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Semi-active damping optimization of vibrational systems using the parametric dominant pole algorithm

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Abstract

We consider the problem of determining an optimal semi-active damping of vibrating systems. For this damping optimization we use a minimization criterion based on the impulse response energy of the system. The optimization approach yields a large number of Lyapunov equations which have to be solved. In this work we propose an optimization approach that works with reduced systems which are generated using the parametric dominant pole algorithm. This optimization process is improved with a modal approach while the initial parameters for the parametric dominant pole algorithm are chosen in advance using residual bounds. Our approach calculates a satisfactory approximation of the impulse response energy while providing a significant acceleration of the optimization process. Numerical results illustrate the effectiveness of the proposed algorithm.

1 Introduction

Consider a vibrational system described by

$$M\ddot{q}(t) + C_u\dot{q}(t) + Kq(t) = B_2u(t) + E_2w(t), \tag{1}$$
$$y(t) = C_2\dot{q}(t), \tag{2}$$
$$z(t) = H_1q(t), \tag{3}$$

where the matrices M and K (called mass and stiffness, respectively) are real, symmetric positive definite matrices of order n. The matrices $B_2 \in \mathbb{R}^{n\times p}$ and $C_2 \in \mathbb{R}^{p\times n}$ are the control and control velocity matrices, respectively, while $E_2 \in \mathbb{R}^{n\times q}$ represents the primary excitation matrix. The internal damping C_u is usually taken to be a small multiple of the critical damping denoted by C_{crit}, that is (see, e.g., [6, 7, 24]),

$$C_u = \alpha_c C_{crit}, \quad \text{where} \quad C_{crit} = 2M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2}. \tag{4}$$

The state variables are contained in the coordinate vector $q \in \mathbb{R}^n$ and $z \in \mathbb{R}^s$ is the performance output which is described by the constant matrix $H_1 \in \mathbb{R}^{s\times n}$. If one is also interested in the velocities, the output z can include an additional part which corresponds to the velocities, but in this paper only the states are of interest. The vectors $u \in \mathbb{R}^p$ and $w \in \mathbb{R}^q$ are the control and primary excitation (i.e. noise) inputs, respectively.

We consider the case of a negative linear feedback corresponding to a linear damper of the form

$$u(t) = -Gy(t), \tag{5}$$

where $G \in \mathbb{R}^{p\times p}$ is a diagonal matrix $G = \operatorname{diag}(g_1, g_2, \ldots, g_p)$, called damping matrix. The non-negative entries g_i represent the friction coefficients of the corresponding dampers. These coefficients are in the following referred to as gain parameters g_i for $i = 1, \ldots, p$ and can be constant or variable with time within feasible margins, e.g., $g_i(t) \in [g_i^-, g_i^+]$ for $i = 1, \ldots, p$. The internal damping given by (4) with $\alpha_c > 0$ stabilizes the system which helps during the damping optimization.

By using the control u given in (5) together with $C_2 = B_2^T$, we obtain from equations (1)-(3)

$$M\ddot{g}(t) + C\dot{q}(t) + Kq(t) = E_2 w(t), \tag{6}$$

$$z(t) = H_1 q(t), \tag{7}$$

where $C := C_u + B_2 G B_2^T$. The usual assumption $C_2 = B_2^T$ gives a positive definite damping matrix C such that the system is asymptotically stable. The matrices B_2 and G constitute the semi-active damping part of the above second order system. More details regarding stability and the given model can be found in [9, 10].

The transfer function matrix of (6) - (7), obtained from the Laplace transform, is given by

$$F(s) = H_1(s^2 M + sC + K)^{-1} E_2, \quad s \in \mathbb{C}. \tag{8}$$

With the substitutions $x_1 = q$ and $x_2 = \dot{q}$ we obtain a representation of our vibrating system as the first order differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Ew(t), \\ z(t) &= Hx(t), \end{aligned} \tag{9}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A(G) = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}(C_u + B_2 G B_2^T) \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ M^{-1}E_2 \end{bmatrix}, \tag{10}$$

$$\text{and} \quad H = \begin{bmatrix} H_1 & 0 \end{bmatrix}. \tag{11}$$

The main problem is to determine "the best" damping matrix G which will minimize the output z under the influence of the input w.

Our aim will be the construction of an efficient algorithm that optimizes the damping matrix G. Then one can improve performance by a switching strategy which will ensure at least the same performance. The switching strategy can be in an on-off mode, which arises in problems of a semi-active damping, see [9, 10] for more details.

In a damping optimization one can consider different criteria (see, e.g., [10]). For our purposes we will consider the impulse response energy which corresponds to the system's 2-norm. In the frequency domain we define a cost function J_2 in terms of the transfer function matrix (8) via

$$J_2 = \int_{-\infty}^{+\infty} \operatorname{tr}\left(F(j\omega)^* F(j\omega)\right) \, d\omega. \tag{12}$$

In terms of the impulse as an input, for the single-input single-output case, this criterion can be written as

$$\int_0^{+\infty} \|z_D(t)\|_2^2 \, dt, \tag{13}$$

where z_D is the impulse response of system (9) obtained with $w(t)$ as the Dirac impulse.

Using the standard theory (see, e.g., [10, 11, 27]), it can be shown that J_2 can be expressed in the form

$$J_2 = \operatorname{tr}\left(E^T P E\right), \tag{14}$$

which is much more convenient for numerical computations. There, P is the solution of the following Lyapunov equation

$$A^T P + PA = -H^T H, \tag{15}$$

with the matrices A, H given in (10)-(11).

This leads to an optimization problem of determining the optimal matrix G such that

$$\operatorname{tr}\left(E^T P E\right) \quad \to \quad \min. \tag{16}$$

Damping optimization using the given criterion requires solving the Lyapunov equation (15) numerous times, which in general could be inefficient, as well as memory and time consuming. This is even the case when state of the art numerical methods for large-scale Lyapunov equations (see, e.g., the surveys [5, 23]) are employed. Thus, our aim is to introduce an approach which calculates approximations of our second-order system such that the optimization process is significantly accelerated.

There exist a number of different methods for calculating the approximated system. A review of different methods for this dimension reduction can be found in [1, 2, 4]. In [6, 7], the approximation is based on dimension reduction of second-order systems for optimizing a passive damping. In this paper we use another strategy based on an interpolatory eigenvalue based approach from [22] in order to optimize the semi-active damping efficiently. This approaches uses a small number of selected eigenvalues and -vector corresponding to the second order system (6)-(7). Since these can be computed efficiently by the dominant pole algorithm [18, 21], this can be a computational advantage compared to other model reduction techniques for dynamical systems. Moreover, for parameter dependent systems, it enables to reuse certain data for different gain parameters.

In the next section, we will at first briefly describe the main ideas of the class of dominant pole algorithms, including also the parametric dominant pole algorithm which we plan to use in damping optimization. Afterwards, a strategy for determining good initial parameters for the parametric dominant pole algorithms, and a complete gain optimization routine using the proposed reduction techniques is presented. Section 3 illustrates the effectiveness of the method by virtue of numerical examples and Section 4 concludes.

2 Parametric dominant pole algorithm for damping optimization

Since our optimization criterion can be written in terms of the transfer function (12), our aim is to have a good approximation of it. For this purpose we will use an approximation obtained with the dominant pole algorithm. The transfer function can be expressed as a function of eigenvalues (called poles) of the corresponding quadratic eigenvalue problem. If the algebraic equals the geometric multiplicity for all eigenvalues, the transfer function (8) can be represented as

$$F(s) = \sum_{i=1}^{2n} \frac{R_i}{s - \lambda_i} \tag{17}$$

with the residues

$$R_i = (H_1 x_i)(y_i^* E_2)\lambda_i \in \mathbb{C}^{s \times q}.$$

There, $\lambda_i \in \mathbb{C}$, $x_i,\ y_i \in \mathbb{C}^n \backslash \{0\}$ are eigenvalues, right and left eigenvectors of the quadratic eigenvalue problem

$$(\lambda_i^2 M + \lambda_i C + K)x_i = 0, \quad y_i^*(\lambda_i^2 M + \lambda_i C + K) = 0, \quad i = 1, \ldots, 2n. \tag{18}$$

For a proper approximation of the transfer function $F(s)$ from (17) we will need the concept of dominant poles. There are different definitions of dominant poles, and we will use one which is well suited from our point of view regarding semi-active damping optimization.

Definition 1 (Dominant Poles [15, 16, 18, 21])**.** *For the transfer function $F(s)$ in* (17)*, a pole λ_i will be called dominant if*

$$\frac{\|R_i\|}{\mathcal{R}e(\lambda_i)} > \frac{\|R_j\|}{\mathcal{R}e(\lambda_j)} \quad \forall j \neq i.$$

Regarding the underlying eigenvalue problem (18)*, $x_i,\ y_i$ are then also referred to as dominant right and left eigenvectors.*

First we would like to emphasize that the importance of the input-output behavior is, on the one hand, included in the residue $\|R_i\|$. On the other hand, if some pole is close to the imaginary axis (real part of a pole is small) it can have a greater influence on the energy, but if $\|R_i\|$ is very small (due to the input and output matrices) this influence might not be that important which is included in the fraction $\frac{\|R_i\|}{\mathcal{R}e(\lambda_i)}$. Furthermore, from the Bode magnitude plot $(\omega, \|F(i\omega)\|)$ it can be seen that peaks occur at frequencies close to the imaginary parts of the dominant poles of $F(s)$ [20] which also justifies the importance of the dominant poles.

In order to have a good approximation of the transfer function (8) we will approximate it by its dominant terms. That is, using the k most dominant poles, the transfer function is approximated as

$$F(s) \approx F_k(s) = \sum_{i=1}^{k} \frac{R_i}{s - \lambda_i}. \tag{19}$$

The approximation $F_k(s)$ is also referred to as transfer function modal equivalent [16, Section IV] of $F(s)$ corresponding to the k eigentriples $(\lambda_i,\ x_i,\ y_i)$. The reduced transfer function $F_k(s)$ corresponds to the reduced order model

$$Y^* M X \ddot{q}_k(t) + Y^* C X \dot{q}_k(t) + Y^* K X q_k(t) = Y^* E_2 w(t), \tag{20}$$

$$z(t) = H_1 X q_k(t), \tag{21}$$

where $q(t) \approx Xq_k(t)$ and $X,\ Y \in \mathbb{C}^{n\times k}$ span the eigenspaces associated with the k selected dominant eigenvalues of the quadratic eigenvalue problem. Since we obtain matrices of the dimension k in the reduced system, we will call the parameter k reduced dimension. Note that in our setting M, C and K are symmetric positive definite matrices and it holds that if x_i is a right eigenvector for an eigenvalue λ_i, then $y_i = \overline{x_i}$ is a left eigenvector for λ_i and vice-versa. Since eigenvalues come in complex conjugate pairs, the space spanned by the right eigenvectors is equal to the space spanned by the left eigenvectors, thus $Y = X$ can be chosen. For the reduced system we need therefore only right eigenvectors and the original system (6)-(7) can be approximated by the following reduced-order model:

$$\begin{aligned} X^*MX\ddot{q}_k(t) + X^*CX\dot{q}_k(t) + X^*KXq_k(t) &= X^*E_2w(t), \\ z(t) &= H_1Xq_k(t). \end{aligned} \tag{22}$$

The calculation of dominant poles can be performed by using an efficient structure-exploiting, specially tailored eigenvalue algorithm which will be described in the next subsection.

2.1 Subspace accelerated dominant pole algorithm for second order systems

In this section we will summarize an algorithm based on approaches from [18, 21] which calculates the dominant poles and the corresponding eigenvectors using the second-order structure of the multi-input multi-output system. The algorithm we are interested in is the subspace accelerated quadratic MIMO dominant pole algorithm (SAQMDP) [18, 21] which is a generalization of the MIMO dominant pole algorithm (SAMDP) [15, 16] to second order systems. The main ingredient in the derivation of dominant pole algorithms is the observation that, for any transfer function matrix $F(s)$, the function

$$f(s) := (\sigma_{\max}(F(s)))^{-1} \to 0 \tag{23}$$

as s approaches an eigenvalue of the quadratic eigenvalue problem. The basic idea beyond dominant pole algorithms is then to formally apply a Newton scheme to the above function. This gives a sequence of approximate eigenvectors that convergence to the right and left eigenvectors corresponding to a dominant pole of $F(s)$. Storing these approximate eigenvectors obtained in each iteration in subspaces, i.e., adding the concept of subspace acceleration, gives the subspace accelerated dominant pole algorithm [20, 19]. The incorporation of the second order structure leads to SAQMDP [18, 21] which is illustrated in Algorithm 1. In the following we briefly describe the main steps of Algorithm 1.

Step 2 The Newton process applied to (23) requires the computation of the right and left singular vectors u, z corresponding to the largest singular value of $F(s^{(1)})$. See [18, 19] for detailed calculations. We refer to u, z as direction vectors in the following. This step is simplified for square transfer functions matrices (number of inputs equals the number of outputs) [19] and can be neglected for single-input single-output systems [20, 21].

Step 4 The adjoint linear systems of equations are intrinsically also a part of the Newton step. The results $v^{(j)},\ w^{(j)}$ are approximate right and left eigenvectors. It is assumed that we are able to solve these large and sparse linear systems by direct solvers.

Algorithm 1 SAQMDP

Require: System matrices M, C, K, E_2, H_1 defining (6),(7), initial value $s^{(1)}$ for frequency, tolerance $0 < \tau \ll 1$, number of wanted poles k_{wanted}.

Ensure: Dominant poles $S = \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$ and corresponding right and left eigenvectors $X = [x_1, \ldots, x_k], Y = [y_1, \ldots, y_k]$.

1: Set $V = W = X = Y = [\ ]$, $k = 0$, $j = 1$.

2: Compute right and left singular vectors $u,\ z$ corresponding to $\sigma_{\max}(F(s^{(j)}))$.

3: **while** $k < k_{\text{wanted}}$ **do**

4: Find $v^{(j)},\ w^{(j)}$ by solving

$$(s^{(j)^2} M + s^{(j)} C + K) v^{(j)} = E_2 z, \quad (s^{(j)^2} M + s^{(j)} C + K)^* w^{(j)} = H_1 u.$$

5: Orthogonally expand V and W by $v^{(j)}$ and $w^{(j)}$, respectively.

6: Compute most dominant eigentriple $(\theta^{(j)},\ \widetilde{x}^{(j)},\ \widetilde{y}^{(j)})$ of the projected system:

$$W^* M V,\ W^* C V,\ W^* K V,\ W^* E_2,\ H_1 V.$$

7: Compute approximate eigenvectors $x^{(j)} = V \widetilde{x}^{(j)}$, $y^{(j)} = W \widetilde{y}^{(j)}$ and the residuals

$$r_R^{(j)} = (\theta^{(j)^2} M + \theta^{(j)} C + K) x^{(j)}, \quad r_L^{(j)} = (\theta^{(j)^2} M + \theta^{(j)} C + K)^H y^{(j)}.$$

8: **if** $\max(\|r_R^{(j)}\|, \|r_L^{(j)}\|) < \tau$ **then**

9: Set $k = k + 1$.

10: Augment $X = [X, x_k := x^{(j)}],\ Y = [Y, y_k := y^{(j)}],\ S = \operatorname{diag}\left(S, \lambda_k := \theta^{(j)}\right)$.

11: Deflate $(\lambda_k,\ x_k,\ y_k)$.

12: Compute right and left singular vectors $u,\ z$ corresponding to $\sigma_{\max}(F(\theta^{(j)}))$.

13: **end if**

14: Perform a restart if necessary.

15: Set $s^{(j+1)} = \theta^{(j)}$, $j = j + 1$.

16: **end while**

Step 5–7 These steps belong to the subspace acceleration phase. The results $v^{(j)},\ w^{(j)}$ are used as orthogonal expansion for the spaces spanned by V and W and a Petrov-Galerkin projection is performed onto the second order system. The dominant pole $\theta^{(j)}$ and its corresponding eigenvectors $\widetilde{x}^{(j)},\ \widetilde{y}^{(j)}$ of the resulting small system in Step 6 can be computed by direct methods for eigenvalue problems. In our implementation, the `quadeig` routine [13] worked satisfactory for this task. These small vectors are lifted up into the n-dimensional space in Step 7 to serve as approximate eigenvectors of the large-scale problem. The residuals are computed to enable an accuracy assessment of the current approximate eigentriplet.

Step 8–12 If the norms of the right and left eigenvalue residuals $\|r_R^{(j)}\|, \|r_L^{(j)}\|$ are small enough, the triple $(x_k := x^{(j)},\ y_k := y^{(j)},\ \lambda_k := \theta^{(j)})$ is accepted as approximate dominant eigentriplet. The eigenvector and eigenvalue matrices X, Y, S are augmented. If the eigenvalue is complex, X and Y can be augmented by $[\mathcal{R}ex_k,\ \mathcal{I}mx_k]$ and $[\mathcal{R}ey_k,\ \mathcal{I}my_k]$ to avoid some of the involved complex arithmetic operations. In order to prevent detecting this found eigentriplet again by the algorithm, it is deflated from the process. There are various ways for this, e.g. by restricting the input and output matrices E_2, H_1 and we point towards the relevant literature [19, 21] for details. After that, new direction vectors $u,\ z$ corresponding to $F(\theta^{(j)})$ are computed as in Step 2.

Step 14 As the iteration proceeds, the column dimensions of V and W increase and so does the numerical effort for the orthogonal expansions and the solution of the small eigenvalue problem. Hence, $V,\ W$ should be reduced to a smaller column dimension if they become too large. This is usually referred to as restart. Optionally, new direction vectors $u,\ z$ can be computed after the restart has been performed.

Recall from the above discussion that due to the symmetry properties of M, C, K in our application for the eigenvectors it holds $y = \overline{x}$ for a complex eigenvalue λ. However, SAQMDP cannot be simplified according to this observation in a straightforward way because in general $v^{(j)} \neq \overline{w^{(j)}}$ unless $E_2 = H_1^T$. If convergence to an eigenvalue occurs, it holds that $v^{(j)} \to x$ and $w^{(j)} \to \overline{x}$. Hence, a single matrix X is sufficient to store the found eigenvectors. In order to get the best result we propose to always select the eigenvector with the smallest residual to expand X. I.e., if $\|r_R^{(j)}\| < \|r_L^{(j)}\|$ then $x_k := x^{(j)}$, $y_k = \overline{x^{(j)}}$, but $x_k := \overline{y^{(j)}}$, $y_k := y^{(j)}$ otherwise.

The given matrices defining (6)–(7) are assumed to be real and it is possible to exploit this realness also in SAQMDP, e.g., by keeping the spaces V and W in real arithmetic. This amounts to applying similar algorithmic modifications as proposed in [25] for the generalized linear eigenvalue problem. In the remainder, when we mention SAQMDP, we always imply that both realness and symmetry of the quadratic eigenvalue problems are exploited as mentioned above.

2.2 Parametric dominant pole algorithm

Our aim is to obtain an accurate reduced system for various gains since we would like to optimize these gains for the corresponding configuration of the system matrices. In the given parametric system (6), the matrix C depends on the semi-active damping parameters and, in order to emphasize this dependence, it is in the following denoted by $C(g)$ with $g = (g_1, g_2, \ldots, g_p)$. The main idea in the parametric dominant pole algorithm is to compute approximations acquired with the dominant pole algorithm for a small number of selected parameters and to obtain an 'interpolatory' reduced model for other values in g.

In [22] it is shown that with such an interpolatory based approach, the original and reduced model have the same behavior near the dominant poles and the parametric reduced model obtained with this parametric dominant pole algorithm satisfies a Hermite interpolation property.

Proposition 1 ([22, Theorem 2]). *Let $x \in \operatorname{span}\{X\}$, $y \in \operatorname{span}\{Y\}$ for an eigentriple $(\lambda,\ x,\ y)$ of* (18) *for a given g. Moreover, let $F(s)$, $\widehat{F}(s)$ denote the original and reduced* *transfer function matrices of* (6)-(7)*, where* $\widehat{F}(s)$ *is obtained using* (20)*. Assume that* λ *is simple and that* $y^*x \neq 0$*,* $y^*x - |\lambda|^2 M^{-1} y^* B_2 B_2^T x \neq 0$*. Then it holds*

$$\lim_{s \to \lambda} |s-\lambda|^2 \frac{\partial F}{\partial g} = \lim_{s \to \lambda} |s-\lambda|^2 \frac{\partial \widehat{F}}{\partial g}.$$

Proof. The proof can be carried out exactly as for [22, Theorem 2] by rewriting the second order system (6)-(7) into a first order system (9) and exploiting the structure of the matrices A, E, H. □

The interpolation property is a desirable and important property for gain optimization since a reduced model which is a good approximation of the original model for various values of gains g is required. Hence, using the parametric dominant pole algorithm can be advantageous over other model reduction approaches that do not have the Hermite interpolation property.

There are different implementations for the interpolatory dominant pole algorithm (for more details see [22]) and we will summarize an approach based on SAQMDP for the calculation of the reduced system. An algorithm called continuation SAQMDP is illustrated in Algorithm 2 and computes reduced models that will enable an efficient semi-active damping optimization.

The selected values of parameters for which we will calculate dominant poles will be called initial parameters and are denoted by $g^{(1)}, g^{(2)}, \ldots, g^{(m)}$. For these values the reduced transfer function approximates the original transfer function in the sense of (19) and the reduced model satisfies interpolation properties as shown in [22].

Algorithm 2 continuation SAQMDP

Require: System matrices defining (6),(7), initial value for frequency $s^{(1)}$, initial parameters $g^{(1)}, g^{(2)}, \ldots, g^{(m)}$, number of wanted poles k_{wanted} for the given setting of parameters.
Ensure: Right eigenvectors X corresponding to found dominant poles.

1: Set $X = [\ ]$.
2: **for** $j = 1, \ldots, m$ **do**
3: Use SAQMDP for computing k dominant poles and corresponding right eigenvectors $X^{(j)}$, where the damping part $C(g^{(j)})$ is determined by the gains $g^{(j)}$.
4: Merge the eigenvectors $X^{(j)}$ to X.
5: **end for**
6: $X = \mathrm{orth}(X)$.

In Step 6 of Algorithm 2 we introduce an additional orthogonalization. There, a clever orthogonalization routine should be employed to ensure that numerically linearly dependent columns in X are discarded and the matrix X has full column rank such that reduced system will have regular coefficient matrices. It is noteworthy that in principle, also several other model reduction techniques [8] can be used in Step 3 to provide the matrices $X^{(j)}$ for a given $g^{(j)}$. However, as we are interested in the dominant poles of the systems transfer function, the application of SAQMDP is chosen here. Note that in order to improve the convergence to dominant poles in step 3, we can use the most dominant pole from the previous run of SAQMDP as new initial value for the frequency. Using the obtained eigenvectors collected in the matrix X from Algorithm 2, the reduced system can be formed as in (22).

It is important to point out that the mass, damping and stiffness matrices of the reduced system are again symmetric, positive definite and thus, the reduced system is also stable as is the original system.

Next, we draw special attention to one particular setting of initial parameters, namely the case with initial gain $g^{(1)} = 0$.

2.3 Modal approximation

In the case of a zero gain ($g = 0$), the system is only damped by the internal damping term. In order to simplify the analysis of this particular case, we transform the system using the so-called modal coordinates. For that purpose let Φ denote the nonsingular matrix that simultaneously diagonalizes M and K, i.e.,

$$\Phi^T K \Phi = \Omega^2 = \operatorname{diag}\left(\omega_1^2, \ldots, \omega_n^2\right) \quad \text{and} \quad \Phi^T M \Phi = I, \tag{24}$$

where $0 < \omega_1 \leq \omega_2 \leq \ldots \leq \omega_n$. The columns of the matrix Φ (usually called modal matrix) are the undamped eigenvectors of the system $M\ddot{q} + Kq = 0$ and the positive numbers $\omega_1, \omega_2, \ldots, \omega_n$ are the associated eigenvalues, and thus, they are called undamped eigenfrequencies. Furthermore, the matrix Φ also diagonalizes the internal damping as $\Phi^T C_u \Phi = \alpha\Omega$ with $\alpha = 2\alpha_c$.

By using (24) and substituting $q(t) = \Phi\widehat{q}(t)$, from the equations (6)-(7) we obtain the vibrational system:

$$\ddot{\widehat{q}}(t) + \Phi^T B_2 G B_2^T \Phi \dot{\widehat{q}}(t) + \Omega^2 \widehat{q}(t) = \Phi^T E_2 w(t), \tag{25}$$

$$z(t) = H_1 \Phi \widehat{q}(t). \tag{26}$$

With $\widehat{x}_1 = \Omega\widehat{q}$ and $\widehat{x}_2 = \dot{\widehat{q}}$, similar as in the introduction, we obtain the following first order system of differential equations:

$$\begin{aligned} \dot{\widehat{x}}(t) &= \widehat{A}\widehat{x}(t) + \widehat{E}w(t), \\ z(t) &= \widehat{H}\widehat{x}(t), \end{aligned} \tag{27}$$

where

$$\widehat{x} := \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix}, \quad \widehat{A} := \begin{bmatrix} 0 & \Omega \\ -\Omega & -\alpha\Omega - \Phi^T B_2 G B_2^T \Phi \end{bmatrix}, \quad \widehat{E} := \begin{bmatrix} 0 \\ \Phi^T E_2 \end{bmatrix},$$

and $\widehat{H} := \begin{bmatrix} H_1 \Phi \Omega^{-1} & 0 \end{bmatrix}$. The minimization criterion (16) leads to the Lyapunov equation

$$\widehat{A}^T \widehat{P} + \widehat{P}\widehat{A} = -\widehat{H}^T \widehat{H}. \tag{28}$$

A Lyapunov equation of similar structure was investigated in the papers [6, 7] where the authors were simplifying the solutions of Lyapunov equations using dimension reduction techniques in order to efficiently calculate its approximation.

For formulating the criterion (16) in terms of the solution of the Lyapunov equation (28), a direct calculation shows that

$$P = U^* \widehat{P} U, \quad \text{where} \quad U = \begin{bmatrix} \Phi\Omega^{-1} & 0 \\ 0 & \Phi \end{bmatrix},$$

and the matrices P and $\widehat{P}$ are solutions of the Lyapunov equations (15) and (28), respectively. Furthermore, if we uniformly partition the matrices $\widehat{P}$, E as

$$\widehat{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ M^{-1}E_2 \end{bmatrix},$$

the impulse response energy given in (16) is

$$J_2 = \mathrm{tr}\left(E^T P E\right) = \mathrm{tr}\left(E^T U^* \widehat{P} U E\right) = \mathrm{tr}\left(E_2^T \Phi P_{22} \Phi^T E_2\right).$$

Now, we will construct an approximation for the given impulse response energy using eigenvectors of the undamped system which will correspond to the initial gain equal to zero. Eigenvectors of the system corresponding to the zero gain are equal to columns of the matrix Φ, i.e. the matrix X from (22) corresponding to the zero gain contains columns of the matrix Φ.

Approaches that use undamped eigenvectors for forming the reduced system are usually called modal approximations [7, 12, 26].

In order to decide which undamped eigenvectors should be chosen, we will compare the reduced system with the system in modal coordinates determined by equations (25)-(26), especially for the criterion (16).

Here, the right-hand side of the corresponding Lyapunov equation has a special structure which has a strong influence on the solution. The main question is: *in terms of the criterion* (16)*, which undamped eigenvectors should be chosen in this particular case?*

For the answer we consider the structure of the right-hand side in the Lyapunov equation (28) which is equal to $\widehat{H}^T \widehat{H}$ where $\widehat{H} = \left[H_1 \Phi \Omega^{-1} \; 0\right]$. More precisely,

$$\widehat{H}^T \widehat{H} = \begin{bmatrix} \Omega^{-1}\Phi^T H_1^T H_1 \Phi \Omega^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Furthermore, if we denote ϕ_i to be the ith column of Φ we obtain

$$\Phi\Omega^{-1} = \begin{bmatrix} \frac{1}{\omega_1}\phi_1 & \frac{1}{\omega_2}\phi_2 & \cdots & \frac{1}{\omega_n}\phi_n \end{bmatrix}$$

and a direct multiplication reveals that the upper diagonal block has the following elements

$$(\Omega^{-1}\Phi^T H_1^T H_1 \Phi \Omega^{-1})_{ij} = \frac{1}{\omega_i \omega_j} \phi_i^T H_1^T H_1 \phi_j.$$

The obtained elements are quadratic in $\frac{1}{\omega_i}$ and since the ω_i are ordered increasingly, for maintaining the largest elements in the matrix $\widehat{H}^T \widehat{H}$ in the case of the zero gain, we take the eigenvectors (columns of matrix Φ) such that in the right hand side of the Lyapunov equation the r largest numbers are maintained. Altogether this means that in (22)

$$X = \Phi(:, 1:r) \tag{29}$$

is chosen. With this the reduced Lyapunov equation will include the upper diagonal $r \times r$ block of the right hand-side $(\Omega^{-1}\Phi^T H_1^T H_1 \Phi \Omega^{-1})$. Note that truncating with eigenvectors that correspond to the smallest undamped eigenvalues gives us the eigenvalues of the system with zero gain closest to the imaginary axis. This also shows the importance of the poles close to imaginary axis, especially for the small gains.

In the case where the eigenvectors are equal to the undamped eigenvectors, that is for $X = \Phi(\,:\,,1:r)$, we obtain from equations (22) the reduced system

$$\begin{aligned} \ddot{q}_r(t) + \Phi(\,:\,,1:r)^* C \Phi(\,:\,,1:r)\dot{q}_r(t) + \Omega_k q_r(t) &= \Phi(\,:\,,1:r)^* E_2 w(t), \\ z(t) &= H_1 \Phi(\,:\,,1:r) q_r(t), \end{aligned}$$

where $\Omega_r = \operatorname{diag}(\omega_1, \omega_2, \ldots, \omega_r)$. The corresponding Lyapunov equation for the approximated system is

$$\widetilde{A}^T Q + Q\widetilde{A} = -\widetilde{H}^T \widetilde{H}, \tag{30}$$

where

$$\widetilde{A} = \begin{bmatrix} 0 & \Omega_r \\ -\Omega_r & -\alpha\Omega_r - X^T B_2 G B_2^T X \end{bmatrix}, \quad \widetilde{H} = \begin{bmatrix} H_1 X \Omega_r^{-1} & 0 \end{bmatrix}.$$

Hence, the impulse response energy J_2 can be approximated by

$$J_2 \approx \operatorname{tr}\left(E_2^T X Q_{22} X^T E_2\right) \tag{31}$$

with Q_{22} being the lower diagonal block of the solution Q of the Lyapunov equation (30).

The modal approximation plays an important role in engineering applications. In our case it allows us to conclude, directly from criterion (16), which eigenvectors are important for a gain equal to zero.

Furthermore, modal approximation can be easily incorporated into our approach which uses continuation SAQMDP by taking the initial parameter $g^{(1)}$ equal to zero.

In the optimization procedure this requires insignificant computations since these eigenvectors have to be computed only once in the process and this is achieved efficiently using iterative eigenvalue solvers for generalized eigenvalue problems.

The usage of modal approximations in our approach will be discussed in the next subsection.

2.4 Gain optimization

In the numerical examples we will illustrate that a few different values of parameters $g^{(1)}, g^{(2)}, \ldots, g^{(m)}$ are often sufficient to obtain satisfactory approximations of the optimal damping. Furthermore, by changing the gains for small or even moderate values, the same eigenvectors can provide good approximations of the eigenspaces. Thus, the dimension of the reduced system is usually much smaller than the full dimension. In this section we will show how this can be determined in advance.

Furthermore, in many applications we observe that, even with eigenvectors that correspond to the zero initial gain, we can obtain sufficiently accurate approximations of the original system in the sense of gain optimization. Thus, we include the eigenspace corresponding to the zero gain in the first approximation.

We would like to efficiently determine initial parameters $g^{(1)}, g^{(2)}, \ldots, g^{(m)}$ and at this moment it is not clear how one can choose initial gains. Thus, we will derive a corresponding residual bounds which will provide additional information. In this setting, the main question is: if we have computed an (approximate) eigenspace for the gain $g^{(i)}$, what should be the next gain $g^{(i+1)}$ for which it is required to compute a new eigenspace?

First, recall that the gain $g^{(i)} = (g_1, g_2, \ldots, g_p)$ determines the damping part $C(g^{(i)}) = C_u + B_2 G B_2^T$ where $G = \operatorname{diag}(g_1, g_2, \ldots, g_p)$. Let us assume that for the gain $g^{(i)}$, using Algorithm 1, we have calculated k dominant right eigenvectors $x^{(1)}, x^{(2)}, \ldots, x^{(k)}$ and corresponding eigenvalues $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(k)}$ for a residual tolerance τ:

$$\|({\theta^{(j)}}^2 M + \theta^{(j)}(C_u + B_2 \operatorname{diag}(g_1, g_2, \ldots, g_p)\, B_2^T) + K)x^{(j)}\| < \tau, \quad \forall j = 1, \ldots, k. \tag{32}$$

The residual corresponding to these approximate eigenpairs for the system with gain $g^{(i)} + \Delta g^{(i)}$ is, using (32),

$$\begin{aligned}
\|({\theta^{(j)}}^2 M + &\theta^{(j)} C(g^{(i)} + \Delta g^{(i)}) + K)x^{(j)}\| \\
&= \|({\theta^{(j)}}^2 M + \theta^{(j)}(C_u + B_2 \operatorname{diag}(g_1 + \delta g_1, g_2 + \delta g_2, \ldots, g_p + \delta g_p)\, B_2^T) + K)x^{(j)}\| \\
&< \tau + \|\theta^{(j)} B_2 \operatorname{diag}(\delta g_1, \delta g_2, \ldots, \delta g_p)\, B_2^T x^{(j)}\| \\
&\le \tau + \|\theta^{(j)} B_2\| \|\operatorname{diag}(\delta g_1, \delta g_2, \ldots, \delta g_p)\| \|B_2^T x^{(j)}\|, \quad \forall j = 1, \ldots, k.
\end{aligned}$$

Our aim is to determine for which perturbation $\Delta g^{(i)} = (\delta g_1, \delta g_2, \ldots, \delta g_p)$ the same subspace will be also good up to fixed new tolerance ν.

In general,

$$\tau \ll \|({\theta^{(j)}}^2 M + \theta^{(j)} C(g^{(i)} + \Delta g^{(i)}) + K)x^{(j)}\| \le \nu, \quad \forall j = 1, \ldots, k$$

because the approximate eigenpairs $(\theta^{(j)},\, x^{(j)})$ correspond to the system with gain $g^{(i)}$. We would like to determine $\Delta g^{(i)}$ such that the same eigenvectors are also good for $g^{(i)} + \Delta g^{(i)}$ for a prescribed tolerance ν. Since $\tau \ll \nu$ we neglect τ in the formula above and we have that

$$\begin{aligned}
\|({\theta^{(j)}}^2 M + \theta^{(j)} C(g^{(i)} + \Delta g^{(i)}) + K)x^{(j)}\| &< \tau + \|\theta^{(j)} B_2\| \|\|\Delta g^{(i)}\|\| \|B_2^T x^{(j)}\| \\
&\lessapprox \|\theta^{(j)} B_2\| \|\Delta g^{(i)}\| \|B_2^T x^{(j)}\| \le \nu, \quad \forall j = 1, \ldots, k.
\end{aligned}$$

Thus, if

$$\|\Delta g^{(i)}\| \le \frac{\nu}{\|\theta^{(j)} B_2\| \|B_2^T x^{(j)}\|}, \quad \forall j = 1, \ldots, k.$$

holds, the eigenspace obtained for $g^{(j)}$ will also give proper approximation for $g^{(i)} + \Delta g^{(i)}$. A measure that takes all computed approximate eigenpairs into account can be formulated by using the mean average over the upper bounds for all eigenpairs. The upper bound for acceptable change of $g^{(j)}$ associated to all eigenpairs is determined by

$$r_j := \frac{1}{k} \sum_{j=1}^{k} \frac{\nu}{\|\theta^{(j)} B_2\| \|B_2^T x^{(j)}\|}.$$

Another possibility would be a criterion which includes the term $\min\limits_j \dfrac{\nu}{\|\theta^{(j)} B_2\| \|B_2^T x^{(j)}\|}$. In practice this might be too pessimistic since we can have very few indexes j where $\frac{\nu}{\|\theta^{(j)} B_2\| \|B_2^T x^{(j)}\|}$ is large while in the other cases this is quite small.

For including the obtained bound in the algorithm for the gain optimization, we introduce

$$K(g^{(j)}, r_j) := \{g : \|g - g^{(j)}\| \leq r_j\},$$

where r_j is defined in (33).

Note that, if we assume that all dampers have the same gains ($G = gI_p$), then we can obtain a sharper bound for $\Delta g^{(i)} \in \mathbb{R}$ as

$$|\Delta g^{(i)}| \leq \frac{\nu}{\|\theta^{(j)} B_2 B_2^T x^{(j)}\|}, \quad \forall j = 1, \dots, k$$

and use

$$r_j = \frac{1}{k} \sum_{j=1}^{k} \frac{\nu}{\|\theta^{(j)} B_2 B_2^T x^{(j)}\|} \tag{33}$$

in this case. The efficiency of our gain optimization algorithm can be improved by using the above bounds. For the given gain g we check if $g \in K(g^{(j)}, r_j)$ for some j. If this is the case, the same subspace can be used for the approximation of the function J_2. If not, this gain is added to the set of gains and the subspace X is enriched by new eigenvectors. Algorithm 3 illustrates this approach for the gain optimization.

Algorithm 3 Computation of optimal gains

Require: System matrices; initial value for frequency $s^{(1)}$; tolerance ν for determination of initial parameters; number of wanted poles k_{wanted} for each setting of parameters.
Ensure: Approximate optimal gains.
1: Set $j = 1$, initial parameter $g^{(1)} = 0$ and X as in (29) and calculate r_j .
2: Find optimal gains by using an appropriate optimization procedure (e.g., the Nelder-Mead algorithm). Evaluate the function value of J_2 at the found g as in steps 3 to 11:
3: **if** $g \in \bigcup_{i=1}^{j} K(g^{(j)}, r_j)$ **then**
4: Calculate reduced system using current X as in (22).
5: Compute function value at gain g for the reduced system.
6: **else**
7: Set $j = j + 1$, $g^{(j)} = g$.
8: Calculate eigenvectors $X^{(j)}$ for the initial parameter $g^{(j)}$ using SAQMDP (Algorithm 1).
9: Merge the eigenvectors $X^{(j)}$ to X ($X = \mathrm{orth}(X)$) and calculate r_j from (33).
10: Calculate reduced system using the augmented X as in (22).
11: Compute function value at gain g for the reduced system .
12: **end if**

Note that the calculation of the eigenvectors corresponding to the zero initial gain should be carried out only once for the whole optimization procedure. Furthermore, including the eigenvectors of the undamped system additionally stabilizes the optimization procedure. In particular, due to the occasional slow convergence of the SAQMDP for some configurations, we might end up with a very small reduced dimension which yields a reduced system which is not accurate enough. With the zero initial gain we, however, always end up with reduced dimensions that are greater or equal to the number of wanted poles.

3 Numerical experiments

In this section we illustrate the efficiency of the introduced approximation techniques. For that purpose we will compare the new approach with the optimization without reduction. In both approaches the Lyapunov equations are solved by the Hammarling's algorithm [14] implemented in the MATLAB® function `lyapchol` based on the SLICOT routines `SB03OD` and `SG03BD` [3].

In these examples, the computations have been carried out on a compute server using 4 Intel Xeon @2.67 GHz CPUs with 8 cores per CPU and 1 TB RAM. Results were calculated in MATLAB Version 7.11.0.584 (R2010b) 64-bit.

Example 1. We will consider an n-mass oscillator or oscillator ladder which describes the mechanical system of n masses and $n+1$ springs (see e.g. [7]). The mathematical model is given by (1)-(3), where the mass and stiffness matrices are

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n),$$

$$K = \begin{pmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & & & -k_n & k_n+k_{n+1} \end{pmatrix}.$$

We will consider the following configuration for the mass and stiffness values:

$$n = 1800; \quad k_i = 2, \quad \forall i; \qquad m_i = \begin{cases} 1500-2i, & i = 1,\ldots,600, \\ 600-i/2, & i = 601,\ldots,999, \\ i-800, & i = 1000,\ldots,n. \end{cases}$$

The internal damping is given by the damping matrix (4) with $\alpha_c = 0.002$. For simplicity G is a diagonal matrix $G = \operatorname{diag}(g,\ g,\ldots,g) \in \mathbb{R}^{p\times p}$ with $p = 6$. The control as well as the control velocity matrix are given by

$$B_2 = [e_j\ e_{j+1}\ e_{j+10}\ e_{j+11}\ e_{j+20}\ e_{j+21}], \tag{34}$$

where $1 \le j+21 \le n$ and e_j is the jth canonical vector. Here, we have introduced an index j since we will consider different configurations for B_2. Furthermore, we are interested in the states in the first third of n-mass oscillator that correspond to the smallest masses, that is, we consider 40 states starting with the 580th state:

$$z(t) = [q_{580}(t)\ q_{581}(t)\ \ldots\ q_{619}(t)]^T.$$

Hence, $H_1 \in \mathbb{R}^{40\times n}$ with

$$H_1(1:40, 580:619) = I_{40\times 40}$$

and all other entries are equal to zero. Since we consider gain optimization for different primary excitation matrices, we define $E_2 \in \mathbb{R}^{n\times 30}$ such that the primary excitation is applied to 30 masses determined by the index k:

$$(E_2)_i = \begin{cases} 1, & k \le i \le k+29, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{where} \quad k+29 \le n. \tag{35}$$

We compare the approximation of the impulse response energy J_2 for different damper positions and different primary excitations, i.e., the following configurations of (k, j) are taken into consideration:

$$\begin{aligned} &\text{for} \quad k = 30 : 200 : 1800, \qquad j = 100 : 200 : 1800, \\ &\qquad \text{exclude indices, where for} \quad (j < 600 \,\&\, k < 600) \,|\, (j > 600 \,\&\, k > 600) \\ &\qquad \text{it holds} \quad |j - 500| < |k - 500|, \end{aligned}$$

with k and j determining B_2 and E_2 as given by (34) and (35), respectively. This yields in total 60 different starting configurations for the gain optimization.

The following parameters required by Algorithms 1–3 are used:

$$k_{wanted} = 72, \quad \nu = 3 \cdot 10^{-4}, \quad \tau = 10^{-12}.$$

The initial value for the frequency $s^{(1)}$ in SAQMDP is taken as the eigenvalue closest to the imaginary axis corresponding to the zero initial gain. For this example this is $-2.505 \cdot 10^{-7} + 1.252 \cdot 10^{-4}i$.

Since we consider different configurations (we especially change the primarily excitation input, the control and the control velocity matrix) we need to sort these configurations. In order to present relative errors more clearly, the different configurations are sorted w.r.t. the magnitude of the relative error in the optimal gain. Figure 1 presents the relative errors of the

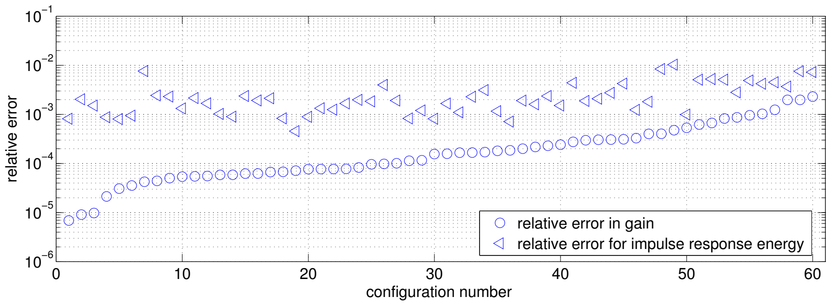

Figure 1: Relative errors for the gain and the energy J_2.

optimal gain calculated with and without dimension reduction by Algorithm 3. The circles denote the relative error of the optimal gain calculated by $|g^0 - g|/g$, where g and g^0 denote the optimal gain calculated with and without dimension reduction, respectively. Similarly, triangles denote the relative errors w.r.t. the approximation of the impulse response energy J_2, where in the case without dimension reduction, J_2 was calculated at the exact optimal gains.

Recall that all gains are constant and equal, thus we can apply Nelder-Mead's method (see e.g. [17]) implemented in the MATLAB function `fminsearch`. The tolerance for this optimization was 10^{-5} and the initial guess was set to 40.

The magnitude of the optimal gains and impulse response energies J_2 can be seen in Figure 2. As it can be seen there, the optimal gain varies between 4 and 46.

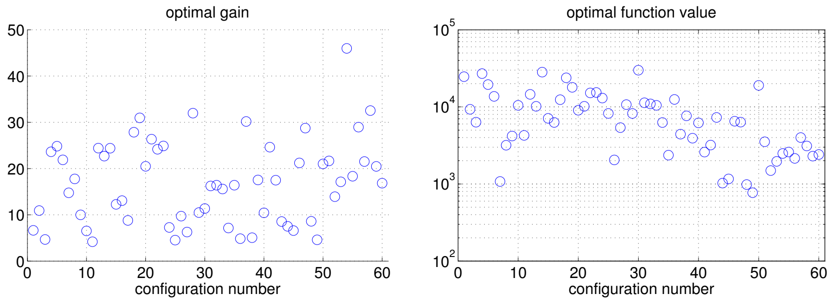

Figure 2: Magnitudes of the optimal gains and impulse response energies.

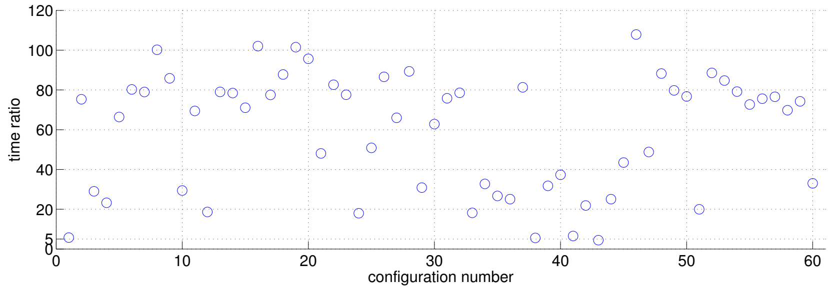

Figure 3: Time ratio between exact and approximation based approach.

In Figure 3 the speed-up in computational time of the optimization process obtained by the new approximation technique is illustrated. More precisely, it shows the ratio between the times required for the gain optimization with and without the novel approximation technique using dimension reduction. Evidently, the new approach requires from 4 to 107 times less computation time which leads to a considerably faster gain optimization process.

Note that in Algorithm 3 we calculate eigenvectors which correspond to $g = 0$ and additional gains which are determined using the residual bound from Section 2.4. Altogether, in the optimization procedure based on the residual bound we had to recalculate eigenvectors for one to four additional gains g and thus the reduced dimension varies between 90 and 216 (the full dimension is 1800). Figure 1 reveals that all relative errors are of order 10^{-2} or smaller. For the tolerance $\nu = 3 \cdot 10^{-4}$, the maximal reduced dimension varies between 5% and 12% of the original dimension. Figure 4 shows the maximal reduced dimensions for the corresponding configurations.

In the following example we will compare quality of the proposed approach on different mechanical structure. Apart from the previous example we will consider two different gains and compare the results when varying geometry of the corresponding control as well as control velocity matrix.

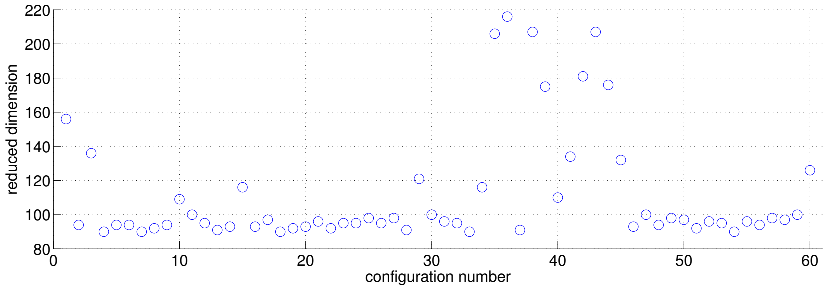

Figure 4: Reduced dimension.

Example 2. In this example we will consider a mass oscillator with $2d+1$ masses and $2d+3$ springs given in [7, Example 2]. This example has two rows of d masses connected with springs where the first row of masses has stiffness k_1 and the second row has stiffnesses k_2. The first masses on the left edge (m_1 and m_{d+1}) are connected to a fixed bound while on the other side of rows the masses (m_d and m_{2d}) are connected to mass m_{2d+1} with a stiffness k_3 connected to a fixed bound.

The mass matrix is a diagonal matrix as in the previous example and the stiffness matrix is defined by

$$K = \begin{bmatrix} K_{11} & & -\kappa_1 \\ & K_{22} & -\kappa_2 \\ -\kappa_1^T & -\kappa_2^T & k_1 + k_2 + k_3 \end{bmatrix},$$

where

$$K_{ii} = k_i \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \kappa_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k_i \end{bmatrix}, \quad i = 1, 2.$$

We will consider the following configuration for the mass and stiffness values:

$$\begin{aligned} d &= 800, \quad n = 2d+1 = 1601; \quad k_1 = 3, \quad k_2 = 1, \quad k_3 = 2; \\ m_i &= 3.98 + 0.02 \cdot i \quad \text{for} \quad i = 1, \dots, 600; \\ m_i &= 34 - 0.03 \cdot i \quad \text{for} \quad i = 601, \dots, 800; \\ m_i &= 23 - 0.01 \cdot i \quad \text{for} \quad i = 801, \dots, 1600; \quad m_{2d+1} = 10. \end{aligned}$$

With this configuration, masses on the first row are smaller on the edges, while on the second row of masses we have masses that decrease in magnitude.

The internal damping is given by the damping matrix (4) with $\alpha_c = 0.005$.

Here, we are interested in 20 states. In particular, 10 masses on the first row of masses starting from the 301st to 310th mass and on the second row of masses starting from the

1201st to 1210th mass. Thus, the matrix $H_1 \in \mathbb{R}^{10\times n}$ has the following form

$$H_1(1:10, 301:310) = I_{10\times 10},$$
$$H_1(1:10, 1201:1210) = I_{10\times 10},$$

and all other entries are equal to zero.

Here, the primary excitation input acts on masses closer to edges of the mechanical structure and the input is stronger on edges, thus we define $E_2 \in \mathbb{R}^{n\times 11}$ with

$$E_2(1:5, 1:5) = \operatorname{diag}(5, 4, 3, 2, 1),$$
$$E_2(801:85, 6:10) = \operatorname{diag}(5, 4, 3, 2, 1),$$
$$E_2(1601, 11) = 10,$$

and all other entries are equal to zero.

In this example we have two different gains g_1 and g_2 determining G which is a diagonal matrix $G = \operatorname{diag}(g_1, g_1, g_2, g_2) \in \mathbb{R}^{p\times p}$ with $p = 4$. The control as well as the control velocity matrix $B_2 \in \mathbb{R}^{n\times 4}$ is determined with indices j and k such that

$$B_2 = [(e_j - e_{j+1})\ (e_{j+10} - e_{j+11})\ (e_k - e_{k+1})\ (e_{k+10} - e_{k+11})].$$

Note that by varying indices j and k we change the geometry of control as well as the control velocity matrix. In particular, we consider the following configuration of indices j and k:

$$j = 50:100:800, \quad k = 900:200:1600$$

which will give 32 different configurations. This means that the first two columns of B_2 determine the control and the control velocity on the first row of masses while the third and fourth column determine the control and the control velocity part applied to the second row of masses.

The following parameters required by Algorithms 1–3 are used:

$$k_{wanted} = 120, \quad \nu = 5\cdot 10^{-3}, \quad \tau = 10^{-12}.$$

The initial value for the frequency $s^{(1)}$ in SAQMDP is taken as the eigenvalue closest to the imaginary axis corresponding to the zero initial gain.

Here, we consider two different gains and, like in the previous example, we apply Nelder-Mead's method, but the tolerance for this optimization was set to 10^{-4} and the initial guess was set to [500, 500].

Figure 5 presents the relative errors of the optimal gain calculated with and without dimension reduction by Algorithm 3. The circles denote the relative error of the optimal gain calculated by $\|g^0 - g\|/\|g\|$, where g and g^0 denote the optimal gain calculated with and without dimension reduction, respectively. We consider different configurations regarding the control velocity matrix, thus, in Figure 5 different configurations (j, k) are sorted w.r.t. the magnitude of the relative error in the optimal gain.

Similarly, triangles denote the relative errors w.r.t. the approximation of the impulse response energy J_2, where in the case without dimension reduction, J_2 was calculated at the exact optimal gains.

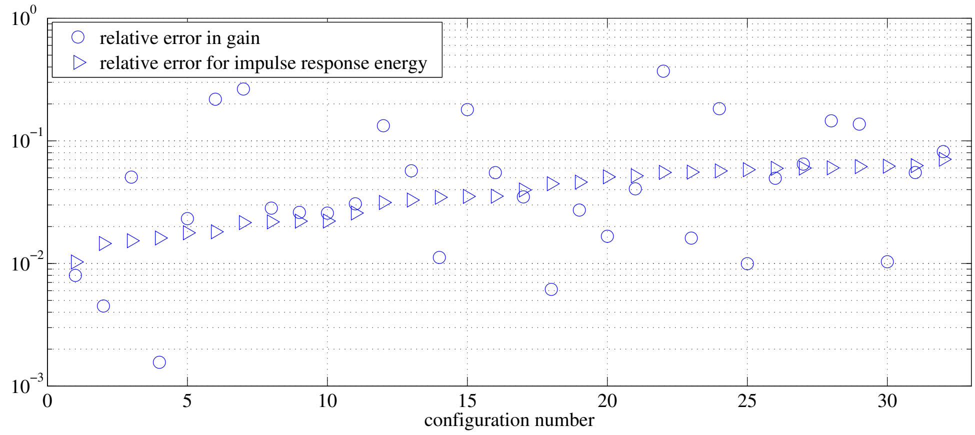

Figure 5: Relative errors for the gain and the energy J_2.

Figure 5 reveals that in this example all relative errors are of order or smaller than 10^{-1}. Moreover, the reduced dimension varies between 152 and 757 (full dimension is 1601) and in this example we needed more additional gains in the optimization process, compared to the previous example. This is partially expected since damping optimization with different gains is more demanding, but in general this also depends on the system configuration.

We would like to note that also in this example at each configuration our approach was faster. For the considered 32 configurations, the whole optimization process algorithm without the approximation technique required 509.61 hours, while with the approximation technique we needed 40.2 hours which lead to a considerably faster gain optimization process.

4 Conclusions

In this paper we have considered optimizing a semi-active damping using a criterion based on the impulse response energy. This optimization problem is a very demanding due to the numerous Lyapunov equations which have to be solved. We have presented an approach which approximates the impulse response energy by efficiently computing approximations of the transfer function. This is achieved by using the dominant pole algorithm for parametric systems with the initial parameters chosen using eigenvalue residual bounds. Numerical experiments confirms the efficiency of our approach in the sense that the optimization process is considerably accelerated and satisfactory approximations of the optimal gain are obtained.

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