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Positive Operators and Stable Truncation

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Abstract

We introduce a notion of balanced truncation for generalized Lyapunov operators and show that it preserves asymptotic stability. The proof relies on the theory of positive mappings and a result by Hans Schneider. Applications of our result can be found in model order reduction of stochastic linear systems.

Keywords. Lyapunov equation, positive operator, balanced truncation, asymptotic stability.

Mathematics Subject Classification (MSC 2010). 15A24, 15B48, 93D05.

1 Introduction

In his inspiring paper [1], *Positive Operators and an Inertia Theorem*, Hans Schneider pointed out a close relationship between inertia theorems for Lyapunov equations and positive operators on the space of Hermitian matrices. Among other things, he showed that Lyapunov's matrix theorem can be extended to the case where a positive operator is added to the Lyapunov operator (see Theorem 2.1). This result turned out to be fundamental e.g. for the analysis of linear stochastic systems, see [2]. In typical applications it is interpreted as a criterion for a system to be asymptotically stable.

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There is another famous result involving the Lyapunov operator (see e.g. [3, 4]), which plays an important role in model order reduction. For an asymptotically stable system of linear ordinary differential equations it roughly says the following (see Theorem 2.2): If the associated Lyapunov operator and its adjoint operator share a block diagonal solution to a certain matrix inequality, then the projected subsystems corresponding to the blocks are asymptotically stable.

It is immediate to formulate an analogous generalized statement for the case, where a positive operator is added to the Lyapunov operator, that is for the class of operators considered in [1]. This is done in Theorem 2.3, which is the main contribution of the present paper. Its proof turns out to be slightly involved and requires some auxiliary results, which are also interesting by themselves (see Section 3).

To a large extent, our investigations are motivated by the problem of model order reduction for stochastic systems, see [5]. In the present paper, however, we omit the discussion of stochastic aspects, since the problem is attractive also from the purely linear algebraic point of view.

2 Setup and statement of the main result

Let $\mathcal{H}^n \subset \mathbb{K}^{n\times n}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) denote the real space of real or complex $n \times n$ Hermitian matrices endowed with the Frobenius inner product $\langle X\,,\,Y\rangle = \operatorname{trace}(XY)$. By $\mathcal{H}^n_+ = \{X \in \mathcal{H}^n \,\big|\, X \geq 0\}$ we denote the closed convex cone of nonnegative definite matrices and by $\operatorname{int}\big(\mathcal{H}^n_+\big)$ its interior, i.e. the open cone of positive definite matrices. The cone $\mathcal{H}^n_+$ induces a partial ordering on $\mathcal{H}^n$. We write $X \geq Y$, if $X - Y \in \mathcal{H}^n_+$ and $X > Y$, if $X - Y \in \operatorname{int}\big(\mathcal{H}^n_+\big)$. If $X, Y \in \mathcal{H}^n_+$, then $\langle X, Y\rangle \geq 0$, where equality implies $XY = 0$.

For $A \in \mathbb{K}^{n\times n}$ we define the Lyapunov operator $\mathcal{L}_A : \mathcal{H}^n \to \mathcal{H}^n$ by

$$\mathcal{L}_A(X) = AX + XA^* \,. \tag{1}$$

If $N = (N^{(1)}, \dots, N^{(\nu)})$ is a ν-tuple of matrices $N^{(j)} \in \mathbb{K}^{n\times n}$, then let $\Pi_N : \mathcal{H}^n \to \mathcal{H}^n$ be defined by

$$\Pi_N(X) = \sum_{j=1}^{\nu} N^{(j)} X (N^{(j)})^* \,. \tag{2}$$

Note that the adjoint operators with respect to $\langle\cdot,\cdot\rangle$ are given by $\mathcal{L}_A^* = \mathcal{L}_{A^*}$ and $\Pi_N^* : X \mapsto \sum_{j=1}^{\nu} (N^{(j)})^* X N^{(j)} = \Pi_{N^*}$, if $N^* := ((N^{(1)})^*, \dots, (N^{(\nu)})^*)$.

If T is an arbitrary linear mapping on a finite dimensional $\mathbb{K}$-vector space, then $\sigma(T)$ denotes the spectrum, $\rho(T) = \max\{|\lambda|; \lambda \in \sigma(T)\}$ the spectral radius, and $\alpha(T) = \max\{\operatorname{Re}\lambda; \lambda \in \sigma(T)\}$ the spectral abscissa. By $\mathbb{C}_-$ we denote the open left half complex plane and by $\overline{\mathbb{C}_-}$ its topological closure.

We now formulate a special version of Schneider's result [1, Lemma 1] (or [6, 7]). For $\Pi_N = 0$ it is known as Lyapunov's matrix theorem, e.g. [8].

Theorem 2.1 *For $\mathcal{L}_A$, Π_N as in (1), (2), the following are equivalent.*

(i) $\sigma(\mathcal{L}_A + \Pi_N) \subset \mathbb{C}_-$.

(ii) $\sigma(\mathcal{L}_A) \subset \mathbb{C}_-$ *and* $\rho(\mathcal{L}_A^{-1}\Pi_N) < 1$.

(iii) $\exists Y > 0 : \exists X > 0$: $(\mathcal{L}_A + \Pi_N)(X) = -Y$.

(iv) $\forall Y > 0 : \exists X > 0$: $(\mathcal{L}_A + \Pi_N)(X) = -Y$.

For simplicity, we call $\mathcal{L}_A + \Pi_N$ stable if $\sigma(\mathcal{L}_A + \Pi_N) \subset \mathbb{C}_-$. Note that $\sigma(\mathcal{L}_A) \subset \mathbb{C}_-$ if and only if $\sigma(A) \subset \mathbb{C}_-$.

From now on, we will assume A and $N^{(j)}$ to be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } N^{(j)} = \begin{bmatrix} N_{11}^{(j)} & N_{12}^{(j)} \\ N_{21}^{(j)} & N_{22}^{(j)} \end{bmatrix} \text{ with } A_{11}, N_{11}^{(j)} \in \mathbb{K}^{r\times r} . \tag{3}$$

Naturally, we define $\mathcal{L}_{A_{11}}, \Pi_{N_{11}} : \mathcal{H}^r \to \mathcal{H}^r$ by

$$\mathcal{L}_{A_{11}}(X) = A_{11}X + XA_{11}^* \quad \text{and} \quad \Pi_{N_{11}}(X) = \sum_{j=1}^{\nu} N_{11}^{(j)} X (N_{11}^{(j)})^* \tag{4}$$

and call $\mathcal{L}_{A_{11}} + \Pi_{N_{11}}$ the *truncated* operator obtained from $\mathcal{L}_A + \Pi_N$.

In the context of model order reduction it is important to have criteria for the truncated operators to be stable. If $\Pi_N = 0$, such a criterion is given by the following result from [3, 4] (see also [9]).

Theorem 2.2 *Let A be as in (3) and assume that there exists a block-diagonal matrix $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2) > 0$ with $\Sigma_1 \in \mathbb{R}^{r\times r}$ and $\sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \emptyset$, so that*

$$\mathcal{L}_A(\Sigma) \le 0 \quad \text{and} \quad \mathcal{L}_{A^*}(\Sigma) \le 0 \tag{5}$$

Then $\sigma(A_{11}) \subset \mathbb{C}_-$ if $\sigma(A) \subset \mathbb{C}_-$.

In view of Theorem 2.1 it is natural to ask, whether in Theorem 2.2 we can also replace $\mathcal{L}_A$ by $\mathcal{L}_A + \Pi_N$. This leads us to our central result.

Theorem 2.3 *Let A and N be as in (3), and assume that there exists a block diagonal matrix $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2) > 0$ with $\Sigma_1 \in \mathbb{R}^{r\times r}$ and $\sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \emptyset$, so that*

$$(\mathcal{L}_A + \Pi_N)(\Sigma) \le 0 \quad \text{and} \quad (\mathcal{L}_A + \Pi_N)^*(\Sigma) \le 0 . \tag{6}$$

Then $\sigma(\mathcal{L}_{A_{11}} + \Pi_{N_{11}}) \subset \mathbb{C}_-$ if $\sigma(\mathcal{L}_A + \Pi_N) \subset \mathbb{C}_-$.

The proof of this theorem is provided in Section 4. It requires a number of auxiliary results that are given in Section 3.

Remark 2.4 **(i)** *If $\sigma(A) \subset \mathbb{C}_-$, then (5) (with possibly $\sigma(\Sigma_1) \cap \sigma(\Sigma_2) \neq \emptyset$) is always satisfied after a suitable similarity transformation of A. In this case, we call the matrix A or the operator $\mathcal{L}_A$* balanced. *More commonly, if $\mathcal{L}_A(\Sigma) = -BB^*$ and $\mathcal{L}_{A^*}(\Sigma) = -C^*C$, then the system (A, B, C) is called balanced. A balancing similarity transformation $(A, B, C) \mapsto (S^{-1}AS, S^{-1}B, CS)$ exists if and only if (A, B) is controllable and (A, C) is observable. In the pure stability analysis, however, we do not need to focus on the matrices B and C.*

(ii) *Analogously, we call the operator $\mathcal{L}_A + \Pi_N$* balanced *if (6) holds. If $\sigma(\mathcal{L}_A + \Pi_N) \subset \mathbb{C}_-$, then there always exists a similarity transformation*

$$(A, N^{(j)}) \mapsto (\tilde{A}, \tilde{N}^{(j)}) = (S^{-1}AS, S^{-1}N^{(j)}S) ,$$

so that $\mathcal{L}_{\tilde{A}} + \Pi_{\tilde{N}}$ is balanced. By Theorem 2.3, truncation of balanced systems preserves asymptotic stability.

(iii) *Theorem 2.3 comprises the deterministic continuous and discrete time case of balanced truncation (see e.g. [9]), where for discrete time, we set $A = -\frac{1}{2}I$. The same is true for stochastic systems (see [5]).*

3 Spectral properties of operators on matrix spaces

For the proof of Theorem 2.3 we first need to recall some properties of the ordered vector space $\mathcal{H}^n$ and the operator $\mathcal{L}_A + \Pi_N$, which have been summarized e.g. in [2]. Many of these results hold in a much more general setting, but we state them here in the form that will be needed later.

Obviously, the operator Π_N is *positive* in the sense that it maps $\mathcal{H}^n_+$ to $\mathcal{H}^n_+$. In fact, by its representation (2), it is a special type of positive operators on $\mathcal{H}^n_+$ called *completely positive* [10]. The operator $T = \mathcal{L}_A + \Pi_N$ is *(completely) resolvent positive* (e.g. [11, 12, 13]), which means that for sufficiently large $s > 0$ the resolvent $(sI - T)^{-1}$ is (completely) positive (actually, $s > \alpha(T)$ is sufficient, as follows from Theorem 2.1). Spectral properties of positive matrices have been analyzed first by Perron [14] and Frobenius [15], whose results later have been extended to general positive operators mainly by Krein and Rutman [16]. The following fundamental theorem is a consequence of these results (see also [17, 12, 18, 2]). In view of our later application, we state it for the adjoint operators Π_N^* and $(\mathcal{L}_A + \Pi_N)^*$.

Theorem 3.1 **(i)** *There exists $V \in \mathcal{H}^n_+ \setminus \{0\}$ so that $\Pi_N^*(V) = \rho(\Pi_N)V$.*

(ii) *There exists $V \in \mathcal{H}^n_+ \setminus \{0\}$ so that $(\mathcal{L}_A + \Pi_N)^*(V) = \alpha(\mathcal{L}_A + \Pi_N)V$.*

We note a simple corollary for the case of semidefinite Y in Theorem 2.1.

Corollary 3.2 *For given $Y \geq 0$ assume that*

$$\exists X > 0 : \ \mathcal{L}_A(X) + \Pi_N(X) \leq -Y \ . \tag{7}$$

Then $\sigma(\mathcal{L}_A + \Pi_N) \subset \overline{\mathbb{C}_-}$.
Moreover, if $\sigma(\mathcal{L}_A+\Pi_N) \not\subset \mathbb{C}_-$ *and* V *is given as in Thereom 3.1(ii), then* $\alpha(\mathcal{L}_A+\Pi_N) = 0$ *and* $YV = VY = 0$.

Proof: Scalar multiplication of (7) with V from Thereom 3.1(ii) yields

$$0 \geq \langle -Y, V\rangle = \langle (\mathcal{L}_A + \Pi_N)(X), V\rangle = \alpha(\mathcal{L}_A + \Pi_N)\langle X, V\rangle \ .$$

Since $\langle X, V\rangle > 0$ we have $\alpha(\mathcal{L}_A + \Pi_N) \leq 0$, i.e. $\sigma(\mathcal{L}_A + \Pi_N) \subset \overline{\mathbb{C}_-}$.
If $\sigma(\mathcal{L}_A + \Pi_N) \not\subset \mathbb{C}_-$ then necessarily $\alpha = 0$ and $\langle Y, V\rangle = 0$ which is equivalent to $YV = VY = 0$. □

In our main result we deal with block matrices. This leads us to mixings of given operators as in the next proposition, which contains a Cauchy-Schwarz-type inequality for the spectral radii of two completely positive operators.

Proposition 3.3 *For* $J = \{1, \ldots, \nu\}$ *and* $j \in J$ *let* $L_j \in \mathbb{K}^{\ell\times\ell}$, $M_j \in \mathbb{K}^{m\times m}$ *and define operators* $\Pi_L : \mathbb{K}^{\ell\times\ell} \to \mathbb{K}^{\ell\times\ell}$, $\Pi_M : \mathbb{K}^{m\times m} \to \mathbb{K}^{m\times m}$ *and* $\Pi_{LM} : \mathbb{K}^{\ell\times m} \to \mathbb{K}^{\ell\times m}$ *via*

$$\Pi_L(X) = \sum_{j=1}^{\nu} L_j X L_j^* \ , \quad \Pi_M(Y) = \sum_{j=1}^{\nu} M_j Y M_j^* \ , \quad \Pi_{LM}(Z) = \sum_{j=1}^{\nu} L_j Z M_j^* \ .$$

Then $\rho(\Pi_{LM})^2 \leq \rho(\Pi_L)\rho(\Pi_M)$.

Proof: (i) We first assume $\rho(\Pi_L) < 1$ and $\rho(\Pi_M) < 1$, and show $\rho(\Pi_{LM}) < 1$.
For $k \in \mathbb{N}$ and a multiindex $a = [a_1, \ldots, a_k] \in J^k$ we set

$$L_a = L_{a_1} L_{a_2} \cdots L_{a_k} \quad \text{and} \quad M_a = M_{a_1} M_{a_2} \cdots M_{a_k} \ .$$

Then

$$\Pi_L^k(X) = \sum_{a\in J^k} L_a X L_a^* \, , \ \ \Pi_M^k(Y) = \sum_{a\in J^k} M_a Y M_a^* \, , \ \ \Pi_{LM}^k(Z) = \sum_{a\in J^k} L_a Z M_a^* \, .$$

By assumption, $\Pi_L^k(X) \to 0$ and $\Pi_M^k(Y) \to 0$ as $k \to \infty$ for arbitrary X and Y. In particular let $X = xx^*$, $Y = yy^*$, and $Z = xy^*$. Then

$$\sum_{a\in J^k} \|L_a x\|_2^2 = \sum_{a\in J^k} x^* L_a^* L_a x = \operatorname{trace} \Pi_L^k(X) \overset{k\to\infty}{\to} 0$$
$$\sum_{a\in J^k} \|M_a y\|_2^2 = \sum_{a\in J^k} y^* M_a^* M_a y = \operatorname{trace} \Pi_M^k(Y) \overset{k\to\infty}{\to} 0$$

and by the triangle inequality and the Cauchy-Schwarz inequality

$$\begin{aligned} \|\Pi_{LM}^k(Z)\|_F &\leq \sum_{a\in J^k} \|L_a x y^* M_a^*\|_F \leq \sum_{a\in J^k} \|L_a x\|_2 \|M_a y\|_2 \\ &\leq \left(\operatorname{trace} \Pi_L^k(X) \operatorname{trace} \Pi_M^k(Y)\right)^{\frac{1}{2}} \overset{k\to\infty}{\to} 0 \ . \end{aligned}$$

Since matrices of the form $Z = xy^*$ span the whole space $\mathbb{K}^{\ell\times m}$ it follows that $\rho(\Pi_{LM}) < 1$.

(ii) For arbitrary $\rho(\Pi_L)$ and $\rho(\Pi_M)$, we let $\varepsilon > 0$ and set

$$\tilde{L}_j = \frac{1}{\sqrt{\rho(\Pi_L)+\varepsilon}} L_j \;, \quad \tilde{M}_j = \frac{1}{\sqrt{\rho(\Pi_M)+\varepsilon}} M_j \;.$$

Then $\Pi_{\tilde{L}} = \frac{1}{\rho(\Pi_L)+\varepsilon}\Pi_L$, $\Pi_{\tilde{M}} = \frac{1}{\rho(\Pi_M)+\varepsilon}\Pi_M$, $\Pi_{\tilde{L}\tilde{M}} = \frac{1}{\sqrt{(\rho(\Pi_M)+\varepsilon)(\rho(\Pi_L)+\varepsilon)}}\Pi_{LM}$. By construction, $\rho(\Pi_{\tilde{L}}) < 1$ and $\rho(\Pi_{\tilde{M}}) < 1$, so that (i) yields $\rho(\Pi_{\tilde{L}\tilde{M}}) < 1$ implying

$$\rho(\Pi_{LM})^2 < (\rho(\Pi_M)+\varepsilon)(\rho(\Pi_L)+\varepsilon) \;.$$

Letting $\varepsilon \to 0$, we obtain $\rho(\Pi_{LM})^2 \le \rho(\Pi_L)\rho(\Pi_M)$. □

We now prove a logarithmic version of Proposition 3.3 for pairs of operators of the form $\mathcal{L}_A + \Pi_N$. By itself, it is an interesting consequence of Theorem 2.1. Recall that $\alpha(T) = \max \operatorname{Re} \sigma(T)$.

Proposition 3.4 *Consider the situation of Proposition 3.3. Moreover let $K_1 \in \mathbb{K}^{\ell\times\ell}$ and $K_2 \in \mathbb{K}^{m\times m}$ and define the linear operators*

$$\begin{aligned} T_1(X) &= K_1X + XK_1^* + \Pi_L(X) \;, \quad T_2(Y) = K_2Y + YK_2^* + \Pi_M(Y) \;, \\ T_{12}(Z) &= K_1Z + ZK_2^* + \Pi_{LM}(Z) \;. \end{aligned}$$

Then $\alpha(T_{12}) \le \frac{\alpha(T_1)+\alpha(T_2)}{2}$.

Proof: Assume that for some μ with $\operatorname{Re}\mu > 0$ and a matrix Z_0 we have

$$T_{12}(Z_0) = \left(\frac{\alpha(T_1)+\alpha(T_2)}{2} + \mu\right) Z_0 \;.$$

We will show that then necessarily $Z_0 = 0$, which proves $\alpha(T_{12}) \le \frac{\alpha(T_1)+\alpha(T_2)}{2}$.

To this end let $\tilde{K}_j = K_j - \frac{1}{2}(\alpha(T_j)+\mu)I$, $j = 1,2$. For the corresponding operators $\tilde{T}_1$, $\tilde{T}_2$, $\tilde{T}_{12}$ with K_1, K_2 replaced by $\tilde{K}_1, \tilde{K}_2$, we easily check that

$$\alpha(\tilde{T}_1) = \alpha(\tilde{T}_2) = -\operatorname{Re}\mu \;, \quad \text{and} \quad \tilde{T}_{12}(Z_0) = 0 \;.$$

The identity $2(AX + XB) = (A+I)X(B+I) - (A-I)X(B-I)$, yields

$$\begin{aligned} \tilde{T}_1(X) &= -\tfrac{1}{2}(\tilde{K}_1 - I)X(\tilde{K}_1 - I)^* + \tfrac{1}{2}(\tilde{K}_1 + I)X(\tilde{K}_1 + I)^* + \Pi_L(X) \\ \tilde{T}_2(Y) &= -\tfrac{1}{2}(\tilde{K}_2 - I)Y(\tilde{K}_2 - I)^* + \tfrac{1}{2}(\tilde{K}_2 + I)Y(\tilde{K}_2 + I)^* + \Pi_M(Y) \\ \tilde{T}_{12}(Z) &= -\tfrac{1}{2}(\tilde{K}_1 - I)Z(\tilde{K}_2 - I)^* + \tfrac{1}{2}(\tilde{K}_1 + I)Z(\tilde{K}_2 + I)^* + \Pi_{LM}(Z) \;. \end{aligned} \tag{8}$$

Since necessarily $\sigma(\tilde{K}_1) \subset \mathbb{C}_-$ and $\sigma(\tilde{K}_2) \subset \mathbb{C}_-$, we can define

$$\Pi_1(X) := (\tilde{K}_1 - I)^{-1}\Big((\tilde{K}_1 + I)X(\tilde{K}_1 + I)^* + 2\Pi_L(X)\Big)(\tilde{K}_1 - I)^{-*}$$

$$\Pi_2(Y) := (\tilde{K}_2 - I)^{-1}\Big((\tilde{K}_2 + I)Y(\tilde{K}_2 + I)^* + 2\Pi_M(Y)\Big)(\tilde{K}_2 - I)^{-*}\,.$$

By construction and Theorem 2.1 there exists $X > 0$ so that

$$0 > 2\tilde{T}_1(X) = (\tilde{K}_1 - I)(-X + \Pi_1(X))(\tilde{K}_1 - I)^*\,,$$

whence also $-X + \Pi_1(X) < 0$. Again by Theorem 2.1, we get $\rho(\Pi_1) < 1$. Analogously, $\rho(\Pi_2) < 1$. From (8) it follows that $\tilde{T}_{12}(Z_0) = 0$ implies

$$Z_0 = (\tilde{K}_1 - I)^{-1}\big((\tilde{K}_1 + I)Z_0(\tilde{K}_2 + I)^* + 2\Pi_{LM}(Z_0)\big)(\tilde{K}_2 - I)^{-*} =: \Pi_{12}(Z_0)\,.$$

But $\rho(\Pi_{12}) < 1$ by Proposition 3.3, so that $Z_0 = 0$. □

The next lemma is based on the notion of the *field of values* (see e.g. [19]) of a linear mapping $T : \mathbb{K}^{m\times n} \to \mathbb{K}^{m\times n}$, where $\mathbb{K}^{m\times n}$ is equipped with the standard scalar product

$$\langle X, Y\rangle = \operatorname{trace} XY^* = \operatorname{trace} Y^*X\,.$$

We define the field of values as $F(T) = \{\langle T(X), X\rangle \mid \langle X, X\rangle = 1\}$.

Lemma 3.5 *For $j = 1, \ldots, \nu$ consider $L^{(j)} = \begin{bmatrix} L_{11}^{(j)} & L_{12}^{(j)} \\ L_{21}^{(j)} & L_{22}^{(j)} \end{bmatrix} \in \mathbb{K}^{\ell\times\ell}$ with $L_{22}^{(j)} \in \mathbb{K}^{\ell_2\times\ell_2}$ and $M^{(j)} = \begin{bmatrix} M_{11}^{(j)} & M_{12}^{(j)} \\ M_{21}^{(j)} & M_{22}^{(j)} \end{bmatrix} \in \mathbb{K}^{m\times m}$ with $M_{11}^{(j)} \in \mathbb{K}^{m_1\times m_1}$.*
Define operators $T : \mathbb{K}^{\ell\times m} \to \mathbb{K}^{\ell\times m}$ and $T_{21} : \mathbb{K}^{\ell_2\times m_1} \to \mathbb{K}^{\ell_2\times m_1}$ via

$$T(X) = \sum_{j=1}^{\nu} L^{(j)} X M^{(j)} \quad \text{and} \quad T_{21}(Y) = \sum_{j=1}^{\nu} L_{22}^{(j)} Y M_{11}^{(j)}\,.$$

Then $F(T_{21} + T_{21}^*) \subset [\min\sigma(T + T^*), \max\sigma(T + T^*)]$.

Proof: The adjoint operators $T^* : \mathbb{K}^{\ell\times m} \to \mathbb{K}^{\ell\times m}$, $T_{21}^* : \mathbb{K}^{\ell_2\times m_1} \to \mathbb{K}^{\ell_2\times m_1}$ are given by

$$T^*(X) = \sum_{j=1}^{\nu} (L^{(j)})^* X (M^{(j)})^* \quad \text{and} \quad T_{21}^*(Y) = \sum_{j=1}^{\nu} (L_{22}^{(j)})^* Y (M_{11}^{(j)})^*\,.$$

By self-adjointness, $F(T + T^*) = [\min\sigma(T + T^*), \max\sigma(T + T^*)]$ (e.g. [20, Fact 8.14.7]). Let $Y \in \mathbb{K}^{\ell_2\times m_1}$ and consider the block matrix $X = \begin{bmatrix} 0 & 0 \\ Y & 0 \end{bmatrix}$ with $\langle X, X\rangle = \langle Y, Y\rangle$. Then $(T + T^*)(X) = \begin{bmatrix} \star & \star \\ (T_{21} + T_{21}^*)(Y) & \star \end{bmatrix}$, so that $\langle X, (T + T^*)(X)\rangle = \langle Y, (T_{21} + T_{21}^*)(Y)\rangle$ and $F(T_{21} + T_{21}^*) \subset F(T + T^*)$. □

Remark 3.6 *We call T_{21} the* lower left block *of T.*

4 Proof of Theorem 2.3

To simplify the presentation we will restrict our attention to the case where $\nu = 1$ and $N = N^{(1)}$, i.e. $\Pi_N : X \mapsto NXN^*$. From the proof it will be easy to see that this is no loss of generality. For convenience we restate Theorem 2.3 for this situation.

Theorem 4.1 *Let $A, N \in \mathbb{K}^{n\times n}$ with $\sigma(\mathcal{L}_A + \Pi_N) \subset \mathbb{C}_-$, and assume that there exists a matrix $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2) > 0$ with $\sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \emptyset$, so that*

$$A\Sigma + \Sigma A^* + N\Sigma N^* \leq 0 \quad \text{and} \quad A^*\Sigma + \Sigma A + N^*\Sigma N \leq 0 \,.$$

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$ be partitioned as Σ. Then

$$\sigma(\mathcal{L}_{A_{11}} + \Pi_{N_{11}}) \subset \mathbb{C}_- \,. \tag{9}$$

Proof: Assume that (9) does not hold. Then by Theorem 3.1 there exists a number $\beta \geq 0$ and a nonzero matrix $V_1 \geq 0$ such that

$$A_{11}^*V_1 + V_1A_{11} + N_{11}^*V_1N_{11} = \beta V_1 \,. \tag{10}$$

We will show that this implies $\beta \in \sigma(\mathcal{L}_A + \Pi_N)$ in contradiction to our assumption.

Since the proof is rather long we arrange it in several subsections.

4.1 Show that $\beta = 0$

Let us choose B such that

$$A\Sigma + \Sigma A^* + N\Sigma N^* = -BB^* \,. \tag{11}$$

With suitable partitioning of B, the left upper block of (11) is

$$A_{11}\Sigma_1 + \Sigma_1A_{11}^* + N_{11}\Sigma_1N_{11}^* = -B_1B_1^* - N_{12}\Sigma_2N_{12}^* \,. \tag{12}$$

By Corollary 3.2 we find that $\beta = 0$ and $B_1^*V_1 = 0$, $N_{12}^*V_1 = 0$.

4.2 Invariance of $\ker V_1$ and $\operatorname{im} V_1$

Without loss of generality, we can assume that V_1 has maximal rank, i.e.

$$\left(\tilde{V}_1 \geq 0 \text{ and } A_{11}^*\tilde{V}_1 + \tilde{V}_1A_{11} + N_{11}^*\tilde{V}_1N_{11} = 0\right) \Rightarrow \operatorname{rank}\tilde{V}_1 \leq \operatorname{rank}V_1 \,. \tag{13}$$

We now observe that $\ker V_1$ is invariant under A_{11} and N_{11} and $\operatorname{im} V_1$ is invariant under A_{11}^* and N_{11}^*. To see this, let $V_1z = 0$. Then

$$0 = z^*\left(A_{11}^*V_1 + V_1A_{11} + N_{11}^*V_1N_{11}\right)z = z^*N_{11}^*V_1N_{11}z \,,$$

whence also $V_1 N_{11} z = 0$, i.e. $N_{11} z \in \ker V_1$. From this, we have

$$0 = (A_{11}^* V_1 + V_1 A_{11} + N_{11}^* V_1 N_{11})\, z = V_1 A_{11} z\ ,$$

implying $A_{11} z \in \ker V_1$. Thus $A_{11} \ker V_1 \subset \ker V_1$ and $N_{11} \ker V_1 \subset \ker V_1$.

Since $\ker V_1 = (\operatorname{im} V_1)^\perp$, it follows further that $\operatorname{im} V_1$ is invariant under A_{11}^* and N_{11}^*.

Let $\operatorname{im} V_1 = \operatorname{im} V_{11}$, with $V_{11}^* V_{11} = I$, $V_1 = V_{11} D_{11} V_{11}^*$, for some $D_{11} > 0$ and $\ker V_1 = \operatorname{im} V_{12}$ with $V_{12}^* V_{12} = I$, so that in particular $V_{11}^* V_{12} = 0$. By the invariance properties, we know that

$$A_{11}^* V_{11} = V_{11} \tilde{A}_{11}^* \quad \text{and} \quad A_{11} V_{12} = V_{12} \tilde{A}_{22} \tag{14}$$

for suitable matrices $\tilde{A}_{11}$ and $\tilde{A}_{22}$. Analogously

$$N_{11}^* V_{11} = V_{11} \tilde{N}_{11}^* \quad \text{and} \quad N_{11} V_{12} = V_{12} \tilde{N}_{22} \tag{15}$$

for suitable matrices $\tilde{N}_{11}$ and $\tilde{N}_{22}$.

Note that

$$\begin{aligned} 0 &= A_{11}^* V_1 + V_1 A_{11} + N_{11}^* V_1 N_{11} \\ &= A_{11}^* V_{11} D_{11} V_{11}^* + V_{11} D_{11} V_{11}^* A_{11} + N_{11}^* V_{11} D_{11} V_{11}^* N_{11} \\ &= V_{11} \left(\tilde{A}_{11}^* D_{11} + D_{11} \tilde{A}_{11} + \tilde{N}_{11}^* D_{11} \tilde{N}_{11} \right) V_{11}^*\ , \end{aligned}$$

whence $\tilde{A}_{11}^* D_{11} + D_{11} \tilde{A}_{11} + \tilde{N}_{11}^* D_{11} \tilde{N}_{11} = 0$ implying that $\sigma(\mathcal{L}_{\tilde{A}_{11}} + \Pi_{\tilde{N}_{11}}) \subset \overline{\mathbb{C}_-}$ by Corollary 3.2.

Moreover, $N_{12}^* V_{11} = 0$ and $B_1^* V_{11} = 0$, because $N_{12}^* V_1 = 0$ and $B_1^* V_1 = 0$.

4.3 A unitary similarity transformation

Now, let us also choose C such that

$$A^* \Sigma + \Sigma A + N^* \Sigma N = -C^* C\ , \tag{16}$$

and consider the unitary transformation matrix $U = \left[\begin{array}{cc|c} V_{11} & V_{12} & 0 \\ \hline 0 & 0 & I \end{array}\right]$. Then

$$\begin{aligned} U^* A U &= \begin{bmatrix} V_{11}^* A_{11} V_{11} & V_{11}^* A_{11} V_{12} & V_{11}^* A_{12} \\ V_{12}^* A_{11} V_{11} & V_{12}^* A_{11} V_{12} & V_{12}^* A_{12} \\ A_{21} V_{11} & A_{21} V_{12} & A_{22} \end{bmatrix} \\ &= \begin{bmatrix} \tilde{A}_{11} & 0 & V_{11}^* A_{12} \\ V_{12}^* A_{11} V_{11} & \tilde{A}_{22} & V_{12}^* A_{12} \\ A_{21} V_{11} & A_{21} V_{12} & A_{22} \end{bmatrix} =: \begin{bmatrix} \tilde{A}_{11} & 0 & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix} = \tilde{A}\ , \end{aligned}$$

$$U^*NU = \begin{bmatrix} \tilde{N}_{11} & 0 & V_{11}^* N_{12} \\ V_{12}^* N_{11} V_{11} & \tilde{N}_{22} & V_{12}^* N_{12} \\ N_{21} V_{11} & N_{21} V_{12} & N_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{N}_{11} & 0 & 0 \\ V_{12}^* N_{11} V_{11} & \tilde{N}_{22} & V_{12}^* N_{12} \\ N_{21} V_{11} & N_{21} V_{12} & N_{22} \end{bmatrix} =: \begin{bmatrix} \tilde{N}_{11} & 0 & 0 \\ \tilde{N}_{21} & \tilde{N}_{22} & \tilde{N}_{23} \\ \tilde{N}_{31} & \tilde{N}_{32} & \tilde{N}_{33} \end{bmatrix} = \tilde{N} \ ,$$

$$U^*B = \begin{bmatrix} V_{11}^* & 0 \\ V_{12}^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} V_{11}^* B_1 \\ V_{12}^* B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ V_{12}^* B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{B}_2 \\ \tilde{B}_3 \end{bmatrix} = \tilde{B} \ ,$$

$$CU = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 & \tilde{C}_3 \end{bmatrix} = \tilde{C} \ , \quad U^*\Sigma U = \begin{bmatrix} \tilde{\Sigma}_{11} & \tilde{\Sigma}_{21}^* & 0 \\ \tilde{\Sigma}_{21} & \tilde{\Sigma}_{22} & 0 \\ 0 & 0 & \tilde{\Sigma}_{33} \end{bmatrix} = \tilde{\Sigma}$$

with $\tilde{\Sigma}_{33} = \Sigma_2$ and $\sigma\left(\begin{bmatrix} \tilde{\Sigma}_{11} & \tilde{\Sigma}_{21}^* \\ \tilde{\Sigma}_{21} & \tilde{\Sigma}_{22} \end{bmatrix}\right) = \sigma(\Sigma_1)$.

Let us write $\tilde{A}_1 = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$, $\tilde{N}_1 = \begin{bmatrix} \tilde{N}_{11} & 0 \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix}$, $\tilde{\Sigma}_1 = \begin{bmatrix} \tilde{\Sigma}_{11} & \tilde{\Sigma}_{21}^* \\ \tilde{\Sigma}_{21} & \tilde{\Sigma}_{22} \end{bmatrix}$ and define

$$T(X) = \tilde{A}_1 X + X \tilde{A}_1^* + \tilde{N}_1 X \tilde{N}_1^* \ .$$

As seen above, $T^*(D_1) = 0$ for $D_1 = \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix}$.

4.4 The field of values of T and its lower left block

By construction $T(\tilde{\Sigma}_1) \le 0$ and $T^*(\tilde{\Sigma}_1) \le 0$, whence also $(T + T^*)(\tilde{\Sigma}_1) \le 0$ implying

$$\sigma(T + T^*) \subset \overline{\mathbb{C}_-} \cap \mathbb{R} \ . \tag{17}$$

Looking at the left upper blocks of

$$\tilde{A}\tilde{\Sigma} + \tilde{\Sigma}\tilde{A}^* + \tilde{N}\tilde{\Sigma}\tilde{N}^* = -\tilde{B}\tilde{B}^* \quad \text{and} \tag{18}$$

$$\tilde{A}^*\tilde{\Sigma} + \tilde{\Sigma}\tilde{A} + \tilde{N}^*\tilde{\Sigma}\tilde{N} = -\tilde{C}^*\tilde{C} \ , \tag{19}$$

we obtain

$$\tilde{\Sigma}_{11}\tilde{A}_{11}^* + \tilde{A}_{11}\tilde{\Sigma}_{11} + \tilde{N}_{11}\tilde{\Sigma}_{11}\tilde{N}_{11}^* = 0 \quad \text{and} \tag{20}$$

$$\begin{aligned} \tilde{A}_{11}^*\tilde{\Sigma}_{11} + \tilde{\Sigma}_{11}\tilde{A}_{11} + \tilde{N}_{11}^*\tilde{\Sigma}_{11}\tilde{N}_{11} &= -\tilde{C}_1^*\tilde{C}_1 - \tilde{N}_{21}^*\tilde{\Sigma}_{22}\tilde{N}_{21} - \tilde{N}_{31}^*\tilde{\Sigma}_{33}\tilde{N}_{31} \\ &\quad - \tilde{\Sigma}_{21}^*\tilde{A}_{21} - \tilde{A}_{21}^*\tilde{\Sigma}_{21} - \tilde{N}_{21}^*\tilde{\Sigma}_{21}\tilde{N}_{11} - \tilde{N}_{11}^*\tilde{\Sigma}_{21}^*\tilde{N}_{21} \ . \end{aligned} \tag{21}$$

Taking the scalar product of (21) with $\tilde{\Sigma}_{11}$, we get

$$
\begin{aligned}
&\langle \tilde{C}_1^*\tilde{C}_1 + \tilde{N}_{21}^*\tilde{\Sigma}_{22}\tilde{N}_{21} + \tilde{N}_{31}^*\tilde{\Sigma}_{33}\tilde{N}_{31}, \tilde{\Sigma}_{11}\rangle \\
&\quad = -\langle \tilde{\Sigma}_{21}^*\tilde{A}_{21} + \tilde{A}_{21}^*\tilde{\Sigma}_{21} + \tilde{N}_{21}^*\tilde{\Sigma}_{21}\tilde{N}_{11} + \tilde{N}_{11}^*\tilde{\Sigma}_{21}^*\tilde{N}_{21}, \tilde{\Sigma}_{11}\rangle \qquad (22) \\
&\quad = -2\langle \tilde{\Sigma}_{21}^*\tilde{A}_{21} + \tilde{N}_{11}^*\tilde{\Sigma}_{21}^*\tilde{N}_{21}, \tilde{\Sigma}_{11}\rangle \\
&\quad = -2\,\mathrm{trace}\left((\tilde{\Sigma}_{21}^*\tilde{A}_{21} + \tilde{N}_{11}^*\tilde{\Sigma}_{21}^*\tilde{N}_{21})\tilde{\Sigma}_{11}\right) \\
&\quad = -2\,\mathrm{trace}\left(\tilde{\Sigma}_{21}^*(\tilde{A}_{21}\tilde{\Sigma}_{11} + \tilde{N}_{21}\tilde{\Sigma}_{11}\tilde{N}_{11}^*)\right) \\
&\quad = 2\langle -\tilde{A}_{21}\tilde{\Sigma}_{11} - \tilde{N}_{21}\tilde{\Sigma}_{11}\tilde{N}_{11}^*, \tilde{\Sigma}_{21}\rangle \ . \qquad (23)
\end{aligned}
$$

The second block in the first column of $\tilde{A}\tilde{\Sigma} + \tilde{\Sigma}\tilde{A}^* + \tilde{N}\Sigma\tilde{N}^* = -\tilde{B}\tilde{B}^*$ is

$$0 = \tilde{\Sigma}_{21}\tilde{A}_{11}^* + \tilde{N}_{21}\tilde{\Sigma}_{11}\tilde{N}_{11}^* + \tilde{N}_{22}\tilde{\Sigma}_{21}\tilde{N}_{11}^* + \tilde{A}_{21}\tilde{\Sigma}_{11} + \tilde{A}_{22}\tilde{\Sigma}_{21}$$

whence

$$-\tilde{A}_{21}\tilde{\Sigma}_{11} - \tilde{N}_{21}\tilde{\Sigma}_{11}\tilde{N}_{11}^* = \tilde{\Sigma}_{21}\tilde{A}_{11}^* + \tilde{N}_{22}\tilde{\Sigma}_{21}\tilde{N}_{11}^* + \tilde{A}_{22}\tilde{\Sigma}_{21} =: T_{21}(\tilde{\Sigma}_{21}) \ . \qquad (24)$$

Note that T_{21} is the lower left block of T in the sense of Remark 3.6. Inserting (24) into (23) and using Lemma 3.5, we obtain

$$0 \le 2\langle T_{21}(\tilde{\Sigma}_{21}), \tilde{\Sigma}_{21}\rangle = \langle (T_{21} + T_{21}^*)(\tilde{\Sigma}_{21}), \tilde{\Sigma}_{21}\rangle \le 0 \ , \qquad (25)$$

since $F(T_{21} + T_{21}^*) \subset [\min \sigma(T + T^*), \max \sigma(T + T^*)] \subset \overline{\mathbb{C}_-} \cap \mathbb{R}$. From (25) and (24) it follows that the right hand side of (23) vanishes, and consequently

$$\tilde{C}_1 = 0 \ , \quad \tilde{N}_{21} = 0 \ , \quad \tilde{N}_{31} = 0 \ . \qquad (26)$$

Moreover, from (25) we obtain

$$(T_{21} + T_{21}^*)(\tilde{\Sigma}_{21}) = 0, \qquad (27)$$

because the quadratic form defined by $T_{21} + T_{21}^*$ is positive semidefinite.

4.5 Reduction to Sylvester equations

Exploiting (26), we find that the second blocks of the first column of (18) and (19), respectively take the forms

$$
\begin{aligned}
0 &= \tilde{\Sigma}_{21}\tilde{A}_{11}^* + \tilde{N}_{22}\tilde{\Sigma}_{21}\tilde{N}_{11}^* + \tilde{A}_{21}\tilde{\Sigma}_{11} + \tilde{A}_{22}\tilde{\Sigma}_{21} = T_{21}(\tilde{\Sigma}_{21}) + \tilde{A}_{21}\tilde{\Sigma}_{11} \\
0 &= \tilde{A}_{22}^*\tilde{\Sigma}_{21} + \tilde{\Sigma}_{21}\tilde{A}_{11} + \tilde{\Sigma}_{22}\tilde{A}_{21} + \tilde{N}_{22}^*\tilde{\Sigma}_{21}\tilde{N}_{11} = T_{21}^*(\tilde{\Sigma}_{21}) + \tilde{\Sigma}_{22}\tilde{A}_{21} \ .
\end{aligned}
$$

Adding these and using (27), we get the homogeneous Sylvester equation

$$0 = \tilde{A}_{21}\tilde{\Sigma}_{11} + \tilde{\Sigma}_{22}\tilde{A}_{21} \ .$$

It follows that $\tilde{A}_{21} = 0$, since all eigenvalues of $\tilde{\Sigma}_{11}$ and $\tilde{\Sigma}_{22}$ are strictly positive. Inserting $\tilde{A}_{21} = \tilde{N}_{21} = 0$ in (24) we see that $T_{21}(\tilde{\Sigma}_{21}) = 0$.

Moreover, the mapping $X \mapsto \tilde{A}_{22}^* X + X\tilde{A}_{22} + \tilde{N}_{22}^* X \tilde{N}_{22}$ has all eigenvalues in $\mathbb{C}_-$. Otherwise, there would exist a non-zero matrix $D_{22} \geq 0$ with

$$A_{22}^* D_{22} + D_{22}\tilde{A}_{22} + \tilde{N}_{22}^* D_{22} \tilde{N}_{22} = 0 \; .$$

But then, with $D = \operatorname{diag}(D_{11}, D_{22})$, we would have

$$\begin{aligned} T^*(D) = &\begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & \tilde{A}_{22} \end{bmatrix}^* \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} + \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & \tilde{A}_{22} \end{bmatrix} \\ &+ \begin{bmatrix} \tilde{N}_{11} & 0 \\ 0 & \tilde{N}_{22} \end{bmatrix}^* \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} \begin{bmatrix} \tilde{N}_{11} & 0 \\ 0 & \tilde{N}_{22} \end{bmatrix} = 0 \; . \end{aligned}$$

Thus $\tilde{V}_1 = [V_{11}, V_{12}]D[V_{11}, V_{12}]^*$ would contradict (13).

Hence, $\alpha(T_{21}) < 0$ by Proposition 3.4, and $T_{21}(\tilde{\Sigma}_{21}) = 0$ implies $\tilde{\Sigma}_{21} = 0$.

Now the third blocks of the first column of (18) and (19), respectively simplify to

$$0 = \tilde{\Sigma}_{33}\tilde{A}_{13}^* + \tilde{A}_{31}\tilde{\Sigma}_{11} \quad \text{and} \quad 0 = \tilde{A}_{13}^*\tilde{\Sigma}_{11} + \tilde{\Sigma}_{33}\tilde{A}_{31} \; .$$

By suitable multiplication with $\tilde{\Sigma}_{11}$ and $\tilde{\Sigma}_{33}$ we can eliminate either $\tilde{A}_{31}$ or $\tilde{A}_{13}$ to obtain the equations

$$0 = \tilde{\Sigma}_{33}^2\tilde{A}_{13}^* - \tilde{A}_{13}^*\tilde{\Sigma}_{11}^2 \quad \text{and} \quad 0 = \tilde{\Sigma}_{33}^2\tilde{A}_{31} - \tilde{A}_{31}\tilde{\Sigma}_{11}^2 \; .$$

By construction $\sigma(\tilde{\Sigma}_{11}) \subset \sigma(\Sigma_1)$ and $\sigma(\Sigma_1) \cap \sigma(\tilde{\Sigma}_{33}) = \emptyset$. Hence, both Sylvester equations are uniquely solvable and $\tilde{A}_{13}^* = \tilde{A}_{31} = 0$.
Finally, with $\tilde{\Sigma}_0 = \operatorname{diag}(\tilde{\Sigma}_{11}, 0, 0)$ we get

$$\tilde{A}\tilde{\Sigma}_0 + \tilde{\Sigma}_0\tilde{A}^* + \tilde{N}\tilde{\Sigma}_0\tilde{N}^* = 0 \; ,$$

contradicting asymptotic stability of the full system. □

5 Conclusions

Balanced truncation is a well-known method for model order reduction of linear control systems. Changing the point of view slightly, we have interpreted it as balancing and truncation of a Lyapunov operator. This formulation immediatedly generalizes to completely resolvent positive operators which are given as the sum of a Lyapunov operator and a completely positive operator. As our main result we have shown that balanced truncation in this more general framework still has the property of preserving asymptotic stability. The result relies essentially on [1] and has been presented in a purely linear algebraic way. However, it has important implications for model order reduction

of stochastic linear systems. For instance, it fills one of the gaps left in [5, 21] and plays a vital role in [22].
As a further generalization one might consider operators which are given as the sum of a Lyapunov operator and an arbitrary positive operator (not necessarily completely positive). Since we have exploited the specific structure of the operator Π_N, our proof does not apply to this case, but we are neither aware of any additional applications that could be covered by such a generalization.

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