization [20, 25], or by adding suitable penalty terms to the cost functional instead of exactly satisfying the state constraints, i.e., Moreau-Yosida regularization [3, 15], virtual control approach [16], barrier methods [11, 23].

Adaptive mesh refinement is particularly attractive for the solution of optimal control problems governed by convection dominated PDEs, since the solution of the governing state PDE or the solution of the associated adjoint PDE may exhibit boundary and/or interior layers, localized regions where the derivative of the PDE solution is large. In this case adaptivity allows local mesh refinement around the layers as needed, thereby achieving a desired residual error bound with as few degrees of freedom as possible. The key point of any adaptive finite element method is an a posteriori error estimator or indicator. In this regard, the dual weighted residual methods are investigated in [2, 9, 10, 27], whereas the residual-type a posteriori error estimators are studied in [13, 14] for the state constrained optimal control problems governed by elliptic equations. In [8], a mixed variational scheme is proposed with a posteriori error estimate by reformulating the state constrained optimal control problem as a constrained minimization problem involving only the state, which is characterized by a fourth order variational inequality. Further, an a posteriori error estimator containing only computable quantities is proposed in [22] for elliptic control problems with control and state constraints. Recent results derived in [29] for unconstrained optimal control problems and in [31] for control constrained optimal control problems governed by convection dominated equations show that the residual based a posteriori error estimators based on discontinuous Galerkin (DG) discretization yield more accurate results since the errors in layers do not propagate into the entire domain [17]. Our goal here is to extend the results in [29, 31] to the state constrained optimal control problems governed by convection diffusion equations, regularized by Moreau-Yosida and Lavrentiev-based techniques.

The remainder of the paper is organized as follows: In the next section, we specify the problem data and we present the optimality systems based on Moreau-Yosida and Lavrentiev regularizations. Section 3 describes the DG discretization of the optimal control problems. We use the symmetric interior penalty Galerkin (SIPG) discretization due to its symmetrical property. This implies that discretization and optimization commute, see, e.g., [30]. A posteriori error estimators of the state constrained optimal control problems regularized by Moreau-Yosida and Lavrentiev-based techniques are given in Section 4. The reliability and efficiency estimates are also derived. Finally, in the last section, the adaptive cycle is described and numerical results are presented to illustrate the performance of adaptive mesh refinement.

2 Optimal control problem

Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\Gamma = \partial\Omega$. We consider the following objective functional

$$\text{minimize } J(y,u) := \frac{1}{2}\|y - y^d\|^2_{L^2(\Omega)} + \frac{\omega}{2}\|u - u^d\|^2_{L^2(\Omega)} \tag{2.1}$$

subject to

$$-\varepsilon\Delta y(x) + \beta(x)\cdot\nabla y(x) + \alpha(x)y(x) = f(x) + u(x) \qquad x \in \Omega, \tag{2.2a}$$
$$y(x) = d \qquad x \in \Gamma, \tag{2.2b}$$
$$y^a(x) \le y(x) \le y^b(x) \qquad x \in \Omega. \tag{2.2c}$$

We make the following assumptions on the functions and parameters in the optimal control problem (2.1)-(2.2) to show the well-posedness of the optimal control problem:

$$f, y^d, u^d \in L^2(\Omega),\ d \in H^{3/2}(\Gamma),\ \beta \in W^{1,\infty}(\Omega)^2,\ \alpha \in L^\infty(\Omega),\ \varepsilon,\omega > 0,$$
$$y^a, y^b \in L^\infty(\Omega) \quad \text{with} \quad y^a \le y^b \ \text{ a.e. in } \Omega, \tag{2.3a}$$

and for $r_0 \ge 0$

$$\alpha(x) - \frac{1}{2}\nabla\cdot\beta(x) \ge r_0 \ge 0 \qquad x \in \Omega. \tag{2.3b}$$

Further, we assume the following condition to use in some estimates proven in [24, 26]:

$$|| -\nabla\cdot\beta(x) + r(x)||_{L^\infty(\Omega)} \le c_* r_0 \quad \text{with} \quad c_* \ge 0. \tag{2.4}$$

Although our error estimators can be formulated for $r_0 = 0$, we require $r_0 > 0$ to prove reliability and efficiency of our estimators. Of course, if $r_0 > 0$, we can always find c_* such that (2.4) holds and the condition (2.4) is more critical if $r_0 = 0$, which is allowed in [24, 26]. In this case, the condition (2.4) holds only for the case $\alpha \equiv \nabla\cdot\beta$. Therefore, we also require $\nabla\cdot\beta \ge 0$ to satisfy the condition (2.3b).

We regularize the state constrained optimal control problem (2.1)-(2.2) by using Moreau-Yosida and Lavrentiev-based regularization techniques. To make the notation easier for the readers, the superscript M will be used to indicate the Moreau-Yosida regularization, whereas the superscript L will be used to indicate the Lavrentiev regularization.

We first define the spaces of state, control and test functions by

$$Y^M = \{y \in H^2(\Omega) : \ y = d \text{ on } \Gamma\}, \quad Y^L = \{y \in H^1(\Omega) : \ y = d \text{ on } \Gamma\},$$
$$U = L^2(\Omega) \quad \text{and} \quad V = H^1_0(\Omega),$$

respectively. Due to regularization issues, the state space is different for each regularization techniques [9, 14]. Furthermore, we define the usual (bi-)linear forms by

$$a(y,v) = \int_\Omega (\varepsilon\nabla y\cdot\nabla v + \beta\cdot\nabla y v + \alpha y v)\, dx, \quad l(v) = \int_\Omega f v\, dx.$$

2.1 Moreau-Yosida-based regularization

We penalize the state constraints with a Moreau-Yosida-based technique by modifying the objective functional $J(y,u)$ in (2.1). The state constrained optimal control problem governed by convection dominated PDEs, regularized by a Moreau-Yosida penalty function is given by

$$\begin{aligned}\text{minimize } J^M(y,u) :=& \frac{1}{2}\|y-y^d\|^2_{L^2(\Omega)} + \frac{\omega}{2}\|u-u^d\|^2_{L^2(\Omega)} \\ &+ \frac{1}{2\delta_M}\|\max\{0,y-y^b\}\|^2_{L^2(\Omega)} + \frac{1}{2\delta_M}\|\min\{0,y-y^a\}\|^2_{L^2(\Omega)}\end{aligned} \tag{2.5}$$

subject to

$$-\varepsilon\Delta y(x) + \beta(x)\cdot\nabla y(x) + \alpha(x)y(x) = f(x) + u(x) \qquad x \in \Omega, \tag{2.6a}$$

$$y(x) = d \qquad x \in \Gamma, \tag{2.6b}$$

where δ_M is the Moreau-Yosida regularization parameter. The max- and min- expressions in the regularized objective functional $J^M(y,u)$ arise from regularizing the indicator function corresponding to the set of admissible states.

The variational formulation corresponding to (2.5)-(2.6) can be written as

$$\begin{aligned}\text{minimize } J^M(y,u) :=& \frac{1}{2}\|y-y^d\|^2_{L^2(\Omega)} + \frac{\omega}{2}\|u-u^d\|^2_{L^2(\Omega)} \\ &+ \frac{1}{2\delta_M}\|\max\{0,y-y^b\}\|^2_{L^2(\Omega)} + \frac{1}{2\delta_M}\|\min\{0,y-y^a\}\|^2_{L^2(\Omega)}\end{aligned} \tag{2.7a}$$

subject to

$$a(y,v) = l(v) + (u,v) \qquad \forall (y,u,v) \in Y^M \times U \times V. \tag{2.7b}$$

The optimal control problem (2.7) has a unique solution $(y,u) \in Y^M \times U$ and $(y,u) \in Y^M \times U$ solves (2.7) if and only if there exists an adjoint variable $p \in H^2(\Omega)$ such that the optimality system

$$a(y,v) = l(v) + (u,v) \qquad \forall v \in V, \tag{2.8a}$$

$$a(\psi,p) = (y-y^d,\psi) + (\sigma^M,\psi) \qquad \forall \psi \in V, \tag{2.8b}$$

$$(\omega(u-u^d),w) + (p,w) = 0 \qquad \forall w \in U \tag{2.8c}$$

is satisfied. The existence of an adjoint variable is proven using standard theory of mathematical programming in Banach spaces, see, e.g., [34]. The multiplier corresponding to the state constraints is

$$\sigma^M = \frac{1}{\delta_M}\left(\max\{0,y-y^b\} + \min\{0,y-y^a\}\right).$$

2.2 Lavrentiev-based regularization

Lavrentiev-based regularization depends on substituting the state constraint (2.2c) by a mixed control-state constraint. Then, the mixed control-state constrained distributed control problem is given by

$$\text{minimize } J^L(y,u) := \frac{1}{2}\|y-y^d\|^2_{L^2(\Omega)} + \frac{\omega}{2}\|u-u^d\|^2_{L^2(\Omega)} \tag{2.9}$$

subject to

$$-\varepsilon\Delta y(x) + \beta(x)\cdot\nabla y(x) + \alpha(x)y(x) = f(x) + u(x) \qquad x\in\Omega, \tag{2.10a}$$

$$y(x) = d \qquad x\in\Gamma \tag{2.10b}$$

with the mixed control-state constraint

$$\delta_L u + y \in \{v\in L^2(\Omega) \ : \ y^a \le v \le y^b\}, \tag{2.10c}$$

where δ_L is the Lavrentiev regularization parameter.

The optimal solution $(y,u)\in Y^L\times U$ of (2.9)-(2.10) is characterized by the existence of an adjoint state $p\in V$ and a multiplier $\sigma^L \in L^2_+(\Omega) = \{v\in L^2(\Omega) : \ v(x)\ge 0,\ x\in\Omega\}$ such that

$$a(y,v) = l(v) + (u,v) \qquad \forall v\in V, \tag{2.11a}$$

$$a(\psi,p) = (y-y^d,\psi) + (\sigma^L,\psi) \qquad \forall\psi\in V, \tag{2.11b}$$

$$(\omega(u-u^d),w) + (p,w) + (\delta_L\sigma^L,w) = 0 \qquad \forall w\in U \tag{2.11c}$$

with the complementary condition

$$\sigma^L = \max\{0,\sigma^L + (\delta_L u + y - y^b)\} + \min\{0,\sigma^L - (y^a - \delta_L u - y)\} \ \text{ a.e. in } \Omega. \tag{2.11d}$$

Further, the complementary condition (2.11d) can also be described as

$$\begin{aligned} &(\sigma^a, y^a - \delta_L u - y) = (\sigma^b, \delta_L u + y - y^b) = 0, \\ &\sigma^L = \sigma^b - \sigma^a, \quad \sigma^a \ge 0, \quad \sigma^b \ge 0, \quad y^a \le \delta_L u + y \le y^b. \end{aligned} \tag{2.12}$$

Following [13, 14], we decompose the adjoint p as

$$p = \bar{p} + \bar{\sigma}^L, \tag{2.13}$$

where a regularized adjoint $\bar{p}\in V$ and a regularized multiplier $\bar{\sigma}^L\in V$ are solutions of

$$a(\psi,\bar{p}) = (y-y^d,\psi) \qquad \forall\psi\in V, \tag{2.14a}$$

$$a(\psi,\bar{\sigma}^L) = (\sigma^L,\psi) \qquad \forall\psi\in V, \tag{2.14b}$$

respectively.

3 Discretization of the Optimal Control Problem

We discretize our optimal control problem using a DG method in which we choose the SIPG discretization for the diffusion and an upwind discretization for the convection. Our notation follows [24, 29, 30, 31, 32, 33].

Let $\{\mathcal{T}_h\}_h$ be a family of shape-regular simplicial triangulations of Ω such that $\overline{\Omega} = \cup_{K\in\mathcal{T}_h}\overline{K}$, $K_i \cap K_j = \emptyset$ for $K_i, K_j \in \mathcal{T}_h$, $i \neq j$. The diameter of an element K and the length of an edge E are denoted by h_K and h_E, respectively.

We split the set of all edges $\mathcal{E}_h$ into the set $\mathcal{E}_h^0$ of interior edges and the set $\mathcal{E}_h^\partial$ of boundary edges so that $\mathcal{E}_h = \mathcal{E}_h^\partial \cup \mathcal{E}_h^0$. Let $\mathbf{n}$ denote the unit outward normal to $\partial\Omega$. We define the inflow boundary

$$\Gamma^- = \{x \in \partial\Omega \,:\, \beta\cdot\mathbf{n}(x) < 0\},$$

and the outflow boundary $\Gamma^+ = \partial\Omega \setminus \Gamma^-$. The boundary edges are decomposed into edges $\mathcal{E}_h^- = \{E \in \mathcal{E}_h \,:\, E \subset \Gamma^-\}$ that correspond to the inflow boundary and edges $\mathcal{E}_h^+ = \mathcal{E}_h^\partial \setminus \mathcal{E}_h^-$ that correspond to the outflow boundary. The inflow and outflow boundaries of an element $K \in \mathcal{T}_h$ are defined by

$$\partial K^- = \{x \in \partial K \,:\, \beta\cdot\mathbf{n}_K(x) < 0\}, \quad \partial K^+ = \partial K \setminus \partial K^-,$$

where $\mathbf{n}_K$ is the unit normal vector on the boundary ∂K of an element K.

Let the edge E be a common edge for two elements K and K^e. For a piecewise continuous scalar function y, there are two traces of y along E, denoted by $y|_E$ from inside K and $y^e|_E$ from inside K^e. The jump and average of y across the edge E are defined by:

$$[\![y]\!] = y|_E\mathbf{n}_K + y^e|_E\mathbf{n}_{K^e}, \quad \{\{y\}\} = \frac{1}{2}\big(y|_E + y^e|_E\big). \tag{3.1}$$

Similarly, for a piecewise continuous vector field ∇y, the jump and average across an edge E are given by

$$[\![\nabla y]\!] = \nabla y|_E\cdot\mathbf{n}_K + \nabla y^e|_E\cdot\mathbf{n}_{K^e}, \quad \{\{\nabla y\}\} = \frac{1}{2}\big(\nabla y|_E + \nabla y^e|_E\big). \tag{3.2}$$

For a boundary edge $E \in K\cap\Gamma$, we set $\{\{\nabla y\}\} = \nabla y$ and $[\![y]\!] = y\mathbf{n}$, where $\mathbf{n}$ is the outward normal unit vector on Γ.

For the spaces of the discrete state, control and test functions, we use piecewise linear functions

$$V_h = Y_h = U_h = \big\{y \in L^2(\Omega) \,:\, y\,|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h\big\}. \tag{3.3}$$

Note that the space Y_h of discrete states and the space of test functions V_h are identical due to the weak treatment of boundary conditions in DG methods.

The DG method used here is based on the upwind discretization for the convection term and on the SIPG discretization for the diffusion term. This leads to the following (bi-)linear forms applied to $y \in Y_h$, $v \in V_h$, $u \in U_h$:

$$\begin{aligned}
a_h(y,v) \quad = \quad & \sum_{K\in\mathcal{T}_h} \int_K \varepsilon \nabla y \cdot \nabla v \, dx - \sum_{E\in\mathcal{E}_h} \int_E \{\{\varepsilon \nabla y\}\} \cdot [\![v]\!] \, ds - \sum_{E\in\mathcal{E}_h} \int_E \{\{\varepsilon \nabla v\}\} \cdot [\![y]\!] \, ds \\
& + \sum_{E\in\mathcal{E}_h} \frac{\sigma\varepsilon}{h_E} \int_E [\![y]\!] \cdot [\![v]\!] \, ds + \sum_{K\in\mathcal{T}_h} \int_K \beta \cdot \nabla y v + \alpha y v \, dx \\
& + \sum_{K\in\mathcal{T}_h} \int_{\partial K^- \setminus \Gamma} \beta \cdot \mathbf{n}(y^e - y) v \, ds - \sum_{K\in\mathcal{T}_h} \int_{\partial K^- \cap \Gamma^-} \beta \cdot \mathbf{n} y v \, ds,
\end{aligned} \tag{3.4a}$$

$$\begin{aligned}
l_h(v) \quad = \quad & \sum_{K\in\mathcal{T}_h} \int_K f v \, dx + \sum_{E\in\mathcal{E}_h^\partial} \frac{\sigma\varepsilon}{h_E} \int_E d\mathbf{n} \cdot [\![v]\!] \, ds - \sum_{E\in\mathcal{E}_h^\partial} \int_E d\{\{\varepsilon \nabla v\}\} \, ds \\
& - \sum_{K\in\mathcal{T}_h} \int_{\partial K^- \cap \Gamma^-} \beta \cdot \mathbf{n} \, d v \, ds
\end{aligned} \tag{3.4b}$$

with the nonnegative real parameter σ being called the penalty parameter. We choose σ to be sufficiently large, independently of the mesh size h and the diffusion coefficient ε to ensure the stability of the DG discretization as described in [21, Sec. 2.7.1] with a lower bound depending only on the polynomial degree. Large penalty parameters decrease the jumps across element interfaces, which can affect the numerical approximation. Further, the DG approximation converges to the continuous Galerkin approximation as the penalty parameter goes to infinity (see, e.g., [6] for details).

3.1 Discretization of Moreau-Yosida-based optimal control problem

The DG discretization of the optimal control problem regularized by Moreau-Yosida (2.7) is given as follows:

$$\begin{aligned}
\text{minimize } J^M(y_h,u_h) = & \frac{1}{2} \sum_{K\in\mathcal{T}_h} \|y_h - y_h^d\|^2_{L^2(K)} + \sum_{K\in\mathcal{T}_h} \frac{\omega}{2} \|u_h - u_h^d\|^2_{L^2(K)} \qquad (3.5a) \\
& + \sum_{K\in\mathcal{T}_h} \frac{1}{2\delta_M} \|\max\{0, y_h - y^b\}\|^2_{L^2(K)} + \frac{1}{2\delta_M} \|\min\{0, y_h - y^a\}\|^2_{L^2(K)}
\end{aligned}$$

subject to

$$a_h(y_h,v_h) = l_h(v_h) + (u_h,v_h) \qquad \forall (y_h,u_h,v_h) \in (Y_h,U_h,V_h). \tag{3.5b}$$

The existence of a solution of (3.5) as well as of Lagrange multipliers follows from standard arguments. The DG discretized optimal control problem (3.5) has a unique solution $(y_h,u_h) \in Y_h \times U_h$. The functions $(y_h,u_h) \in Y_h \times U_h$ solve (3.5) if and only if there exists an adjoint variable $p_h \in V_h$ such that the optimality system

$$a(y_h,v_h) = l_h(v_h) + (u_h,v_h) \qquad \forall v_h \in V_h, \tag{3.6a}$$

$$a(\psi_h,p_h) = (y_h - y_h^d, \psi_h) + (\sigma_h^M, \psi_h) \qquad \forall \psi_h \in V_h, \tag{3.6b}$$

$$(\omega(u_h - u_h^d), w_h) + (p_h,w_h) = 0 \qquad \forall w_h \in U_h \tag{3.6c}$$

is satisfied. The multiplier σ_h^M is

$$\sigma_h^M = \frac{1}{\delta_M}\big(\max\{0, y_h - y^b\} + \min\{0, y_h - y^a\}\big). \tag{3.7}$$

3.2 Discretization of Lavrentiev-based optimal control problem

The DG discretization of the mixed control-state distributed optimal control problem (2.9)-(2.10) is given as follows:

$$\text{minimize } J^L(y_h, u_h) = \frac{1}{2}\sum_{K\in\mathcal{T}_h} \|y_h - y_h^d\|^2_{L^2(K)} + \sum_{K\in\mathcal{T}_h} \frac{\omega}{2}\|u_h - u_h^d\|^2_{L^2(K)} \tag{3.8a}$$

subject to

$$a_h(y_h, v_h) = l_h(v_h) + (u_h, v_h) \quad \text{and} \quad y^a \leq \delta_L u_h + y_h \leq y^b. \tag{3.8b}$$

The optimal solution $(y_h, u_h) \in Y_h \times U_h$ of (3.8) is characterized by the existence of an adjoint state $p_h \in V_h$ and a multiplier $\sigma_h^L \in U_h$ such that

$$a(y_h, v_h) = l(v_h) + (u_h, v_h) \qquad \forall v_h \in V_h, \tag{3.9a}$$

$$a(\psi_h, p_h) = (y_h - y_h^d, \psi_h) + (\sigma_h^L, \psi_h) \qquad \forall \psi_h \in V_h, \tag{3.9b}$$

$$(\omega(u_h - u_h^d), w_h) + (p_h, w_h) + (\delta_L \sigma_h^L, w_h) = 0 \qquad \forall w_h \in U_h \tag{3.9c}$$

with

$$\sigma_h^L = \max\{0, \sigma_h^L + (\delta_L u_h + y_h - y^b)\} - \min\{0, \sigma_h^L - (y^a - \delta_L u_h - y_h)\}. \tag{3.9d}$$

Further, the complementary condition (3.9d) is equivalent to

$$\begin{aligned}
&(\sigma_h^a, y^a - \delta_L u_h - y_h) = (\sigma_h^b, \delta_L u_h + y_h - y^b) = 0, \\
&\sigma_h^L = \sigma_h^b - \sigma_h^a, \quad \sigma_h^a \geq 0, \quad \sigma_h^b \geq 0, \quad y^a \leq \delta_L u_h + y_h \leq y^b.
\end{aligned} \tag{3.10}$$

As in the continuous setting, we decompose the discrete adjoint p_h as

$$p_h = \bar{p}_h + \bar{\sigma}_h^L, \tag{3.11}$$

where a regularized discrete adjoint $\bar{p}_h \in V_h$ and a regularized discrete multiplier $\bar{\sigma}_h^L \in V_h$ are solutions of

$$a(\psi_h, \bar{p}_h) = (y_h - y_h^d, \psi_h) \qquad \forall \psi_h \in V_h, \tag{3.12a}$$

$$a(\psi_h, \bar{\sigma}_h^L) = (\sigma_h^L, \psi_h) \qquad \forall \psi_h \in V_h, \tag{3.12b}$$

respectively.

Remark 3.1 *For our computations we have to replace the state constraints y^a and y^b by using finite dimensional approximations, y_h^a and y_h^b, respectively, either due to interpolation as a finite element function $y_h^a, y_h^b \in V_h$ or by numerical integration. However, we neglect the errors introduced by the discretization of the state constraints. Hence, we take $y^a = y_h^a$ and $y^b = y_h^b$. Although we still obtain satisfactory results, the estimation of this discretization error should be addressed in future work.*

4 The residual-type error estimator

In this section, we introduce our error indicators for optimal control problem regularized by Moreu-Yosida (2.5)-(2.6) and by Lavrentiev (2.9)-(2.10). We measure the error in the state and adjoint by using the norm $|||\cdot|||$ and the semi-norm $|\cdot|$ [24] which are defined by

$$|||y|||^2 = \sum_{K\in\mathcal{T}_h} (\|\varepsilon\nabla y\|^2_{L^2(K)} + r_0\|y\|^2_{L^2(K)}) + \sum_{E\in\mathcal{E}_h} \frac{\sigma\varepsilon}{h_E}\|[\![y]\!]\|^2_{L^2(E)}, \tag{4.1}$$

$$|y|^2_A = |\beta y|^2_* + \sum_{E\in\mathcal{E}_h} (r_0 h_E + \frac{h_E}{\varepsilon})\|[\![y]\!]\|^2_{L^2(E)}, \tag{4.2}$$

where

$$|q|_* = \sup_{v\in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega q\cdot\nabla v dx}{|||v|||} \qquad \text{for } q\in L^2(\Omega)^2. \tag{4.3}$$

The terms $|\beta y|^2_*$ and $h_E\varepsilon^{-1}\|[\![y]\!]\|^2_{L^2(E)}$ of the semi-norm $|\cdot|_A$ will be used to bound the convective derivative, similar to [24, 26]. The other term $r_0 h_E\|[\![y]\!]\|^2_{L^2(E)}$ is related to the reaction term.

Let

$$f_h, y^d_h, u^d_h, r_h \in V_h, \quad \beta_h\in V_h^2$$

denote approximations to the right hand side f, the desired state y^d, the desired control u^d, the reaction term r and the convection β, respectively. We define weights by $\rho_K = \min\{h_K\varepsilon^{-\frac{1}{2}}, r_0^{-\frac{1}{2}}\}$, $\rho_E = \min\{h_E\varepsilon^{-\frac{1}{2}}, r_0^{-\frac{1}{2}}\}$. When $r_0 = 0$, $\rho_K = h_K\varepsilon^{-\frac{1}{2}}$ and $\rho_E = h_E\varepsilon^{-\frac{1}{2}}$ are taken.

Our a posteriori error indicators are given by

$$\eta^z = \Big(\sum_{K\in\mathcal{T}_h} (\eta^z_{R_K})^2 + (\eta^z_{E_K})^2 + (\eta^z_{J_K})^2\Big)^{1/2}, \tag{4.4}$$

where $z\in\{y,p\}$ for Moreau-Yosida regularization and $z\in\{y,\bar{p}\}$ for Lavrentiev regularization.

For $K\in\mathcal{T}_h$, the interior residual terms are defined by

$$\begin{aligned}
(\eta^y_{R_K})^2 &= \rho_K\|f_h + u_h + \varepsilon\Delta y_h - \beta_h\cdot\nabla y_h - \alpha_h y_h\|^2_{L^2(K)},\\
(\eta^p_{R_K})^2 &= \rho_K\|(y_h - y^d_h) + \varepsilon\Delta p_h + \beta_h\cdot\nabla p_h - (\alpha_h - \nabla\cdot\beta_h)p_h\\
&\qquad + \frac{1}{\delta_M}\big(\max\{0, y_h - y^b\} + \min\{0, y_h - y^a\}\big)\|^2_{L^2(K)},\\
(\eta^{\bar{p}}_{R_K})^2 &= \rho_K\|(y_h - y^d_h) + \varepsilon\Delta\bar{p}_h + \beta_h\cdot\nabla\bar{p}_h - (\alpha_h - \nabla\cdot\beta_h)\bar{p}_h\big)\|^2_{L^2(K)}
\end{aligned}$$

and the edge residuals with the terms measuring the jumps

$$\begin{aligned}
(\eta^z_{E_K})^2 &= \frac{1}{2}\sum_{E\in\partial K\setminus\Gamma} \varepsilon^{-\frac{1}{2}}\rho_E\|[\![\varepsilon\nabla z_h]\!]\|^2_{L^2(E)},\\
(\eta^z_{J_K})^2 &= \frac{1}{2}\sum_{E\in\partial K\setminus\Gamma} \big(\frac{\sigma\varepsilon}{h_E} + r_0 h_E + \frac{h_E}{\varepsilon}\big)\|[\![z_h]\!]\|^2_{L^2(E)}\\
&\quad + \sum_{E\in\partial K\cap\Gamma} \big(\frac{\sigma\varepsilon}{h_E} + r_0 h_E + \frac{h_E}{\varepsilon}\big)\|[\![z_h]\!]\|^2_{L^2(E)}.
\end{aligned}$$

Our residual-type a posteriori error estimator of the optimal control problem regularized by the Morau-Yosida-based technique involves estimators of the state y and of the adjoint p, while the estimator of the optimal control problem regularized by the Lavrentiev-based technique involves estimators of the state y and of the regularized adjoint $\bar{p}$.

We use an active set strategy to eliminate the max- and min- functions in the interior residual term $\eta^p_{R_K}$. The active sets are determined by

$$\mathcal{A}_M^+ = \{x \in \Omega \,:\, y - y^b > 0\}, \qquad \mathcal{A}_M^- = \{x \in \Omega \,:\, y - y^a < 0\}. \tag{4.5}$$

Then, $\eta^p_{R_K}$ is defined as

$$\begin{aligned}\eta^p_{R_K} =& \rho_K \|(y_h - y_h^d) + \varepsilon\Delta p_h + \beta_h \cdot \nabla p_h - (\alpha_h - \nabla\cdot\beta_h) p_h \\ &+ \frac{1}{\delta_M}\left(\chi_{\mathcal{A}_M} y - \chi_{\mathcal{A}_M^+} y^b - \chi_{\mathcal{A}_M^-} y^a\right)\|_{L^2(K)},\end{aligned}$$

where $\chi_{\mathcal{A}_M^+}$, $\chi_{\mathcal{A}_M^-}$ and $\chi_{\mathcal{A}_M}$ denote the characteristic functions of $\mathcal{A}_M^+$, $\mathcal{A}_M^-$ and $\mathcal{A}_M = \mathcal{A}_M^+ \cup \mathcal{A}_M^-$, respectively. Since the characteristic functions depend on the function of y, we approximate them by the finite element solution as done in [19]. For $\mu > 0$, let

$$\chi_{\mathcal{A}_M} = \frac{(y_h - y^a)(y_h - y^b)}{h^\mu + (y_h - y^a)(y_h - y^b)}, \quad \chi_{\mathcal{A}_M^+} = \frac{(y_h - y^b)}{h^\mu + (y_h - y^b)}, \quad \chi_{\mathcal{A}_M^-} = \frac{(y_h - y^a)}{h^\mu + (y_h - y^a)}.$$

Finally, we introduce data approximation errors by

$$\theta_L^z = \Big(\sum_{K\in\mathcal{T}_h} (\theta_K^z)^2\Big)^{1/2}, \tag{4.6}$$

where $z \in \{y, p, u\}$ for Moreau-Yosida regularization and $z \in \{y, \bar{p}, u\}$ for Lavrentiev regularization. Data approximation terms are given by

$$\begin{aligned}(\theta_K^y)^2 &= \rho_K^2\left(\|f - f_h\|^2_{L^2(K)} + \|(\beta - \beta_h)\cdot\nabla y_h\|^2_{L^2(K)} + \|(\alpha - \alpha_h) y_h\|^2_{L^2(K)}\right), \\ (\theta_K^p)^2 &= \rho_K^2\Big(\|y_h^d - y^d\|^2_{L^2(K)} + \|(\beta - \beta_h)\cdot\nabla p_h\|^2_{L^2(K)} \\ &\qquad + \|\big((\alpha - \nabla\cdot\beta) - (\alpha_h - \nabla\cdot\beta_h)\big) p_h\|^2_{L^2(K)}\Big), \\ (\theta_K^{\bar{p}})^2 &= \rho_K^2\Big(\|y_h^d - y^d\|^2_{L^2(K)} + \|(\beta - \beta_h)\cdot\nabla \bar{p}_h\|^2_{L^2(K)} \\ &\qquad + \|\big((\alpha - \nabla\cdot\beta) - (\alpha_h - \nabla\cdot\beta_h)\big) \bar{p}_h\|^2_{L^2(K)}\Big), \\ (\theta_K^u)^2 &= \omega\|u^d - u_h^d\|^2_{L^2(K)}.\end{aligned}$$

Throughout this section, we will use the symbols $\lesssim$ and $\gtrsim$ to denote bounds that are valid up to positive constants independent of the local mesh sizes, the diffusion coefficient ε and the penalty parameter σ, provided that $\sigma \geq 1$.

The reliability and efficiency of our estimator are proven provided that the state equation (2.2) has homogeneous boundary conditions, as proven in [24] for a single convection diffusion equation.

We will need the following inequalities a few times along the analysis of the error estimators:

$$\|\max\{0,a\}-\max\{0,b\}\|_{L^2(\Omega)} \leq \|a-b\|_{L^2(\Omega)}, \tag{4.7a}$$

$$\|\min\{0,a\}-\min\{0,b\}\|_{L^2(\Omega)} \leq \|a-b\|_{L^2(\Omega)}. \tag{4.7b}$$

Next, we state the continuous dependence of the solution to the state equation, and of the solution to the adjoint equation.

Lemma 4.1 *Let (2.3) and (2.4) be satisfied with $r_0 > 0$, and let $g \in L^2(\Omega)$. If $y \in V$ solves $a(y,v) = (g,v)$ for all $v \in V$, then*

$$|||y||| + |y|_A \lesssim \|g\|_{L^2(\Omega)}. \tag{4.8}$$

If $q \in V$ solves $a(v,q) = (g,v)$ for all $v \in V$, then

$$|||q||| + |q|_A \lesssim \|g\|_{L^2(\Omega)}. \tag{4.9}$$

Proof. The papers [26, L. 3.1] and [24, L. 4.4] prove the existence of a constant $C > 0$ such that

$$\inf_{y\in H_0^1(\Omega)\setminus\{0\}} \sup_{v\in H_0^1(\Omega)\setminus\{0\}} \frac{a(y,v)}{(|||y|||+|y|_A)|||v|||} \geq C > 0.$$

Since $r_0 > 0$, we have $(g,v) \leq \|g\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \lesssim \|g\|_{L^2(\Omega)}|||v|||$. If $y \in V$ solves $a(y,v) = (g,v)$ for all $v \in V$, then the inf-sup condition implies

$$(|||y||| + |y|_A)|||v||| \lesssim a(y,v) = (g,v) \lesssim \|g\|_{L^2(\Omega)}|||v|||,$$

which is the desired inequality (4.8).

The inequality (4.9) can be proven analogously. □

Remark 4.2 *If $r_0 = 0$, then, since $v \in V = H_0^1(\Omega)$, $(g,v) \leq \|g\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \lesssim \varepsilon^{-1}\|g\|_{L^2(\Omega)}|||v|||$, and therefore the constants in (4.8) and (4.9) would depend on ε^{-1}. The assumption $r_0 > 0$ makes the constant in the estimate $(g,v) \leq \|g\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \lesssim \|g\|_{L^2(\Omega)}|||v|||$ independent of ε, as desired. In the following, we will use the bound $\|v\|_{L^2(\Omega)} \lesssim |||v|||$ a few times, which is possible since $r_0 > 0$. We note that this assumption is also made for analysis of optimal control problems governed by convection dominated equations [1, 12, 17, 28, 29, 31].*

4.1 Reliability estimates of a posteriori error estimators

We first derive the reliability estimate of our a posteriori error estimator of the optimal control problem regularized by the Moreau-Yosida approach.

Theorem 4.3 *Assume that (2.3) and (2.4) are satisfied. Let (y,u,p) and (y_h,u_h,p_h) be the solutions of (2.8) and (3.6), respectively. If the error estimators and the data approximation errors are defined in (4.4) and (4.6), then we have the a posteriori error bound*

$$\begin{aligned} &\|u-u_h\|_{L^2(\Omega)} + |||y-y_h||| + |y-y_h|_A \\ &\qquad + |||p-p_h||| + |p-p_h|_A \lesssim \eta^y + \theta^y + \eta^p + \theta^p + \theta^u. \end{aligned}$$

Proof. We first define the following auxiliary functions. For a given $w \in L^2(\Omega)$, the auxiliary state $y[w] \in H^2(\Omega)$ and the auxiliary adjoint $p[w] \in H^2(\Omega)$ are defined according to

$$a(y[w],v) = (w,v) + l(v) \qquad \forall v \in V, \tag{4.10}$$
$$a(v,p[w]) = (y[w] - y^d, v) + (\sigma^M[w], v) \qquad \forall v \in V. \tag{4.11}$$

From (4.10)-(4.11) and (2.8), we have

$$a(y - y[u_h], v) = (u - u_h, v) \qquad \forall v \in V_h, \tag{4.12}$$
$$a(v, p - p[u_h]) = (y - y[u_h], v) + (\sigma^M - \sigma^M[u_h], v) \qquad \forall v \in V_h. \tag{4.13}$$

Then, application of the continuity results in Lemma 4.1 yields

$$|||y - y[u_h]||| + |y - y[u_h]|_A \lesssim \|u - u_h\|_{L^2(\Omega)}, \tag{4.14}$$
$$|||p - p[u_h]||| + |p - p[u_h]|_A \lesssim \|y - y[u_h]\|_{L^2(\Omega)} + \|\sigma^M - \sigma^M[u_h]\|_{L^2(\Omega)}. \tag{4.15}$$

Now, we will establish a connection between the control u and the adjoint p by using the convexity of the linear quadratic optimal control problem, see, e.g., [18, pp. 1328,1329]). Set

$$\begin{aligned} j(u) = J^M(y,u) :=& \frac{1}{2}\|y - y^d\|^2_{L^2(\Omega)} + \frac{\omega}{2}\|u - u^d\|^2_{L^2(\Omega)} \\ &+ \frac{1}{2\delta_M}\|\max\{0, y - y^b\}\|^2_{L^2(\Omega)} + \frac{1}{2\delta_M}\|\min\{0, y - y^a\}\|^2_{L^2(\Omega)}. \end{aligned}$$

Then,

$$(j'(u), v) = (\omega(u - u^d) + p, v), \tag{4.16}$$
$$(j'(u_h), v) = (\omega(u_h - u_h^d) + p[u_h], v), \tag{4.17}$$

where p and $p[u_h]$ are the solutions of (2.8b) and (4.11), respectively. Hence, we write

$$(j'(u) - j'(u_h), u - u_h) = (\omega(u - u_h), u - u_h) + (\omega(u_h^d - u^d), u - u_h) + (p - p[u_h], u - u_h).$$

The equations (4.10) and (4.11) yield

$$\begin{aligned} (u - u_h, p - p[u_h]) &= (u, p - p[u_h]) - (u_h, p - p[u_h]) \\ &= a(y - y[u_h], p - p[u_h]) \\ &= (y, y - y[u_h]) - (y^d, y - y[u_h]) + (\sigma^M, y - y[u_h]) \\ &\quad -(y[u_h], y - y[u_h]) + (y^d, y - y[u_h]) - (\sigma^M[u_h], y - y[u_h]) \\ &= (y - y[u_h], y - y[u_h]) + (\sigma^M - \sigma^M[u_h], y - y[u_h]). \end{aligned} \tag{4.18}$$

Using the equation (4.18) with the optimality conditions (2.8c) and (3.6c), we obtain

$$
\begin{aligned}
\omega\|u-u_h\|^2_{L^2(\Omega)} &\leq (j'(u)-j'(u_h),u-u_h)\\
&\quad +(\sigma^M[u_h]-\sigma^M,y-y[u_h])+\omega(u^d-u_h^d,u-u_h)\\
&= -(\omega(u_h-u_h^d)+p[u_h],u-u_h)\\
&\quad +(\sigma^M[u_h]-\sigma^M,y-y[u_h])+\omega(u^d-u_h^d,u-u_h)\\
&= (p_h-p[u_h],u-u_h)-(\omega(u_h-u_h^d)+p_h,u-u_h)\\
&\quad +(\sigma^M[u_h]-\sigma^M,y-y[u_h])+\omega(u^d-u_h^d,u-u_h)\\
&= (p_h-p[u_h],u-u_h)+\omega(u^d-u_h^d,u-u_h)\\
&\quad +(\sigma^M[u_h]-\sigma^M,y-y[u_h]).
\end{aligned}
$$

By applying Cauchy-Schwarz and Young's inequalities, we obtain

$$
\begin{aligned}
\|u-u_h\|^2_{L^2(\Omega)} &\lesssim \|p_h-p[u_h]\|^2_{L^2(\Omega)}+\omega\|u^d-u_h^d\|^2_{L^2(\Omega)}\\
&\quad +\|\sigma^M[u_h]-\sigma^M\|^2_{L^2(\Omega)}+\|y-y[u_h]\|^2_{L^2(\Omega)}.
\end{aligned} \tag{4.19}
$$

Then, application of the triangle inequality and the inequalities in (4.7), (4.14) yield

$$
\|\sigma^M[u_h]-\sigma^M\|_{L^2(\Omega)} \leq 2\delta_M\|y[u_h]-y\|_{L^2(\Omega)} \lesssim \|u-u_h\|_{L^2(\Omega)}. \tag{4.20}
$$

The following result, obtained by inserting the estimate (4.20) into (4.19),

$$
\|u-u_h\| \lesssim \|p_h-p[u_h]\|+\theta^u \tag{4.21}
$$

shows the connection between the control and the adjoint.

We will now construct the connection between the adjoint p and the state y. Let $\widetilde{p}$ solve (2.8b) with $y=y_h$. Then, the difference $p[u_h]-\widetilde{p}$ solves

$$
a(v,p[u_h]-\widetilde{p})=(y[u_h]-y_h,v)+(\sigma^M[u_h]-\sigma^M(y_h),v) \quad \forall v\in V,
$$

where

$$
\sigma^M(y_h)=\frac{1}{\delta_M}\big(\max\{0,y_h-y^b\}+\min\{0,y_h-y^a\}\big).
$$

Lemma 4.1 implies

$$
\begin{aligned}
|||p[u_h]-\widetilde{p}|||+|p[u_h]-\widetilde{p}|_A &\lesssim \|y[u_h]-y_h+\sigma^M[u_h]-\sigma^M(y_h)\|_{L^2(\Omega)}\\
&\leq \|y[u_h]-y_h\|_{L^2(\Omega)}+\|\sigma^M[u_h]-\sigma^M(y_h)\|_{L^2(\Omega)}.
\end{aligned}
$$

By the inequality $\|\sigma^M[u_h]-\sigma^M(y_h)\|_{L^2(\Omega)} \leq 2\delta_M\|y[u_h]-y_h\|_{L^2(\Omega)}$ derived from the triangle inequality and the inequalities in (4.7), we obtain

$$
|||p[u_h]-\widetilde{p}|||+|p[u_h]-\widetilde{p}|_A \lesssim \|y[u_h]-y_h\|_{L^2(\Omega)}. \tag{4.22}
$$

Since $\widetilde{p}$ is the solution of (2.8b) with $y=y_h$ and p_h is the solution of (3.6b), we have

$$
|||\widetilde{p}-p_h|||+|\widetilde{p}-p_h|_A \lesssim \eta^p+\theta^p, \tag{4.23}
$$

which is obtained by adapting the notation in [24, Thm. 3.2] for a single convection diffusion equation. Then, combination of (4.22) and (4.23) yields

$$|||p[u_h] - p_h||| + |p[u_h] - p_h|_A \lesssim \eta^p + \theta^p + \|y_h - y[u_h]\|_{L^2(\Omega)}. \tag{4.24}$$

Finally, combining the results (4.14), (4.15), (4.21) and (4.24) with the following reliability estimate obtained by the adaptation of the result in [24, Thm. 3.2]

$$|||y[u_h] - y_h||| + |y[u_h] - y_h|_A \lesssim \eta^y + \theta^y, \tag{4.25}$$

the desired result is obtained.

□

Now, we present the reliability estimate of the error estimator of the optimal control problems regularized by Lavrentiev regularization.

Theorem 4.4 *Assume that (2.3) and (2.4) are satisfied. Let (y, u, p, σ^L) and $(y_h, u_h, p_h, \sigma_h^L)$ be the solutions of (2.11) and (3.9), and let the error estimators, the data approximation errors and the consistency error $e_{c,h}$ be defined in (4.4), (4.6) and (4.29), respectively. Further, if $\bar{p}$ and $\bar{p}_h$ are the regularized adjoints as given by (2.14a) and (3.12a), then we have the a posteriori error bound*

$$\begin{aligned} &\|u - u_h\|_{L^2(\Omega)} + |||y - y_h||| + |y - y_h|_A \\ &\qquad + |||\bar{p} - \bar{p}_h||| + |\bar{p} - \bar{p}_h|_A \lesssim \eta^y + \theta^y + \eta^{\bar{p}} + \theta^{\bar{p}} + \theta^u + e_{c,h}. \end{aligned}$$

Proof. To prove our reliability result, we need the following auxiliary functions. For given $w \in L^2(\Omega)$, we let $y[w] \in H^1(\Omega)$ and $\bar{p}[w] \in H^1(\Omega)$ denote the solutions of

$$a(y[w], v) = (w, v) + l(v) \qquad \forall v \in V, \tag{4.26}$$

$$a(v, \bar{p}[w]) = (y[w] - y^d, v) \qquad \forall v \in V. \tag{4.27}$$

We further introduce an auxiliary discrete state $y_h[u] \in V_h$ as the solution of

$$a(y_h[u], v_h) - (u, v_h) = l(v_h) \qquad \forall v \in V_h. \tag{4.28}$$

The auxiliary states $y[u_h] \in Y^L$ and $y_h[u] \in Y_h$ do not necessarily satisfy the mixed control-state constraints. Therefore, we introduce the consistency error

$$\begin{aligned} e_{c,h} = &\max\{(\sigma_h^b, \delta_L u + y_h[u] - y^b) + (\sigma^b, \delta_L u_h + y[u_h] - y^b), 0\} \\ &+ \max\{(\sigma_h^a, y^a - \delta_L u - y_h[u]) + (\sigma^a, y^a - \delta_L u_h - y[u_h]), 0\}. \end{aligned} \tag{4.29}$$

See, e.g., [14] to derive a computable upper bound.

Now, we will construct the connection between the control u and the regularized adjoint $\bar{p}$. Using (2.11c), (2.13) and (3.9c), (3.11), we find

$$\begin{aligned} \omega\|u - u_h\|_{L^2(\Omega)}^2 &= (\omega u^d - p - \delta_L \sigma^L, u - u_h) + (p_h + \delta_L \sigma_h^L - \omega u_h^d, u - u_h) \\ &= (\bar{p}_h - \bar{p}, u - u_h) + (\bar{\sigma}_h^L - \bar{\sigma}^L, u - u_h) \\ &\quad + (\delta_L(\sigma_h^L - \sigma^L), u - u_h) + (\omega(u^d - u_h^d), u - u_h). \end{aligned} \tag{4.30}$$

The first term on the right-hand side in (4.30) can be split according to

$$(u-u_h,\bar{p}_h-\bar{p}) = (u-u_h,\bar{p}_h-\bar{p}[u_h]) + (u-u_h,\bar{p}[u_h]-\bar{p}). \tag{4.31}$$

The equations (4.26) and (4.27) yield

$$\begin{aligned}(u-u_h,\bar{p}[u_h]-\bar{p}) &= a(y-y[u_h],\bar{p}[u_h]) - a(y-y[u_h],\bar{p}) \\ &= -(y-y[u_h],y-y[u_h]) \leq 0.\end{aligned} \tag{4.32}$$

Using (4.32) with Young's inequality in (4.31), we obtain

$$\begin{aligned}(u-u_h,\bar{p}_h-\bar{p}) &\leq (u-u_h,\bar{p}_h-\bar{p}[u_h]) \\ &\lesssim \|u-u_h\|^2_{L^2(\Omega)} + \|\bar{p}_h-\bar{p}[u_h]\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.33}$$

By (4.28), (3.9a) and (2.11a), (4.26), we find

$$(u-u_h,v_h-v) = a(y_h[u]-y_h,v_h) - a(y-y[u_h],v). \tag{4.34}$$

Then, choosing $v_h = \bar{\sigma}^L_h$ and $v = \bar{\sigma}^L$ in (4.34) and using (2.14b), (3.12b), we obtain

$$\begin{aligned}(u-u_h,\bar{\sigma}^L_h-\bar{\sigma}^L) &= a(y_h[u]-y_h,\bar{\sigma}^L_h) - a(y-y[u_h],\bar{\sigma}^L) \\ &= (\sigma^L_h,y_h[u]-y_h) - (\sigma^L,y_h[u]-y_h).\end{aligned} \tag{4.35}$$

Combining (4.35) with the third term on the right-hand side of (4.30), we obtain

$$\begin{aligned}\delta_L(u-u_h,\sigma^L_h-\sigma^L) + (u-u_h,\bar{\sigma}^L_h-\bar{\sigma}^L) &= (\delta_L u + y_h[u] - (\delta_L u_h + y_h),\sigma^L_h) \\ &\quad -(\delta_L u + y - (\delta_L u_h + y[u_h],\sigma^L).\end{aligned} \tag{4.36}$$

Then, using the complementary conditions (2.12), (3.10) and the definition of the consistency error (4.29), we find

$$\begin{aligned}\delta_L(u-u_h,\sigma^L_h-\sigma^L) &+ (u-u_h,\bar{\sigma}^L_h-\bar{\sigma}^L) \\ &= (\delta_L u + y_h[u] - y^b,\sigma^b_h) + (\delta_L u_h + y[u_h] - y^b,\sigma^b) \\ &\quad +(y^a - \delta_L u - y_h[u],\sigma^a_h) + (y^a - \delta_L u_h - y[u_h],\sigma^a) \\ &\leq e_{c,h}.\end{aligned} \tag{4.37}$$

For the last term on the right-hand side of (4.30), in view of an application of Young's inequality we obtain

$$(\omega(u^d-u^d_h),u-u_h) \lesssim \|u-u_h\|^2_{L^2(\Omega)} + \theta^u. \tag{4.38}$$

Employing the estimates (4.33), (4.37) and (4.38) in (4.30), the result

$$\|u-u_h\|^2_{L^2(\Omega)} \lesssim \|\bar{p}_h-\bar{p}[u_h]\|^2_{L^2(\Omega)} + e_{c,h} + \theta^u \tag{4.39}$$

is obtained.

We will now establish the connection between the regularized adjoint $\bar{p}$ and the state y. Let $\tilde{p}$ and $\bar{p}[u_h]$ solve (2.14a) and (4.27), respectively. Then, the continuity result in Lemma 4.1 yields

$$|||\bar{p}[u_h] - \tilde{p}||| + |\bar{p}[u_h] - \tilde{p}|_A \lesssim \|y[u_h] - y_h\|_{L^2(\Omega)}. \tag{4.40}$$

By adapting the notation from [24, Thm. 3.2] for a single convection diffusion equation, we obtain

$$|||\tilde{p} - \bar{p}_h||| + |\tilde{p} - \bar{p}_h|_A \lesssim \eta^{\bar{p}} + \theta^{\bar{p}}, \tag{4.41}$$

where $\tilde{p}$ is the solution of (2.14a) with $y = y_h$ and $\bar{p}_h$ is the solution of (3.12a). Then, combining (4.40) with (4.41), we obtain

$$|||\bar{p}[u_h] - \bar{p}_h||| + |\bar{p}[u_h] - \bar{p}_h|_A \lesssim \eta^{\bar{p}} + \theta^{\bar{p}} + \|y[u_h] - y_h\|_{L^2(\Omega)}. \tag{4.42}$$

From (4.26)-(4.27) and (2.14a), (3.12a), we have

$$a(y - y[u_h], v) = (u - u_h, v) \qquad \forall v \in V_h, \tag{4.43}$$

$$a(v, \bar{p} - \bar{p}[u_h]) = (y - y[u_h], v) \qquad \forall v \in V_h. \tag{4.44}$$

Then, applying the continuity results in Lemma 4.1 yields

$$|||y - y[u_h]||| + |y - y[u_h]|_A \lesssim \|u - u_h\|_{L^2(\Omega)}, \tag{4.45}$$

$$|||\bar{p} - \bar{p}[u_h]||| + |\bar{p} - \bar{p}[u_h]|_A \lesssim \|y - y[u_h]\|_{L^2(\Omega)}. \tag{4.46}$$

Finally, combining the results (4.45), (4.46), (4.39) and (4.42) with the following reliability estimate obtained by the adaptation of the result from [24, Thm. 3.2],

$$|||y[u_h] - y_h||| + |y[u_h] - y_h|_A \lesssim \eta^y + \theta^y, \tag{4.47}$$

the desired result is obtained. □

4.2 Efficiency estimates of a posteriori error estimators

In this section, we prove the efficiency estimates for both Moreau-Yosida-based and Lavrentiev-based optimal control problems. Efficiency of the estimator means that up to data oscillations it provides a lower bound for the discretization errors.

Theorem 4.5 *Assume that (2.3) and (2.4) are satisfied. Let (y, u, p) and (y_h, u_h, p_h) be the solutions of (2.8) and (3.6), respectively. If the error estimators and the data approximation errors are defined in (4.4) and (4.6), then we have the lower bound*

$$\begin{aligned} \eta^y + \eta^p \quad \lesssim \quad & \|u - u_h\|_{L^2(\Omega)} + |||y - y_h||| + |y - y_h|_A \\ & + |||p - p_h||| + |p - p_h|_A + \theta^y + \theta^p + \theta^u. \end{aligned}$$

Proof. By adapting the notation in [24, Thm. 3.3], we obtain the following efficiency results for a single convection diffusion equation. If $y[u_h]$ is the solution of (2.8a) with $u = u_h$ and y_h is the solution of (3.6a), then

$$\eta^y \lesssim |||y[u_h] - y_h||| + |y[u_h] - y_h|_A + \theta^y. \tag{4.48}$$

Further, if $\widetilde{p}$ is the solution of (2.8b) with $y = y_h$ and p_h is the solution of (3.6b), then

$$\eta^p \lesssim |||\widetilde{p} - p_h||| + |\widetilde{p} - p_h|_A + \theta^p. \tag{4.49}$$

Let $p[u_h]$ be the solution of (2.8b) with $y = y_h$. Then, the inequalities (4.49), (4.15) and (4.7) imply

$$\begin{aligned}
\eta^p &\lesssim |||p[u_h] - p_h||| + |p[u_h] - p_h|_A + \theta^p \\
&\leq |||p - p_h||| + |p - p_h|_A + |||p - p[u_h]||| + |p - p[u_h]|_A + \theta^p \\
&\leq |||p - p_h||| + |p - p_h|_A + \|y - y[u_h]\|_{L^2(\Omega)} + \|\sigma^M - \sigma^M[u_h]\|_{L^2(\Omega)} + \theta^p \\
&\lesssim |||p - p_h||| + |p - p_h|_A + \|u - u_h]\|_{L^2(\Omega)} + \theta^p. \qquad (4.50)
\end{aligned}$$

Analogously to the previous result (4.50), we obtain the following result for the state:

$$\eta^y \lesssim |||y - y_h||| + |y - y_h|_A + \theta^y + \|u - u_h\|_{L^2(\Omega)}. \qquad (4.51)$$

The desired result is obtained by combining the inequalities (4.50) and (4.51).

□

Now, we will state the efficiency estimate of the a posteriori estimator for the optimal control problem regularized by Lavrentiev-based technique. The proof of Theorem 4.6 is derived by applying the same procedure as done in Theorem 4.5. The only difference is the discretization error in the regularized adjoint $\bar{p}$ instead of the discretization error in the adjoint p.

Theorem 4.6 *Assume that (2.3) and (2.4) are satisfied. Let (y,u,p,σ^L) and (y_h,u_h,p_h,σ_h^L) be the solutions of (2.11) and (3.9), and let the error estimators and the data approximation errors are defined in (4.4) and (4.6), respectively. Further, if $\bar{p}$ and $\bar{p}_h$ are the regularized adjoints as given by (2.14a) and (3.12a), then we have the lower bound*

$$\begin{aligned}
\eta^y + \eta^{\bar{p}} \quad \lesssim \quad & \|u - u_h\|_{L^2(\Omega)} + |||y - y_h||| + |y - y_h|_A \\
& + |||\bar{p} - \bar{p}_h||| + |\bar{p} - \bar{p}_h|_A + \theta^y + \theta^{\bar{p}} + \theta^u.
\end{aligned}$$

5 Implementation details

5.1 The adaptive loop

The adaptive procedure consists of successive execution of the steps **SOLVE** → **ESTIMATE** → **MARK** → **REFINE**. The **SOLVE** step is the numerical solution of the optimal control problem with respect to the given triangulation $\mathcal{T}_h$ using the SIPG discretization. For the **ESTIMATE** step, the residual error estimators $(\eta_K^y)^2 + (\eta_K^p)^2$ or $(\eta_K^y)^2 + (\eta_K^{\bar{p}})^2$, $K \in \mathcal{T}_h$, defined in Section 4 are used. In the **MARK** step, the edges and elements for the refinement are specified by using the a posteriori error estimator and by choosing subsets $\mathcal{M}_K \subset \mathcal{T}_h$ such that the following bulk criterion is satisfied for the given marking parameter θ:

$$\theta \sum_{K\in\mathcal{T}_h} (\eta_K^y)^2 + (\eta_K^p)^2 \leq \sum_{K\in\mathcal{M}_K} (\eta_K^y)^2 + (\eta_K^p)^2. \qquad (5.1)$$

Finally, in the **REFINE** step, the marked elements are refined by longest edge bisection, where the elements of the marked edges are refined by bisection.

5.2 Numerical results

We present several numerical results for the state constrained optimal control problems governed by the convection diffusion equations, regularized by Moria-Yosida and Lavrentiev. We use piecewise linear polynomials for the discretization of the state, the adjoint, and the control variables. The discretized optimal control problems are solved by the primal dual active set (PDAS) algorithm as a semi-smooth Newton step, see, e.g., [4]. The penalty parameter in the SIPG is chosen as $\sigma = 6$ on interior edges and 12 on boundary edges. The refinement parameter θ in the bulk criteria has been specified as $\theta = 0.46$.

5.2.1 Example 1

We modified the optimal control problem in [22] governed by a pure elliptic equation into a convection dominated problem. The data of the problem given in polar coordinates are

$$\Omega = (-1,2)^2, \;\; \varepsilon = 10^{-8}, \;\; \beta = (1,2)^T, \;\; \alpha = 1, \;\; \omega = 1 \;\text{ and }\; u_d(x_1,x_2) = 0.$$

The source function $f(x_1,x_2)$ and desired state y_d are chosen so that the solution of the optimal control problem is given by

$$\begin{aligned}
y(r) &= \frac{1}{2\pi\omega}\chi_{r\le 1}\left(\frac{r^2}{4}(\log r - 2) + \frac{r^3}{4} + \frac{1}{4}\right),\\
p(r) &= \frac{1}{2\pi}\chi_{r\le 1}\left(\log r + r^2 - r^3\right),\\
u(r) &= -\frac{1}{2\pi\omega}\chi_{r\le 1}\left(\log r + r^2 - r^3\right),\\
\sigma(r) &= \delta_0,
\end{aligned}$$

where $r = \sqrt{x_1^2 + x_2^2}$, $\forall (x_1,x_2) \in \Omega$. The problem features only a lower state constraint defined by

$$y_a(r) = \frac{1}{2\pi\omega}\left(\frac{1}{4} - \frac{r}{2}\right).$$

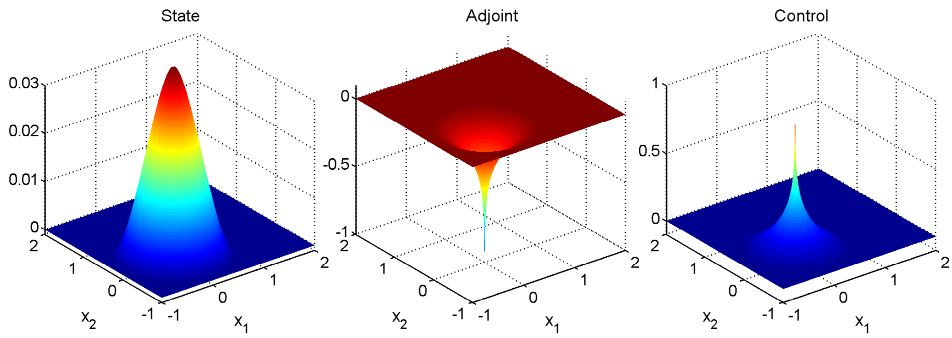

Figure 1: Example 5.2.1: The computed solutions of the state y, the adjoint p and the control u obtained by Moreau-Yosida regularization with $\delta_M = 10^{-4}$ on an adaptively refined mesh (13,308 vertices).

Figure 1 shows the computed solutions of the state y, the adjoint p and the control u obtained by Moreau-Yosida regularization with $\delta_M = 10^{-4}$ on an adaptively generated mesh with 13,308 vertices.

The initial mesh is generated by starting first dividing Ω into 8×8 uniform squares and then dividing each square into two triangles. We ensured that the point $x = (0,0)$ can not be a vertex of any mesh obtained from refinement. Here, we only give the adaptively generated mesh for the Lavrentiev-based technique in Figure 2 since both adaptive meshes obtained for the regularized problems are almost the same. The reason is that the multiplier $\sigma(r)$ is not dominant. So, the missing of $\bar{\sigma}^L$ in the error estimator is not important. The singularity at $(0,0)$, which appears in the adjoint and control equations is identified by the adaptive mesh as shown in Figure 2.

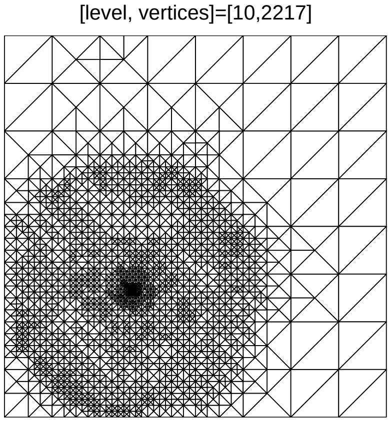

Figure 2: Example 5.2.1: Adaptively generated mesh obtained by Lavrentiev regularization with $\delta_L = 10^{-4}$ (2,217 vertices).

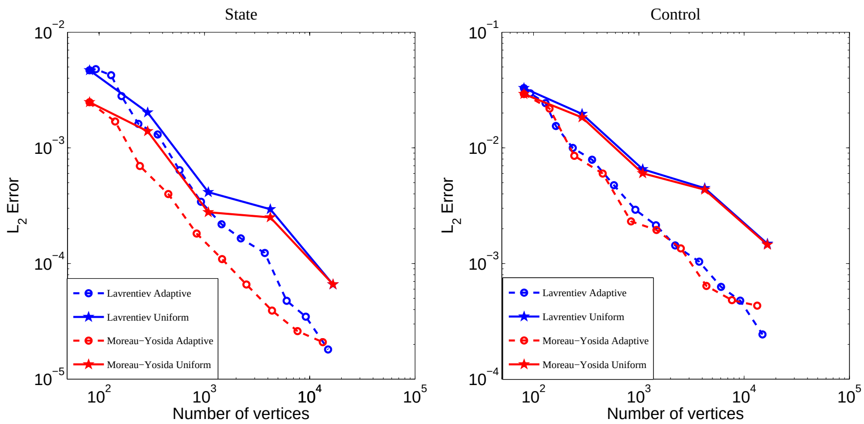

Figure 3: Example 5.2.1: L^2 errors in the state and the control obtained by Lavrentiev-based ($\delta_L = 10^{-4}$) and Moreau-Yosida-based ($\delta_M = 10^{-4}$) regularization techniques.

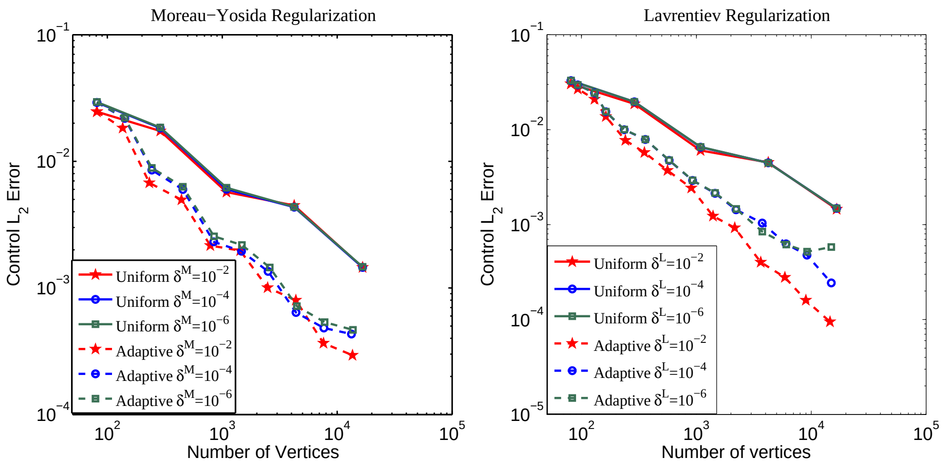

Figure 4: Example 5.2.1: L^2 errors in the control u on uniformly adaptively refined meshes obtained by Lavrentiev regularization with $\delta_L = 10^{-2}, 10^{-4}, 10^{-6}$ and Moreau-Yosida regularization with $\delta_M = 10^{-2}, 10^{-4}, 10^{-6}$.

Figure 3 illustrates the L^2 errors in the state and the control for both regularization techniques with regularization parameters $\delta_L = \delta_M = 10^{-4}$. The adaptive refinements lead to better approximate solutions than uniform refinements. Figure 4 shows the effect of the regularization parameters, i.e., δ_L and δ_M, on adaptively and uniformly refined meshes.

5.2.2 Example 2

This example is a modification of the one given in [13, 14]. Hoppe et al. have solved the state constrained optimal control problem governed by elliptic equation directly in [13] and with Lavrentiev type regularization in [14]. Here, we have modified it to become a diffusion convection reaction equation. The setup of the problem is as follows:

$$\Omega = (-2,2)^2, \ \ \epsilon = 10^{-6}, \ \ \beta = (1,2)^T, \ \ \alpha = 1, \ \ \omega = 0.1, \ \ y_b = 0 \ \text{ and } \ f(x_1,x_2) = 0.$$

The desired state $y_d(r)$ and desired control $u_d(r)$ are defined as

$$y_d(r) = y(r) + \epsilon \Delta p(r) + \beta \cdot \nabla p(r) - \alpha p(r) + \sigma(r) \ \text{ and } \ u_d(r) = u(r) + \omega^{-1} p(r),$$

where $r = \sqrt{x_1^2 + x_2^2}$, $\forall (x_1,x_2) \in \Omega$.

The state $y(r)$, adjoint $p(r)$, control $u(r)$ and multiplier $\sigma(r)$ represent the exact optimal solution of the state control problem according to

$$y(r) = \frac{-1}{\sqrt{\epsilon}} r^{\frac{4}{3}} \gamma_1(r), \quad p(r) = \frac{1}{\sqrt{\epsilon}} \gamma_2(r) \left(r^4 - \frac{3}{2} r^3 + \frac{9}{16} r^2 \right),$$

$$u(r) = -\epsilon \Delta y(r) + \beta \cdot \nabla y(r) + \alpha y(r) \ \text{ and } \ \sigma(r) = \begin{cases} 0, & r < 0.75, \\ 0.1, & \text{otherwise,} \end{cases}$$

where

$$\gamma_1(r) = \begin{cases} 1, & r < 0.25, \\ -192(r-0.25)^5 + 240(r-0.25)^4 - 80(r-0.25)^3 + 1, & 0.25 < r < 0.75, \\ 0, & \text{otherwise}, \end{cases}$$

$$\gamma_2(r) = \begin{cases} 1, & r < 0.75, \\ 0, & \text{otherwise}. \end{cases}$$

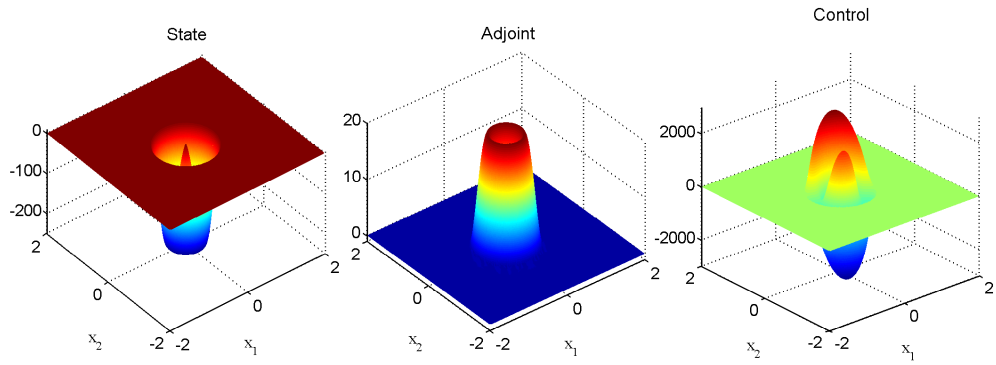

Figure 5: Example 5.2.2: The computed solutions of the state y, the adjoint p and the control u obtained by using the Moreau-Yosida-based technique with $\delta_M = 10^{-3}$ on an adaptively refined mesh (12,937 vertices).

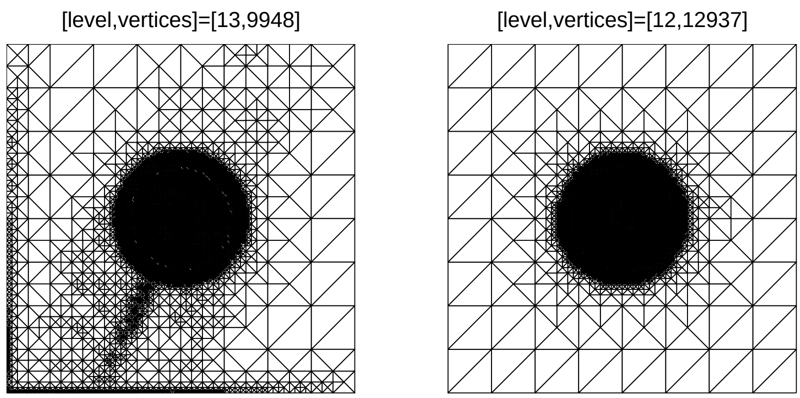

Figure 6: Example 5.2.2: Adaptively generated meshes obtained by Lavrentiev regularization with $\delta_L = 10^{-4}$ (9,948 vertices, left) and Moreau-Yosida regularization with $\delta_M = 10^{-3}$ (12,937 vertices, right).

Figure 5 shows the computed solutions of the state y, the adjoint p and the control u obtained by using the Moreau-Yosida-based technique with $\delta_M = 10^{-3}$ on an adaptively generated mesh with 12,937 vertices.

The initial mesh is generated by starting first dividing Ω into 8×8 uniform squares and then dividing each square into two triangles. Figure 6 displays the adaptive meshes obtained by the Lavrentiev-based technique with $\delta_L = 10^{-4}$ and the Moreau-Yosida-based technique with $\delta_M = 10^{-3}$. All refinement occurs in the center of the region for the Moreau-Yosida regularization, while there occur extra refinements outside of the circle for Lavrentiev regularization since the error estimator in (4.4) of the Lavrentiev regularized optimal control problem does not contain the regularized multiplier $\bar{\sigma}^L$. However, both regularization techniques save substantial computing time as shown in Figure 7.

Figure 7 illustrates the benefit of adaptive versus uniform refinement by showing L^2 errors in the state, the adjoint and the control. The effect of the regularization parameters, i.e., δ_L and δ_M, on adaptively and uniformly refined meshes shown in Figure 8. Independent of the regularization parameters, i.e., δ_L and δ_M, the adaptively refinements work well.

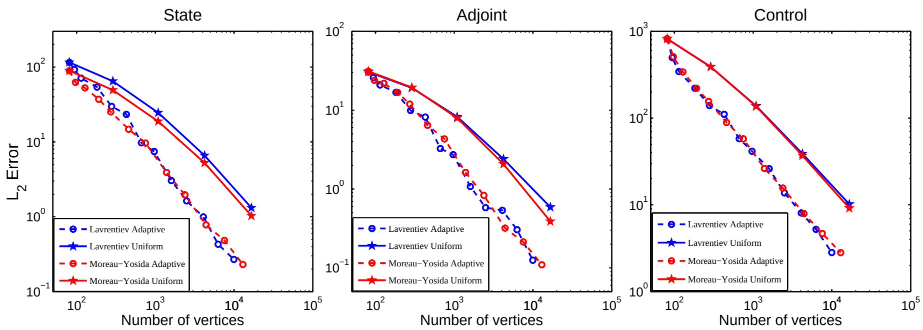

Figure 7: Example 5.2.2: L^2 errors in the state, the adjoint and the control obtained by Lavrentiev-based ($\delta_L = 10^{-4}$) and Moreau-Yosida-based ($\delta_M = 10^{-3}$) regularization techniques.

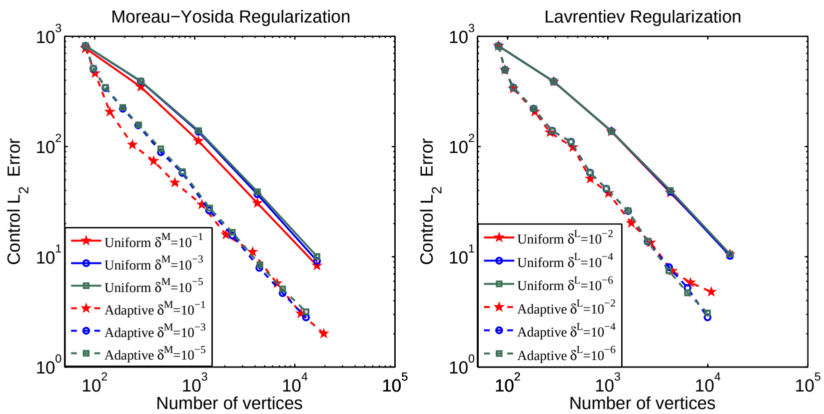

Figure 8: Example 5.2.2: L^2 errors in the control u on uniformly adaptively refined meshes obtained by Lavretiev regularization with $\delta_L = 10^{-2}, 10^{-4}, 10^{-6}$ and Moreau-Yosida regularization with $\delta_M = 10^{-1}, 10^{-3}, 10^{-5}$.

6 Conclusions

In this paper, we study a posteriori error estimates of the symmetric interior penalty Galerkin (SIPG) method for the state constrained optimal control problems governed by convection diffusion equations, regularized by Moreau-Yosida-based and Lavrentiev-based regularization techniques. Piecewise linear polynomials are used to discretize the unknown variables. Reliability and efficiency estimates are derived for both regularization techniques. The numerical results show that the adaptive refinements are superior to uniform refinements independent of the regularization parameters, i.e., δ_L, δ_M. Future work will include the extension of our results to time dependent problems and problems with nonlinear reaction terms.

Acknowledgements

The authors would like to thank Martin Stoll for helpful discussions about this research. The first author is partially supported for this work by Forschungszentrum Dynamische Systeme (CDS): Biosystemtechnik, Otto-von-Guericke-Universität Magdeburg.

References

[1] Becker, R., Vexler, B.: Optimal control of the convection-diffusion equation using stabilized finite element methods. Numer. Math. **106**(3), 349–367 (2007)

[2] Benedix, O., Vexler, B.: A posteriori error estimation and adaptivity for elliptic optimal control problems with state constraints. Comput. Optim. App. **44**(1), 3–25 (2009)

[3] Bergounioux, M., Haddou, M., Hintermueller, M., Kunisch, K.: A comparison of interior–point methods and a moreau–yosida based active set strategy for constrained optimal control problems. SIAM J. on Optimization **11**, 495–521 (2000)

[4] Bergounioux, M., Ito, K., Kunisch, K.: Primal-dual strategy for constrained optimal control problems. SIAM J. Control Optim. **37**(4), 1176–1194 (1999)

[5] Bergounioux, M., Kunisch, K.: On the structure of lagrange multipliers for state-constrained optimal control prolems. Systems and Control Letters **48**, 169–176 (2003)

[6] Cangiani, A., Chapman, J., Georgoulis, E.H., Jensen, M.: On local super-penalization of interior penalty discontinuous Galerkin methods. CoRR **abs/1205.5672** (2012)

[7] Casas, E.: Control of an elliptic problem with pointwise state constraints. SIAM J. Control Optim. **24**(6), 1309–1318 (1986)

[8] Gong, W., Yan, N.: A mixed finite element scheme for optimal control problems with pointwise state constraints. J. Sci. Comput. **46**, 182–203 (2011)

[9] Günther, A., Hinze, M.: A posteriori error control of a state constrained elliptic control problem. J. Numer. Math. **16**, 307–322 (2008)

[10] Hintermüller, M., Hoppe, R.H.W.: Goal-oriented adaptivity in pointwise state constrained optimal control of partial differential equations. SIAM J. Control Optim. **48**, 5468–5487 (2010)

[11] Hinze, M., Schiela, A.: Discretization of interior point methods for state constrained elliptic optimal control problems: optimal error estimates and parameter adjustment. Comput. Optim. App. **48**, 581–600 (2011)

[12] Hinze, M., Yan, N., Zhou, Z.: Variational discretization for optimal control governed by convection dominated diffusion equations. J. Comp. Math. **27**(2-3), 237–253 (2009)

[13] Hoppe, R.H.W., Kieweg, M.: A posteriori error estimation of finite element approximations of pointwise state constrained distributed control problems. J. Numer. Math. **17**(3), 219–244 (2009)

[14] Hoppe, R.H.W., Kieweg, M.: Adaptive finite element methods for mixed control-state constrained optimal control problems for elliptic boundary value problems. Comput. Optim. Appl. **46**(3), 511–533 (2010)

[15] Ito, K., Kunisch, K.: Semi-smooth Newton methods for state-constrained optimal control problems. Systems and Control Letters **50**, 221–228 (2003)

[16] Krumbiegel, K., Rösch, A.: A virtual control concept for state constrained optimal control problems. Comput. Optim. Appl. **43**(2), 213–233 (2009)

[17] Leykekhman, D., Heinkenschloss, M.: Local error analysis of discontinuous Galerkin methods for advection-dominated elliptic linear-quadratic optimal control problems. SIAM J. Numer. Anal. **50**(4), 2012–2038 (2012)

[18] Li, R., Liu, W., Ma, H., Tang, T.: Adaptive finite element approximation for distributed elliptic optimal control problems. SIAM J. Control Optim. **41**(5), 1321–1349 (2002)

[19] Liu, W.B., Yan, N.N.: A posteriori error estimators for a class of variational inequalities. J. Sci. Comput. **35**, 361–393 (2000)

[20] Meyer, C., Prüfert, U., Tröltzsch, F.: On two numerical methods for state-constrained elliptic control problems. Optim. Methods Softw. **22**(6), 871–899 (2007)

[21] Rivière, B.: Discontinuous Galerkin methods for solving elliptic and parabolic equations. Theory and implementation, *Frontiers in Applied Mathematics*, vol. 35. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (2008)

[22] Rösch, A., Wachsmuth, D.: A-posteriori error estimates for optimal control problems with state and control constraints. Numer. Math. **120**, 733–762 (2012)

[23] Schiela, A.: Barrier methods for optimal control problems with state constraints. SIAM J. Optim. **20**(2), 1002–1031 (2009)

[24] Schötzau, D., Zhu, L.: A robust a-posteriori error estimator for discontinuous Galerkin methods for convection-diffusion equations. Appl. Numer. Math. **59**(9), 2236–2255 (2009)

[25] Tröltzsch, F.: Regular Lagrange multipliers for control problems with mixed pointwise control-state constraints. SIAM J. Optim. **15**(2), 616–634 (2005)

[26] Verfürth, R.: Robust a posteriori error estimates for stationary convection-diffusion equations. SIAM J. Numer. Anal. **43**(4), 1766–1782 (2005)

[27] Wollner, W.: A posteriori error estimates for a finite element discretization of interior point methods for an elliptic optimization problem with state constraints. Comput. Optim. App. **47**(1), 133–159 (2010)

[28] Yan, N., Zhou, Z.: A priori and a posteriori error analysis of edge stabilization Galerkin method for the optimal control problem governed by convection-dominated diffusion equation. J. Comput. Appl. Math. **223**(1), 198–217 (2009)

[29] Yücel, H., Heinkenschloss, M., Karasözen, B.: An adaptive discontinuous Galerkin method for convection dominated distributed optimal control problems. Tech. rep., Department of Computational and Applied Mathematics, Rice University, Houston, TX 77005–1892 (2012)

[30] Yücel, H., Heinkenschloss, M., Karasözen, B.: Distributed optimal control of diffusion-convection-reaction equations using discontinuous Galerkin methods. In: Numerical Mathematics and Advanced Applications 2011, pp. 389–397. Springer-Verlag Berlin Heidelberg (2013)

[31] Yücel, H., Karasözen, B.: Adaptive symmetric interior penalty galerkin (SIPG) method for optimal control of convection diffusion equations with control constraints. Optimization (2013)

[32] Yücel, H., Stoll, M., Benner, P.: A discontinous Galerkin method for optimal control problems governed by a system of convection-diffusion PDEs with nonlinear reaction terms. MPI Magdeburg Preprint MPIMD/13-10 (2013)

[33] Yücel, H., Stoll, M., Benner, P.: Discontinuous Galerkin finite element methods with shock-capturing for nonlinear convection dominated models. Computers & Chemical Engineering **58**, 278–287 (2013)

[34] Zowe, J., Kurcyusz, S.: Regularity and stability for the mathematical programming problem in Banach spaces. Appl. Math. Optim. **5**(1), 49–62 (1979)

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