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On Model Reduction of Periodic Descriptor Systems Exploiting the Generalized Inverses of Periodic Matrix Pairs

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Abstract

In this paper, we establish a model reduction technique for periodic discrete-time descriptor systems exploiting the generalized inverses of the periodic singular matrix pairs associated with the systems. We compute the generalized inverses of periodic singular matrix pairs to implement a structure preserving iterative method for the solution of the periodic projected Lyapunov equations that arise in analysis and modelling of periodic discrete-time descriptor systems. We extend the Smith method to solve the large scale projected periodic discrete-time algebraic Lyapunov equations in lifted form. A low-rank version of this method is also presented, which avoids the explicit lifted formulation and works directly with the period matrix coefficients. Moreover, we consider an application of the Lyapunov solvers in balanced truncation model reduction of periodic discrete-time descriptor systems. Numerical results are given to illustrate the efficiency and accuracy of the proposed methods.

Keywords: periodic descriptor systems, lifted state space representation, periodic projected Lyapunov equations, alternating direction implicit method, Smith iteration, model order reduction.

1 Introduction

Periodic systems and control theory have received a lot of attention in the last few decades because they have wide applications in many areas of science and engineering, specially in the areas where the periodic control is deserved, such as aerospace realm, control of industrial processes and communication systems, modeling of periodic time-varying filters and networks [15, 20, 19, 34].

In this paper, we consider linear time-varying (LTV) discrete-time descriptor systems of order $\propto = (n_0, n_1, \ldots, n_{K-1})$, as

$$E_k x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k, \quad k \in \mathbb{Z}, \tag{1}$$

where $E_k \in \mathrm{R}^{\mu_k \times n_{k+1}}$, $A_k \in \mathrm{R}^{\mu_k \times n_k}$, $B_k \in \mathrm{R}^{\mu_k \times m_k}$, $C_k \in \mathrm{R}^{p_k \times n_k}$ are the system matrices, $x_k \in \mathrm{R}^{n_k}$ is the (generalized) state or descriptor vector, $u_k \in \mathrm{R}^{m_k}$ is the control input, and $y_k \in \mathrm{R}^{p_k}$ is the output. The system matrices are periodic with a period $K \geq 1$, and $\sum_{k=0}^{K-1} \mu_k = \sum_{k=0}^{K-1} n_k = n$, $\sum_{k=0}^{K-1} m_k = m$ as well as $\sum_{k=0}^{K-1} q_k = q$. If all E_k are nonsingular, then (1) can be transformed into a periodic standard system.

Efficient numerical methods for computing poles and zeros, $\mathbb{L}_\infty$-norm, minimal and balanced realizations have been developed for such systems [6, 38, 39, 40]. All these methods are restricted to problems of small or medium size because they are based on the reduction of the periodic pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ to a periodic Kronecker-like form [37]. In the last decade, attention has been devoted to the iterative solution of large-scale sparse Lyapunov equations using the alternating directions implicit (ADI) method [17, 21], the Smith method [9, 21, 26], and Krylov subspace methods [11, 24]. All these methods have also been generalized to projected Lyapunov equations [30, 31]. On the other hand, an extension of the Smith method and the block-Arnoldi based Krylov subspace method

to standard periodic Lyapunov equations has been presented in [14]. These methods cannot be directly applied to the projected periodic Lyapunov equations.

Analysis and reduced order modeling of such systems may require to invert the periodic matrix pairs associated with these systems in some appropriate sense. For square and invertible systems, one can explicitly formulate these inverses. For non-square systems, explicit formulation of these inverse may not be always possible [37]. The inversion formulas for periodic systems in standard form have been considered first in [12, 22, 23]. A special form of the generalized inverses, known as *(1,2)-inverses*, of periodic systems in the descriptor form has been considered in [37], which works on the corresponding lifted form of the associated system pencil of the periodic system.

In this paper, we discuss the computation of the generalized inverses of periodic discrete-time descriptor systems using the left and right deflating projectors associated with the eigenstructures of the periodic singular matrix pairs. This technique has been implemented in [31] for continuous-time descriptor system to compute the solution of the corresponding projected Lyapunov equations. We will generalize the idea of [31] for the discrete periodic setting and use those periodic inverses to compute the solutions of the periodic projected Lyapunov equations. Moreover, we reformulate the Smith method to solve the large projected periodic discrete-time algebraic Lyapunov equations in lifted form. The block diagonal structure of the periodic solutions is preserved in every Smith iteration step which is one of the challenging task in many of the iterative computations in the periodic setting. It should be noted that generalized versions of the ADI method and the Smith method have been proposed in [2] for the solution of projected periodic Lyapunov equations. But, the methods fail to preserve the block diagonal structure during the iteration, and it is only achieved upon convergence.

The rest of the paper is organized as follows. In Section 2, we briefly review discrete-time periodic descriptor systems and their cyclic lifted representations. We also study the causal and noncausal decomposition of the periodic descriptor systems. In Section 3, we introduce the periodic matrix equations under consideration and review the different techniques for the solutions of those matrix equations associated to the causal and non-causal systems. We also discuss the challenges of iterative techniques used in computing the structure preserving solution of those matrix equations. In Sections 4, we discuss the generalized inverses of periodic discrete-time descriptor systems using the left and right deflating projectors associated with the eigenstructures of the periodic singular matrix pairs. Iteratively solving of the causal and noncausal lifted Lyapunov equations exploiting the generalized inverses is discussed in Sections 5 and 6, respectively. Low-rank versions of these methods are also presented, which avoid the explicit lifted formulation and work directly with the period matrix coefficients. A balanced truncation model reduction method for periodic descriptor systems is considered in Section 7. Section 8 contains numerical examples that illustrate the properties of the described iterative methods for projected Lyapunov equations and their application to model reduction. Some conclusions are given in Section 9.

2 Periodic descriptor systems

Analysis and modeling of periodic discrete-systems are often described by an analogous time-invariant representation of the periodic systems, known as *lifted representation* [4, 7, 38], which uses the input-state-output behavior of the system over time intervals of length K, rather then 1. Using the lifting isomorphism one can exploit the theory of time-invariant systems for the analysis and control of periodic systems, provided that the results achieved can be easily re-interpreted in a periodic framework. The lifted representation of discrete-time periodic descriptor systems plays an important role in extending many theoretical results for descriptor systems to the periodic setting.

2.1 Cyclic lifted representation of periodic systems

We consider here the *cyclic lifted representation* which was introduced first for standard periodic systems in [20]. The essence of the cyclic lifted system is putting inputs, states and outputs of the original LTV descriptor system at cyclic places of those of the lifted LTI system.

The *cyclic lifted representation* of the periodic descriptor system (1) is given by

$$\mathcal{E}\mathcal{X}_{k+1} = \mathcal{A}\mathcal{X}_k + \mathcal{B}\mathcal{U}_k, \quad \mathcal{Y}_k = \mathcal{C}\mathcal{X}_k, \tag{2}$$

where

$$\mathcal{E} = \operatorname{diag}(E_0, E_1, \ldots, E_{K-1}), \quad \mathcal{B} = \operatorname{diag}(B_0, B_1, \ldots, B_{K-1}),$$

$$\mathcal{A} = \begin{bmatrix} 0 & \cdots & 0 & A_0 \\ A_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & A_{K-1} & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & \cdots & 0 & C_0 \\ C_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & C_{K-1} & 0 \end{bmatrix}. \tag{3}$$

The descriptor vector, system input and output of (2) are related to those of (1) via

$$\mathcal{X}_k = \begin{bmatrix} x_1 \\ \vdots \\ x_{K-1} \\ x_0 \end{bmatrix}, \quad \mathcal{U}_k = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{K-1} \end{bmatrix}, \quad \mathcal{Y}_k = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{K-1} \end{bmatrix},$$

respectively. The transfer function of the lifted system (2) can be rewritten as

$$\mathcal{H}(z) = \mathcal{C}(z\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}. \tag{4}$$

The cyclic lifted system (2) describes the eigenstructure and system dynamics of the LPTV discrete-time descriptor system (1). Regularity of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ implies the regularity of the cyclic matrix pair $(\mathcal{E}, \mathcal{A})$ [16, 27, 15]. The

reverse argument also holds true. Stability of the cyclic lifted system (2) is defined by the regularity of the cyclic matrix pencil $z\mathcal{E} - \mathcal{A}$, defined as [27]

$$z\begin{bmatrix} E_0 & & & \\ & E_1 & & \\ & & \ddots & \\ & & & E_{K-1} \end{bmatrix} - \begin{bmatrix} & & & A_0 \\ A_1 & & & \\ & \ddots & & \\ & & A_{K-1} & \end{bmatrix}. \tag{5}$$

The cyclic matrix pencil $z\mathcal{E} - \mathcal{A}$ is said to be *regular* when $\det(z\mathcal{E} - \mathcal{A}) \neq 0$, for any $z \in \mathbb{C}$. The cyclic lifted system (2) is *asymptotically stable* iff $z\mathcal{E} - \mathcal{A}$ is regular and all its finite eigenvalues lie inside the unit circle. System (1) is *asymptotically stable* if and only if the corresponding cyclic lifted system (2) is *asymptotically stable*.

2.2 Decomposition of periodic descriptor systems

The structure and periodic nature of the matrices E_k, for $k = 0, 1, \ldots, K-1$, play an important role in the analysis and modelling of the periodic descriptor system (1). In the descriptor setting, the matrices E_k in (1) are singular for $k = 0, 1, \ldots, K-1$. Hence, the transformation of system (1) to a standard system is not possible by an explicit inversion of E_k. However, one should avoid the explicit inversion of E_k, even if the E_k in (1) are invertible, due to numerical instability. In this case, we proceed by separating the causal and the noncausal parts of the periodic descriptor system (1). The canonical structures of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ can be used to find such a decomposition [6, 37]. The periodic Kronecker canonical form [33, 28] of the matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ for $k = 0, 1, \ldots, K-1$, can be represented as

$$U_k E_k V_{k+1} = \begin{bmatrix} I_{n_{k+1}^f} & 0 \\ 0 & E_k^b \end{bmatrix}, \; U_k A_k V_k = \begin{bmatrix} A_k^f & 0 \\ 0 & I_{n_k^\infty} \end{bmatrix}, \tag{6}$$

where U_k, V_k are nonsingular, $V_K = V_0$, $A_{k+K-1}^f A_{k+K-2}^f \cdots A_k^f = J_k$ is an $n_k^f \times n_k^f$ Jordan matrices corresponding to finite eigenvalues, $E_k^b E_{k+1}^b \cdots E_{k+K-1}^b = N_k$ is an $n_k^\infty \times n_k^\infty$ nilpotent Jordan matrix corresponding to infinite eigenvalues, and $n_k = n_k^f + n_k^\infty$, $\mu_k = n_{k+1}^f + n_k^\infty$.

Let ν_k be the nilpotency index of N_k for $k = 0, 1, \ldots, K-1$ such that $N_k^{\nu_k - 1} \neq 0$ and $N_k^{\nu_k} = 0$. Then the index ν of the set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ and also of the periodic descriptor system (1) is defined by

$$\nu = \max\{\nu_0, \nu_1, \ldots, \nu_{K-1}\}.$$

Using (6), we can decompose the periodic states x_k, periodic input matrices B_k, and the

periodic output matrices C_k, for $k = 0, 1, \ldots, K-1$. For every $k \in \mathbb{Z}$, let

$$x_k = V_k \begin{bmatrix} x_k^f \\ x_k^\infty \end{bmatrix}, \qquad U_k B_k = \begin{bmatrix} B_k^f \\ B_k^\infty \end{bmatrix}, \qquad C_k V_k = \begin{bmatrix} C_k^f & C_k^\infty \end{bmatrix}.$$

We then decompose the periodic descriptor system (1) into causal (forward) and noncausal (backward) periodic subsystems

$$\begin{array}{rcl@{\qquad}rcl} x_{k+1}^f &=& A_k^f x_k^f + B_k^f u_k, & y_k^f &=& C_k^f x_k^f, \\ E_k^\infty x_{k+1}^\infty &=& x_k^\infty + B_k^\infty u_k, & y_k^\infty &=& C_k^\infty x_k^\infty, \end{array}$$

respectively, with $y_k = y_k^f + y_k^\infty$. The left and right spectral projectors $P_l(k)$ and $P_r(k)$, for $k = 0, 1, \ldots, K-1$, can be represented as [1, 6, 30],

$$P_l(k) = U_k^{-1} \begin{bmatrix} I_{n_{k+1}^f} & 0 \\ 0 & 0 \end{bmatrix} U_k, \quad P_r(k) = V_k \begin{bmatrix} I_{n_k^f} & 0 \\ 0 & 0 \end{bmatrix} V_k^{-1}, \tag{7}$$

respectively.

Remark 1. The index of the periodic descriptor system (1) and the index of the lifted system (2) are closely related to each other. The index η of the lifted pencil $\lambda\mathcal{E} - \mathcal{A}$ is defined as the nilpotency index of the matrix

$$E_\infty = \begin{bmatrix} 0 & E_1^b & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & E_{K-1}^b \\ E_0^b & 0 & \cdots & 0 \end{bmatrix}.$$

Taking into account that $E_\infty^K = \operatorname{diag}(N_1, \ldots, N_{K-1}, N_0)$, the index of $\lambda\mathcal{E} - \mathcal{A}$ can be bounded as $\eta \leq K\nu$ [6, 30].

3 Periodic Matrix Equations and their Solution

Stability analysis and model reduction of periodic systems are strongly related to the matrix equations associated with the systems. For disctere-time periodic descriptor systems, they are well known as *generalized projected periodic discrete-time algebraic Lyapunov equations* (PPDALEs). It has been shown in [2, 6] that the periodic Gramians of the asymptotically stable discrete-time descriptor systems (1) satisfy the PPDALEs with special right-hand sides. The causal and noncausal reachability Gramians X_k and $\hat{X}_k$ are the unique symmetric, positive semidefinite periodic solutions of the PPDALEs

$$\begin{array}{rcl} A_k X_k A_k^T - E_k X_{k+1} E_k^T &=& -P_l(k) B_k B_k^T P_l(k)^T, \\ X_k &=& P_r(k) X_k P_r(k)^T, \end{array} \tag{8}$$

and

$$\begin{aligned} A_k\hat{X}_kA_k^T - E_k\hat{X}_{k+1}E_k^T &= Q_l(k)B_kB_k^TQ_l(k)^T, \\ \hat{X}_k &= Q_r(k)\hat{X}_kQ_r(k)^T, \end{aligned} \tag{9}$$

respectively, where $X_K = X_0$, $\hat{X}_K = \hat{X}_0$. Note that $P_l(k), P_r(k)$, for $k = 0, 1, \ldots, K-1$, are the spectral projectors onto the k-th left and right deflating subspaces of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ corresponding to the finite eigenvalues [1, 6], and $Q_l(k) = I - P_l(k)$ and $Q_r(k) = I - P_r(k)$ in (9).

Similarly, the causal and noncausal observability Gramians Y_k and $\hat{Y}_k$ are the unique symmetric, positive semidefinite periodic solutions of the PPDALEs

$$\begin{aligned} A_kY_kA_k^T - E_kY_{k+1}E_k^T &= -P_l(k)B_kB_k^TP_l(k)^T, \\ Y_k &= P_r(k)Y_kP_r(k)^T, \end{aligned} \tag{10}$$

and

$$\begin{aligned} A_k\hat{Y}_kA_k^T - E_k\hat{Y}_{k+1}E_k^T &= Q_l(k)B_kB_k^TQ_l(k)^T, \\ \hat{Y}_k &= Q_r(k)\hat{Y}_kQ_r(k)^T, \end{aligned} \tag{11}$$

respectively, where $Y_K = Y_0$, $\hat{Y}_K = \hat{Y}_0$.

The numerical solution of (8) has been considered in [6] for time-varying matrix coefficients. The method proposed there is based on an initial reduction of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ to the generalized periodic Schur form [13, 37] and on solving the resulting generalized periodic Sylvester and Lyapunov equations. As a result, the method is computationally expensive and not suitable for large scale problems.

An efficient approach which works with the cyclic lifted representation of (1) and the corresponding lifted form of (8) has been considered in [1]. Following the work of [1], the PPDALEs (8) and (9) are equivalent to the following projected lifted discrete-time algebraic Lyapunov equation (PLDALE)

$$\mathcal{A}\mathcal{X}\mathcal{A}^T - \mathcal{E}\mathcal{X}\mathcal{E}^T = -\mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T, \quad \mathcal{X} = \mathcal{P}_r\mathcal{X}\mathcal{P}_r^T, \tag{12}$$

$$\mathcal{A}\hat{\mathcal{X}}\mathcal{A}^T - \mathcal{E}\hat{\mathcal{X}}\mathcal{E}^T = \mathcal{Q}_r\mathcal{B}\mathcal{B}^T\mathcal{Q}_r^T, \quad \hat{\mathcal{X}} = \mathcal{Q}_r\hat{\mathcal{X}}\mathcal{Q}_r^T, \tag{13}$$

respectively, where $\mathcal{E}$, $\mathcal{A}$ and $\mathcal{B}$ are as in (3), and

$$\mathcal{X} = \mathrm{diag}(X_1, \ldots, X_{K-1}, X_0), \quad \hat{\mathcal{X}} = \mathrm{diag}(\hat{X}_1, \ldots, \hat{X}_{K-1}, \hat{X}_0).$$

In that case, the projectors in lifted forms are given by

$$\begin{aligned} \mathcal{P}_l &= \mathrm{diag}(P_l(0), P_l(1), \ldots, P_l(K-1)), & \mathcal{Q}_l &= I - \mathcal{P}_l, \\ \mathcal{P}_r &= \mathrm{diag}(P_r(1), \ldots, P_r(K-1), P_r(0)), & \mathcal{Q}_r &= I - \mathcal{P}_r. \end{aligned}$$

The matrices $\mathcal{X}$ and $\hat{\mathcal{X}}$ are called *the causal and noncausal reachability Gramians* of the lifted system (2). A similar result can also be stated for the causal and noncausal observability Gramians [1].

In practice, one should avoid these direct methods for large-scale problems because the computational complexity for solving a Lyapunov equation of the form (12) or (13) using direct methods is at least of order ($\mathcal{O}(Kn_{\max}^3)$), where $n_{\max} = \max(n_k)$, and they require extensive storage. Therefore, iterative methods have been developed for the solutions of such equations.

3.1 Revised Iterative Methods for Solutions of PLDALEs

Iteratively solving (8) and (9) using their corresponding lifted structures, i.e., (12) and (13), has been considered in [2]. A generalized version of the alternating direction implicit (ADI) method and the Smith method is proposed there for the solutions of (12) and (13), respectively. Note that, for the descriptor system (1), the matrix $\mathcal{E} = \operatorname{diag}(E_0, \ldots, E_{K-1})$ is singular. However, both the ADI and Smith iterations fail to converge for the resulting Lyapunov equations since the iteration operator of the ADI and the Smith iterations, in this case, does not have spectral radius less than one. This problem has been circumvented by considering a *generalized Cayley transformation* given by

$$\mathfrak{C}(\mathcal{E}, \mathcal{A}) = \lambda(\mathcal{A} - \mathcal{E}) - (\mathcal{A} + \mathcal{E}) \tag{14}$$

see, e.g., [18]. This transformation transfers the PLDALE (12) to an equivalent projected continuous-time algebraic Lyapunov equation (PCALE)

$$\mathbf{E}\mathcal{G}^{cr}\mathbf{A}^T + \mathbf{A}\mathcal{G}^{cr}\mathbf{E}^T = -2\mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T, \quad \mathcal{G}^{cr} = \mathcal{P}_r\mathcal{G}^{cr}\mathcal{P}_r^T, \tag{15}$$

where $\lambda\mathbf{E} - \mathbf{A} = \lambda(\mathcal{A} - \mathcal{E}) - (\mathcal{A} + \mathcal{E})$ is the Cayley-transformed pencil. The finite eigenvalues of $\lambda\mathcal{E} - \mathcal{A}$ lying inside the unit circle are mapped to the finite eigenvalues of $\lambda\mathbf{E} - \mathbf{A}$ in the open left half-plane, and the eigenvalue of $\lambda\mathcal{E} - \mathcal{A}$ at infinity is mapped to $\lambda = 1$.

The solutions of the PLDALE (12) and the PCALE (15) are identical and have the block diagonal structure. Therefore, it is desired that the iterative solution of (15) will have the block diagonal structure analogous to (12) at each ADI iteration step. Unfortunately, the case is not so. It is observed that the generalized ADI method does not preserve the block diagonal structure at every ADI iteration step due to the specific structure of the matrices $\mathbf{E}$ and $\mathbf{A}$, although the approximate Gramian is block diagonal [2]. Therefore, we use the concept of generalized inverses of periodic matrix pairs associated with the periodic system to preserve the block diagonal structure of the approximate solution at each iteration step.

4 Generalized inverses of periodic matrix pairs

For periodic systems, the inversion problem has been introduced in [12, 22] by exploiting the concept of left and right invertibility of system (1) in the standard case (i.e.,

with $E_k = I_{n_{k+1}}$). A generalization of that concept, which computes the generalized inverses of periodic descriptor systems via the corresponding lifted representation has been considered in [37]. A special class of generalized inverse, called *reflexive generalized inverse*, of the system pencil has been proposed in [31] to find the solutions of projected continuous-time algebraic Lyapunov equations using Krylov subspace methods, and also in [3] for computing a partial realization for descriptor systems. Similar inverses have been proposed in [36] to compute generalized inverses of rational matrices in descriptor state-space representation, and named there *(1,2)-inverse* as they satisfy the first two Moore-Penrose conditions. Details of these definitions can be found in [5]. This paper generalizes the idea of [31] for an analogous representation of the generalized inverses associated with the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$. Note that in the periodic setting we do not have any matrix pencil, but a set of periodic matrix pairs. Analogous to [31], we can find the *reflexive generalized inverses* for E_k with respect to the projectors $P_l(k)$ and $P_r(k)$ as

$$\bar{E}_k = V_{k+1} \begin{bmatrix} I_{n_{k+1}^f} & 0 \\ 0 & 0 \end{bmatrix} U_k, \quad \bar{A}_k = V_k \begin{bmatrix} (A_k^f)^{-1} & 0 \\ 0 & I_{n_k^\infty} \end{bmatrix} U_k, \tag{16}$$

for $k = 0, 1, \ldots, K-1$. For nonsingular A_k, the exact inverse of A_k is equal to its reflexive generalized inverse for each $k = 0, 1, \ldots, K-1$. The main advantage of this approach is that the resulting system preserves the block sparsity of the lifted system matrices. Moreover, the generalized inverses follow the relations

$$\bar{E}_k E_k \bar{E}_k = \bar{E}_k, \quad E_k \bar{E}_k = P_l(k), \quad \bar{E}_k E_k = P_r(k+1), \tag{17}$$

for $k = 0, 1, \ldots, K-1$. These reflexive generalized inverses of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ will be exploited in the next two sections to find the block diagonal approximate solutions of (12) and (13) using the Smith iterative method.

5 Generalized Smith method for causal PLDALEs

Consider again the PLDALE (12), i.e.,

$$\mathcal{A}\mathcal{X}\mathcal{A}^T - \mathcal{E}\mathcal{X}\mathcal{E}^T = -\mathcal{P}_l \mathcal{B}\mathcal{B}^T \mathcal{P}_l^T, \quad \mathcal{X} = \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T.$$

Multiplying the above equation from the left and right by $\bar{\mathcal{E}}$, and $(\bar{\mathcal{E}})^T$, we get

$$\mathcal{P}_r \mathcal{X} \mathcal{P}_r^T - \bar{\mathcal{E}}\mathcal{A}\mathcal{X}\mathcal{A}^T(\bar{\mathcal{E}})^T = \bar{\mathcal{E}}\mathcal{P}_l \mathcal{B}\mathcal{B}^T \mathcal{P}_l^T (\bar{\mathcal{E}})^T, \ \mathcal{X} = \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T, \tag{18}$$

where $\bar{\mathcal{E}}\mathcal{E}{=}\mathcal{P}_r$ by the definition of the reflexive generalized inverse, and $\bar{\mathcal{E}} = \mathrm{diag}(\bar{E}_0, \bar{E}_1, \cdots, \bar{E}_{K-1})$. Equation (18) can be written in a more usual form as

$$\mathcal{X} - (\bar{\mathcal{E}}\mathcal{A})\mathcal{X}(\bar{\mathcal{E}}\mathcal{A})^T = \mathcal{P}_r \bar{\mathcal{E}}\mathcal{B}\ (\mathcal{P}_r \bar{\mathcal{E}}\mathcal{B})^T, \tag{19}$$

where $\mathcal{X} = \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T$. Note that in the above representation, we use the relation $\mathcal{P}_r \bar{\mathcal{E}} = \bar{\mathcal{E}} \mathcal{P}_l$. Clearly, $\mathcal{P}_l$ and $\mathcal{P}_r$ are the spectral projectors onto the invariant subspace of the matrix $\bar{\mathcal{E}}\mathcal{A}$ corresponding to eigenvalues inside the unit circle. Then the unique solution of (19) can be obtained using the generalized Smith method [21, 30] and is given by

$$\mathcal{X}_i = \sum_{\ell=0}^{i} (\bar{\mathcal{E}}\mathcal{A})^\ell \mathcal{P}_r \bar{\mathcal{E}}\mathcal{B}\ \mathcal{B}^T \bar{\mathcal{E}}^T \mathcal{P}_r^T ((\bar{\mathcal{E}}\mathcal{A})^T)^\ell. \tag{20}$$

Therefore, the Cholesky factor $\mathcal{R}_i$, where $\mathcal{X}_i = \mathcal{R}_i \mathcal{R}_i^T$, is given by

$$\mathcal{R}_i = [\ \mathcal{P}_r \bar{\mathcal{E}}\mathcal{B},\ (\bar{\mathcal{E}}\mathcal{A})\, \mathcal{P}_r \bar{\mathcal{E}}\mathcal{B}, \ldots,\ (\bar{\mathcal{E}}\mathcal{A})^i\, \mathcal{P}_r \bar{\mathcal{E}}\mathcal{B}\ ]. \tag{21}$$

Remark 2. At each iteration step i, the Smith iteration (21) does not preserve an analogous block diagonal structure as given in (12). Note that $\mathcal{X} = \mathrm{diag}(X_1, \ldots, X_{K-1}, X_0)$, and $X_k = R_k R_k^T$ for $k = 0, 1, \ldots, K-1$. Hence, we demand to compute the block diagonal Cholesky factor $\mathcal{R}_i = \mathrm{diag}(R_{1,i}, \ldots, R_{K-1,i}, R_{0,i})$ at each iteration step i of (21). Unfortunately, the iterations do not result so. This is because in each iteration step i, except for the first iteration, in the right side of (21) we have a different block cyclic matrix.

5.1 Structure Preserving Solutions of PLDALEs

The problem of preserving the block diagonal structure at the iterative computation of the Cholesky factor $\mathcal{R}_i$ can be circumvented by introducing a cyclic permutation matrix in each iteration step i of (21). Consider the permutation matrix Π

$$\Pi = \begin{bmatrix} 0 & I_{n_1} & \cdots & 0 & 0 \\ \vdots & 0 & I_{n_2} & & 0 \\ & & \ddots & & \vdots \\ 0 & & & 0 & I_{n_{K-1}} \\ I_{n_0} & 0 & \cdots & & 0 \end{bmatrix};\ \Pi_{\mathrm{i}} = \Pi^{\mathrm{i}};\ \mathrm{i} = 1, 2, \ldots, \tag{22}$$

We introduce a permutation matrix Π_{i} for each iteration step i in the computation of (20), where Π_{i} changes in a cyclic manner by a backward block-row shift at each iteration step. For an example, suppose that $K = 3$, and $k = 0, 1, 2$. Then for $i = 1, \Pi_{\mathrm{i}} = \Pi$ is given by

$$\Pi = \begin{bmatrix} 0 & I_{n_1} & 0 \\ 0 & 0 & I_{n_2} \\ I_{n_0} & 0 & 0 \end{bmatrix}. \tag{23}$$

For $i = 2$, we get

$$\Pi_2 = \begin{bmatrix} 0 & 0 & I_{n_2} \\ I_{n_0} & 0 & 0 \\ 0 & I_{n_1} & 0 \end{bmatrix}, \tag{24}$$

which is just a backward shift of the first block-row of Π_1 in (23). Clearly $\Pi_3 = \Pi_0 = \mathrm{I}$. One can also prove this using the proposed relation as

$$\Pi_0 = \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & I_{n_1} & 0 \\ 0 & 0 & I_{n_2} \end{bmatrix}, \tag{25}$$

which is nothing but a backward shift of the first block-row of Π_2. One nice property of this permutation matrix is that it satisfies the periodicity property, i.e., $\Pi_{\mathrm{K+k}} = \Pi_{\mathrm{k}}$; $\mathrm{k} = 0, 1, \ldots, \mathrm{K}-1$.

Hence (20) takes the new form

$$\mathcal{X}_i = \sum_{\ell=0}^{i} (\bar{\mathcal{E}}\mathcal{A})^{\ell} \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B} \Pi^{\ell} \, (\Pi^{\ell})^{\mathrm{T}} \mathcal{B}^{\mathrm{T}} \bar{\mathcal{E}}^{\mathrm{T}} \mathcal{P}_{\mathrm{r}}^{\mathrm{T}} ((\bar{\mathcal{E}}\mathcal{A})^{\mathrm{T}})^{\ell}.$$

Therefore, in this new representation the Cholesky factor $\mathcal{R}_i$ has the form

$$\mathcal{R}_i = [\, \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B} \Pi, \; (\bar{\mathcal{E}}\mathcal{A}) \, \mathcal{P}_{\mathrm{r}} \bar{\mathcal{E}} \mathcal{B} \Pi^2, \ldots, \; (\bar{\mathcal{E}}\mathcal{A})^{\mathrm{i}} \, \mathcal{P}_{\mathrm{r}} \bar{\mathcal{E}} \mathcal{B} \Pi^{\mathrm{i}} \,] \tag{26}$$

Note that the Smith iteration (26) preserves the block diagonal structure in the computation of the Cholesky factor $\mathcal{R}_i$, where $\mathcal{R}_i = \mathrm{diag}(R_{1,i}, \ldots, R_{K-1,i}, R_{0,i})$ at each iteration step i, $i = 1, 2, \ldots$, and $X_k = R_k R_k^T$ for $k = 0, 1, \ldots, K-1$. The whole iterative process can be summarized in Algorithm 1.

Algorithm 1 Generalized Smith method for causal PLDALE

Input: $\mathcal{A}, \bar{\mathcal{E}}, \mathcal{B}, \mathcal{P}_r$, cyclic permutation matrix Π.
Output: A low-rank Cholesky factor $\mathcal{R}_i$ such that $\mathcal{X}_i = \mathcal{R}_i \mathcal{R}_i^T$.

for $i = 1$ **do**
  $W_1 = \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}$
  $Z_1 = W_1$
  $\mathcal{R}_1 = Z_1$
end for
for $i = 2, 3$ to $\ldots$ **do**
  $W_i = (\bar{\mathcal{E}}\mathcal{A}) W_{i-1}$
  $Z_i = W_i \Pi^{\mathrm{i}-1}$
  $\mathcal{R}_i = [\mathcal{R}_{i-1}, \; Z_i]$
  Compute the rank-revealing QR decomposition $[\mathcal{V}_i, Q_i, r_n] = \mathrm{RRQR}(\mathcal{R}_i^T, \tau)$;
  Update $\mathcal{R}_i = Q_i \mathcal{V}_i^T [\, I_{r_n}, \, 0\,]^T$;
end for

When the column ranks of the B_k matrices, i.e., m_k, $k = 0, 1, \ldots, K-1$, are big, $\mathcal{R}_i$ may face rank deficiency, because, in each k-th iteration step, Algorithm 1 will add as many more columns as in $\mathcal{B}$ to the previous $\mathcal{R}_i$. Hence, we propose the rank-revealing QR decomposition (RRQR) [8] of $\mathcal{R}_i$ with tolerance τ to compute a low-rank factor of $\mathcal{R}_i$.

The approximate solution satisfies $\mathcal{R}_i = \mathcal{P}_r\mathcal{R}_i$ in every iteration step i. Algorithm 1 is to be stopped as soon as the *normalized residual norm* given by

$$\eta(\mathcal{R}_i) = \frac{\|\mathcal{A}\mathcal{R}_i\mathcal{R}_i^T\mathcal{A}^T - \bar{\mathcal{E}}\mathcal{R}_i\mathcal{R}_i^T\bar{\mathcal{E}}^T + \mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T\|_F}{\|\mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T\|_F}$$

satisfies the condition $\eta(\mathcal{R}_i) < tol$ with a user-defined tolerance tol or a stagnation of residual norms is observed.

5.2 Cyclic computations of causal PLDALEs

In fact, the iteration (26) implemented in Algorithm 1 not only proves that the computed Cholesky factors $\mathcal{R}_i$ stay block diagonal at each iteration step i, it also enables us to rewrite (21) in such a way that one can directly compute the periodic Cholesky factors for different k, $k = 0, 1, \ldots, K-1$. From simple algebraic manipulation of (26), we observe that the periodic matrices E_k, A_k, and B_k appear in a cyclic manner in the computation of the periodic Cholesky factors $R_{k,i}$ in every iteration step i for different values of k, $k = 0, 1, \ldots, K-1$. Observing these cyclic relations and handing them technically, we can compute the periodic Cholesky factors $R_{k,i}$, $k = 0, 1, \ldots, K-1$, $i = 1, 2, \ldots$, directly. We represent some of those computations in the following.

For $i = 1$ and $k = 0, 1, \ldots, K-1$, we get

$$\begin{aligned} R_{0,1} &= P_r(1)\bar{E}_0B_0, \quad \% \ \bar{E}_0 = \bar{E}_K, B_0 = B_K \\ R_{1,1} &= P_r(2)\bar{E}_1B_1, \\ &\vdots \\ R_{K-1,1} &= P_r(0)\bar{E}_{K-1}B_{K-1}, \quad \% \ P_r(0) = P_r(K) \end{aligned}$$

For $i = 2$ and $k = 0, 1, \ldots, K-1$, we get

$$\begin{aligned} R_{0,2} &= \bar{E}_0A_0P_r(0)\bar{E}_{K-1}B_{K-1} \\ R_{1,2} &= \bar{E}_1A_1P_r(1)\bar{E}_0B_0 \\ &\vdots \\ R_{K-1,2} &= \bar{E}_{K-1}A_{K-1}P_r(K-1)\bar{E}_{K-2}B_{K-2}. \end{aligned}$$

For $i = 3$ and $k = 0, 1, \ldots, K-1$, we get

$$\begin{aligned} R_{0,3} &= \bar{E}_0 A_0 \bar{E}_{K-1} A_{K-1} P_r(K-1) \bar{E}_{K-2} B_{K-2} \\ R_{1,3} &= \bar{E}_1 A_1 \bar{E}_K A_K P_r(K) \bar{E}_{K-1} B_{K-1} \\ &\vdots \\ R_{K-1,3} &= \bar{E}_{K-1} A_{K-1} \bar{E}_{K-2} A_{K-2} P_r(K-2) \bar{E}_{K-3} B_{K-3}, \end{aligned}$$

and so on. The whole computation is summarized in Algorithm 2. Note that in the above computations and also in Algorithm 2, we use the periodicity of the coefficient matrices and that of the projectors. Here, $P_r(K) = P_r(0)$, $P_r(K-1) = P_r(-1)$, $\bar{E}_K = \bar{E}_0$, $\bar{E}_{-1} = \bar{E}_{K-1}$, and the similar for others. For the easy index setting, we consider $k = 1, 2, \ldots, K$ in Algorithm 2, and obviously $X_K = X_0 = R_K R_K^T = R_0 R_0^T$. It should be also noted that in Algorithm 2, $R_{k,j}$ means the computed R_k at the jth iteration steps. Finally, $R_{k,j}^b$ collocates all these iterative counterparts for an individual k, where $k = 1, 2, \ldots, K$. That means for $k = 1$, we compute $R_{1,j}^b = [R_{1,1}, R_{1,2}, \ldots, R_{1,J}]$, and similarly the others.

One can define a stoping criteria for Algorithm 2 by setting a *normalized residual norm* given by

$$\eta(\rho_k) = \frac{\|A_k R_k R_k^T A_k^T - E_k R_{k+1} R_{k+1}^T E_k^T + P_l(k) B_k B_k^T P_l(k)^T\|_F}{\|P_l(k) B_k B_k^T P_l(k)^T\|_F}, \tag{27}$$

which satisfy the condition $\eta(\rho_k) < tol$ for $k = 0, 1, \ldots, K-1$, where tol is a user predefined tolerance. A similar computation can also be stated for the causal periodic observability Gramians Y_k of (10).

6 Smith method for noncausal PLDALEs

Consider again the PLDALE (13), i.e.,

$$\mathcal{A}\hat{\mathcal{X}}\mathcal{A}^T - \mathcal{E}\hat{\mathcal{X}}\mathcal{E}^T = \mathcal{Q}_r \mathcal{B}\mathcal{B}^T \mathcal{Q}_r^T, \quad \hat{\mathcal{X}} = \mathcal{Q}_r \hat{\mathcal{X}} \mathcal{Q}_r^T.$$

For nonsingular $\mathcal{A}$, this equation is equivalent to the PLDALE

$$\begin{aligned} \hat{\mathcal{X}} - (\mathcal{A}^{-1}\mathcal{E})\hat{\mathcal{X}}(\mathcal{A}^{-1}\mathcal{E})^T &= \mathcal{Q}_r \mathcal{A}^{-1}\mathcal{B}\mathcal{B}^T \mathcal{A}^{-T} \mathcal{Q}_r^T, \\ \hat{\mathcal{X}} &= \mathcal{Q}_r \hat{\mathcal{X}} \mathcal{Q}_r^T. \end{aligned} \tag{28}$$

In this case, the relation $\mathcal{Q}_r \mathcal{A}^{-1}\mathcal{E} = \mathcal{A}^{-1}\mathcal{E}\mathcal{Q}_r$ holds true [30], and such an equation can be solved by the Smith method [26] given by

$$\begin{aligned} \hat{\mathcal{X}}_1 &= \mathcal{Q}_r \mathcal{A}^{-1}\mathcal{B}\mathcal{B}^T \mathcal{A}^{-T} \mathcal{Q}_r^T, \\ \hat{\mathcal{X}}_i &= \mathcal{Q}_r \mathcal{A}^{-1}\mathcal{B}\mathcal{B}^T \mathcal{A}^{-T} \mathcal{Q}_r^T + (\mathcal{A}^{-1}\mathcal{E})\hat{\mathcal{X}}_{i-1}(\mathcal{A}^{-1}\mathcal{E})^T. \end{aligned} \tag{29}$$

Algorithm 2 Smith method for cyclic computation of causal PLDALEs.

Input: (E_k, A_k, B_k), spectral projectors $P_r(k)$ for $k = 1, \ldots, K$.

Output: Low-rank periodic Cholesky factor R_k such that $X_k = R_k R_k^T$.

for $i = 1 : K$ **do**
  $R_{k,1} = P_r(k+1)\bar{E}_k B_k$ % note that $P_r(K+1) = P_r(1)$
end for
for $i = 1 : K$ **do**
  $P_{k,1} = I_{n_k}$ % initialization of a cyclic matrix
end for
for $j = 2, 3, \ldots$ **do**
  for $k = 1 : K$ **do**
    $m = \mathrm{mod}(k+1, K)$
    $P_{k,j} = P_{k,j-1}\bar{E}_{m-j}\, A_{m-j}$
    $R_{k,j} = P_{k,j}\, P_r(m-j)\, \bar{E}_{m-j-1}\, A_{m-j-1}$
  end for
end for
for $k = 1 : K$ **do**
  for $j = 1, 2, 3, \ldots$ **do**
    if $j = 1$ **then**
      $R^b_{k,j} = R_{k,j}$
    else
      $R^b_{k,j} = [R^b_{k,j-1} \quad R_{k,j}]$
    end if
  end for
  $R_k = \mathrm{RRQR}(R^b_{k,j}, \tau_k)$
end for

Note that in the noncausal case we do not need to compute the generalized inverses of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$, since (29) requires only the inversion of the cyclic lifted matrix $\mathcal{A}$. In that case $\mathcal{Q}_r$ is the spectral projector onto the invariant subspace of the matrix $\mathcal{A}^{-1}\mathcal{E}$ corresponding to the zero eigenvalues. Then $\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{E} = \mathcal{A}^{-1}\mathcal{E}\mathcal{Q}_r$ is nilpotent with the nilpotency index ν, where ν is the index of the periodic descriptor system (1). In this case, after ν iterations we obtain

$$\hat{\mathcal{X}}_\nu = \sum_{i=0}^{\nu-1} (\mathcal{A}^{-1}\mathcal{E})^i \mathcal{Q}_r \mathcal{A}^{-1}\mathcal{B}\mathcal{B}^T \mathcal{A}^{-T} \mathcal{Q}_r^T ((\mathcal{A}^{-1}\mathcal{E})^T)^i = \hat{\mathcal{X}}. \tag{30}$$

Therefore, the Cholesky factor $\hat{\mathcal{R}}$ of the solution $\hat{\mathcal{X}} = \hat{\mathcal{R}}\hat{\mathcal{R}}^T$ of (28) and also of the

PLDALE (13) takes the form

$$\hat{\mathcal{R}} = [\, \mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}, \mathcal{A}^{-1}\mathcal{E}\,\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}, \ldots, (\mathcal{A}^{-1}\mathcal{E})^{\nu-1}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}]. \tag{31}$$

The generalized Smith iteration discussed above does not preserve the block diagonal structure at every iteration step in the computation of the Cholesky factor $\hat{\mathcal{R}}$ [2], but the approximate Gramian $\hat{\mathcal{X}}_i = \hat{\mathcal{R}}_i\hat{\mathcal{R}}_i^T$ computed there has block diagonal structure at each iteration step, $i = 0, 1, \ldots, \nu-1$. By introducing a cyclic permutation matrix in each iteration step, one can easily preserve the block diagonal structure at every iteration step in the computation of the Cholesky factor $\hat{\mathcal{R}}$ using relation (31).

6.1 Structure preserving solutions for noncausal PLDALEs

Let us again consider a cyclic permutation matrix $\hat{\mathcal{P}}$ of the form

$$\hat{\mathcal{P}} = \begin{bmatrix} 0 & & \cdots & 0 & I_{n_0} \\ I_{n_1} & & & & 0 \\ & \ddots & & & \vdots \\ 0 & & & I_{n_{K-1}} & 0 \end{bmatrix}; \ \hat{\mathcal{P}}_i = \hat{\mathcal{P}}^i; \ i = 1, 2, \ldots, \nu. \tag{32}$$

Like in the causal case, we introduce a permutation matrix $\hat{\mathcal{P}}_i$ for each iteration step i in the computation of (30), where the permutation matrix $\hat{\mathcal{P}}_i$ changes at each iteration step in a cyclic manner by a forward block-row shift. Similar to the causal case, the permutation matrix is periodic with a periodicity K, i.e., $\hat{\mathcal{P}}_{K+k} = \hat{\mathcal{P}}_k$, $k = 0, 1, \ldots, K-1$. Then (30) gets the new form

$$\hat{\mathcal{X}}_\nu = \sum_{i=0}^{\nu-1} (\mathcal{A}^{-1}\mathcal{E})^i \mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\hat{\mathcal{P}}_{i+1}\hat{\mathcal{P}}_{i+1}^T\mathcal{B}^T\mathcal{A}^{-T}\mathcal{Q}_r^T((\mathcal{A}^{-1}\mathcal{E})^T)^i = \hat{\mathcal{X}}. \tag{33}$$

Therefore, the Cholesky factor $\hat{\mathcal{R}}$ has the form

$$\hat{\mathcal{R}} = [\, \mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\hat{\mathcal{P}}, \mathcal{A}^{-1}\mathcal{E}\,\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\hat{\mathcal{P}}^2, \ \ldots, \ (\mathcal{A}^{-1}\mathcal{E})^{\nu-1}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\hat{\mathcal{P}}^\nu \,]. \tag{34}$$

It can be verified that each factor inside (34) preserves the block diagonal structure analogous to the solution of (28). The computation of this factor is presented in Algorithm 3.

We note that if the index η is unknown, then Algorithm 3 can be stopped as soon as $\|W_i\|_F \leq \epsilon$ or $\|W_i\|_F/\|\hat{R}_i\|_F \leq \epsilon$ with the machine precision ϵ. Thus, for systems of low index, the solution of (9) can be obtained with few computations.

Remark 3. In order to guarantee that the second equation in (13) (and also in (28)) is satisfied in finite precision arithmetic, we have to project W_i onto the image of $\mathcal{Q}_r$ by pre-multiplication with $\mathcal{Q}_r$.

Algorithm 3 Generalized Smith method for noncausal PLDALEs

Input: $\mathcal{A}, \mathcal{E}, \mathcal{B}$, spectral projector $\mathcal{Q}_r$, cyclic permutation matrix $\hat{\mathcal{P}}$.
Output: Low-rank factor $\hat{\mathcal{R}}_i$ such that $\hat{\mathcal{X}}_i = \hat{\mathcal{R}}_i \hat{\mathcal{R}}_i^T$.

for $i = 1$ **do**
  $W_1 = \mathcal{Q}_r \mathcal{A}^{-1} \mathcal{B}$
  $Z_1 = W_1 \hat{\mathcal{P}}$
  $\hat{\mathcal{R}}_1 = Z_1$
end for
for $i = 2, 3, \ldots, \nu$ **do**
  $W_i = \mathcal{A}^{-1} \mathcal{E} W_{i-1}$
  $Z_i = W_i \hat{\mathcal{P}}_i$
  $\hat{\mathcal{R}}_i = [\hat{\mathcal{R}}_{i-1},\ Z_i]$
end for

The generalized Smith iteration preserves the block diagonal structure at every iteration step in Algorithm 3. Clearly, at the ith iteration step $\hat{\mathcal{R}}_i$ has the block diagonal structure $\hat{\mathcal{R}}_i = \operatorname{diag}(\hat{R}_{1,i}, \ldots, \hat{R}_{K-1,i}, \hat{R}_{0,i})$, where $\hat{R}_{k,i}$ stands for the periodic Cholesky factors of $\hat{X}_{k,i} = \hat{R}_{k,i} \hat{R}_{k,i}^T$ for different values of k, $k = 0, 1, \ldots, K-1$, at the ith iteration step. Since $\hat{\mathcal{X}} = \hat{\mathcal{R}}_\nu \hat{\mathcal{R}}_\nu^T$, where $\hat{\mathcal{R}}_\nu = \operatorname{diag}(\hat{R}_1, \ldots, \hat{R}_{K-1}, \hat{R}_0)$, one can easily read off the periodic solutions $\hat{X}_k = \hat{R}_k \hat{R}_k^T$ of (13) from the block diagonal structure of $\hat{\mathcal{R}}_i$ for different values of k.

6.2 Cyclic computations of periodic noncausal Cholesky factors

In fact, the iteration (34) implemented in Algorithm 3 not only proves that the computed Cholesky factors $\mathcal{R}_i$ stay block diagonal at each iteration step i, it also enables us to rewrite (31) in such a way that one can directly compute the periodic Cholesky factors for different k, $k = 0, 1, \ldots, K-1$. From simple algebraic manipulation of (34), we observe that the periodic matrices E_k, A_k, and B_k appear in a cyclic manner in the computation of the periodic Cholesky factors $\hat{R}_{k,i}$ in every iteration step i for different values of k, $k = 0, 1, \ldots, K-1$. Observing these cyclic relations and handing them technically, we can compute the periodic Cholesky factors $\hat{R}_{k,i}$, $k = 0, 1, \ldots, K-1$, $i = 1, 2, \ldots$, directly. We represent some of those computations in the following.

For $i = 1$ and $k = 0, 1, \ldots, K-1$, we get

$$\begin{aligned} \hat{R}_{0,1} &= Q_r(0) A_0^{-1} B_0 \\ \hat{R}_{1,1} &= Q_r(1) A_1^{-1} B_1 \\ &\vdots \\ \hat{R}_{K-1,1} &= Q_r(K-1) A_{K-1}^{-1} B_{K-1}. \end{aligned}$$

For $i = 2$ and $k = 0, 1, \ldots, K-1$, we get

$$\begin{aligned} \hat{R}_{0,2} &= A_0^{-1} E_0 Q_r(1) A_1^{-1} B_1 \\ \hat{R}_{1,2} &= A_1^{-1} E_1 Q_r(2) A_2^{-1} B_2 \\ &\vdots \\ \hat{R}_{K-1,2} &= A_{K-1}^{-1} E_{K-1} Q_r(K) A_K^{-1} B_K. \end{aligned}$$

For $i = 3$ and $k = 0, 1, \ldots, K-1$, we get

$$\begin{aligned} \hat{R}_{0,3} &= A_0^{-1} E_0 A_1^{-1} E_1 Q_r(2) A_2^{-1} B_2 \\ \hat{R}_{1,3} &= A_1^{-1} E_1 A_2^{-1} E_2 Q_r(3) A_3^{-1} B_3 \\ &\vdots \\ \hat{R}_{K-1,3} &= A_{K-1}^{-1} E_{K-1} A_K^{-1} E_K Q_r(K+1) A_{K+1}^{-1} B_{K+1}, \end{aligned}$$

and so on.

The whole computation is summarized in Algorithm 4. Similar to the causal case, we consider the periodicity of the coefficient matrices and that of the projectors in Algorithm 4. Here $Q_r(K) = Q_r(0)$, $E_K = E_0$, etc. We also consider $k = 1, 2, \ldots, K$ in Algorithm 4, and $\hat{X}_K = \hat{X}_0 = \hat{R}_K \hat{R}_K^T = \hat{R}_0 \hat{R}_0^T$. In Algorithm 4, $\hat{R}_{k,j}$ means the computed $\hat{R}_k$ at the jth iteration step. Finally, $\hat{R}_{k,j}^b$ collocates all these iterative counterparts for an individual k, where $k = 1, 2, \ldots, K$. That means for $k = 1$, we compute

$$\hat{R}_{1,j}^b = [\hat{R}_{1,1}, \hat{R}_{1,2}, \ldots, \hat{R}_{1,J}],$$

and similarly the others.

When the index of the system, i.e., η, is unknown, we propose the rank-revealing QR decomposition (RRQR) [8] of $R_{k,j}^b$ with tolerance τ to truncate redundant columns in the iterations of Algorithm 4. In that case, we assume that after the Jth iteration we have the exact computation of the periodic Cholesky factors $\hat{R}_k$ satisfying relation $\|\hat{R}_{k,j}\|_F \leq \epsilon$ or $\|\hat{R}_{k,j}\|_F / \|\hat{R}_k\|_F \leq \epsilon$, where ϵ is the machine precision, and then, $\hat{X}_k = \hat{R}_k \hat{R}_k^T$ are the periodic solutions of (9) for $k = 0, 1, \ldots, K-1$.

Remark 4. In order to guarantee that the second equation in (9) is satisfied in finite precision arithmetic, we need to project $R_{k,j}$ onto the image of Q_r by pre-multiplication with Q_r.

Remark 5. As the A_k can well be singular in the discrete-time case, Algorithm 4 is restricted in that case.

Algorithm 4 Generalized Smith method for noncausal PLDALEs.

Input: (E_k, A_k, B_k), spectral projectors $Q_r(k)$ for $k = 1, \ldots, K$.
Output: Low-rank periodic Cholesky factor $\hat{R}_k$ such that $\hat{X}_k = \hat{R}_k \hat{R}_k^T$.

for $k = 1 : K$ **do**
  $\hat{R}_{k,1} = Q_r(k) A_k^{-1} B_k$ % note that $\hat{R}_{K,1} = R_{0,1}$
end for
for $k = 1 : K$ **do**
  $P_{k,1} = I_{n_k}$ % initialization of a cyclic matrix
end for
for $j = 2 : \nu$ **do**
  for $k = 1 : K$ **do**
    $m = \mathrm{mod}(j + 1, K)$
    $P_{k,j} = P_{k,j-1} A_{m+k}^{-1} E_{m+k}$
    $\hat{R}_{k,j} = P_{k,j}\, Q_r(k + m + 1)\, A_{k+m+1}^{-1}\, B_{k+m+1}$
  end for
end for
for $k = 1 : K$ **do**
  for $j = 1 : \nu$ **do**
    if $j = 1$ **then**
      $\hat{R}_{k,j}^b = \hat{R}_{k,j}$
    else
      $\hat{R}_{k,j}^b = [\hat{R}_{k,j-1}^b \quad \hat{R}_{k,j}]$
    end if
  end for
  $\hat{R}_k = \mathrm{RRQR}(\hat{R}_{k,j}^b, \tau)$
end for

Remark 6. The causal and noncausal observabiliy Gramians of the periodic descriptor system (1) can also be determined from the corresponding PLDALEs that are dual to the PLDALE (12) and (13), see [1] for details. Applying Algorithm 1 and Algorithm 3 (and also their corresponding cyclic reformulations Algorithm 2 and Algorithm 4) to these equations, we find, respectively, the low-rank Cholesky factors L_k of the causal observability Gramians $Y_k \approx L_k L_k^T$ and the Cholesky factor $\tilde{Y}_k$ of the noncausal observability Gramians $\hat{Y}_k = \hat{L}_k \hat{L}_k^T$.

7 Application to model order reduction

Model order reduction (MOR) is an approach, where a large dynamical system is approximated by a reduced-order model which can be fast and efficiently simulated and

which has nearly the same response characteristics as compared to the original large model. For the periodic descriptor system (1), a reduced-order model of dimension $\mathbf{r} = (r_0, r_1, \ldots, r_{K-1})$ has the form

$$\tilde{E}_k \tilde{x}_{k+1} = \tilde{A}_k \tilde{x}_k + \tilde{B}_k u_k, \quad \tilde{y}_k = \tilde{C}_k \tilde{x}_k, \qquad k \in \mathbb{Z}, \tag{35}$$

where $\tilde{E}_k \in \mathbb{R}^{\gamma_k \times r_{k+1}}$, $\tilde{A}_k \in \mathbb{R}^{\gamma_k \times r_k}$, $\tilde{B}_k \in \mathbb{R}^{\gamma_k \times m_k}$, $\tilde{C}_k \in \mathbb{R}^{q_k \times r_k}$ are K-periodic matrices, $\sum_{k=0}^{K-1} \gamma_k = \sum_{k=0}^{K-1} r_k = r$, and $r \ll n$. It is also important that the reduced-order model preserves physical properties of the original system such as regularity and stability, and that the approximation error is small.

In this section, we present a generalization of a balanced truncation model reduction method to periodic descriptor systems. Balanced truncation for periodic standard discrete-time system and periodic descriptor systems has been considered in [7, 35], and [6], respectively.

Definition 7. *A realization* (E_k, A_k, B_k, C_k) *of a periodic descriptor system* (1) *is called* balanced *if*

$$X_k = Y_k = \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{X}_k = \hat{Y}_{k+1} = \begin{bmatrix} 0 & 0 \\ 0 & \Theta_k \end{bmatrix},$$

where $\Sigma_k = \mathrm{diag}(\sigma_{k,1}, \ldots, \sigma_{k,n_k^f})$, $\Theta_k = \mathrm{diag}(\theta_{k,1}, \ldots, \theta_{k,n_k^\infty})$, $k = 0, \ldots, K-1$.

Consider that the set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is periodic stable, and the Cholesky factors of the causal and noncausal Gramians satisfy

$$\begin{aligned} X_k &= R_k R_k^T, & Y_k &= L_k L_k^T, \\ \hat{X}_k &= \hat{R}_k \hat{R}_k^T, & \hat{Y}_k &= \hat{L}_k \hat{L}_k^T. \end{aligned}$$

Then the *causal* and *noncausal Hankel singular values* of the periodic descriptor system (1) are defined as

$$\begin{aligned} \sigma_{k,j} &= \sqrt{\lambda_j(X_k E_{k-1}^T Y_k E_{k-1})} = \zeta_j(L_k^T E_{k-1} R_k), \\ \theta_{k,j} &= \sqrt{\lambda_j(\hat{X}_k A_k^T \hat{Y}_{k+1} A_k)} = \zeta_j(\hat{L}_{k+1}^T A_k \hat{R}_k), \end{aligned}$$

respectively, where $\lambda_j(.)$ and $\zeta_j(.)$ denote the eigenvalues and singular values of the corresponding product matrices. For a balanced system, truncation of states related to the small causal Hankel singular values does not change system properties essentially. Unfortunately, we can not do the same for the noncausal Hankel singular values because truncation of small non-zero noncausal Hankel singular values may lead the system to become unstable with respect to small perturbations [10].

Let us consider the singular value decompositions of the product matrices

$$\begin{aligned} L_k^T E_{k-1} R_k &= [U_{k,1}, U_{k,2}] \begin{bmatrix} \Sigma_{k,1} & \\ & \Sigma_{k,2} \end{bmatrix} [V_{k,1}, V_{k,2}]^T, \\ \hat{L}_{k+1}^T A_k \hat{R}_k &= U_{k,3} \Theta_k V_{k,3}^T, \end{aligned} \tag{36}$$

where $[U_{k,1}, U_{k,2}]$, $[V_{k,1}, V_{k,2}]$, $U_{k,3}$ and $V_{k,3}$ are orthogonal,

$$\Sigma_{k,1} = \mathrm{diag}(\sigma_{k,1}, \ldots, \sigma_{k,r_k^f}), \quad \Sigma_{k,2} = \mathrm{diag}(\sigma_{k,r_k^f+1}, \ldots, \sigma_{k,n_k^f}),$$

with $\sigma_{k,1} \geq \cdots \geq \sigma_{k,r_k^f} > \sigma_{k,r_k^f+1} \geq \ldots \geq \sigma_{k,n_k^f}$, and $\Theta_k = \mathrm{diag}(\theta_{k,1}, \ldots, \theta_{k,r_k^\infty})$ is non-singular for $k = 0, 1, \ldots, K-1$. Note that the number of non-zero noncausal Hankel singular values of (1) can be estimated by the following relation

$$r_k^\infty = \mathrm{rank}(\hat{L}_{k+1}^T A_k \hat{R}_k) \leq \min(\eta m, \eta q, n_k^\infty),$$

where η is the index of the lifted pencil $\lambda\mathcal{E} - \mathcal{A}$ [29, 2]. Defining the projection matrices [6] as

$$\begin{aligned} S_{k,r} &= [\, L_{k+1} U_{k+1,1} \Sigma_{k+1,1}^{-1/2}, \;\; \hat{L}_{k+1} U_{k,3} \Theta_k^{-1/2} \,] \in \mathbb{R}^{\mu_k \times \gamma_{k+1}}, \\ T_{k,r} &= [\, R_k V_{k,1} \Sigma_{k,1}^{-1/2}, \;\; \hat{R}_k V_{k,3} \Theta_k^{-1/2} \,] \in \mathbb{R}^{n_k \times r_k}, \end{aligned}$$

with $r_k = r_k^f + r_k^\infty$ and $\gamma_{k+1} = r_{k+1}^f + r_k^\infty$, we compute the reduced-order system (35) as

$$\tilde{E}_k = S_{k,r}^T E_k T_{k+1,r}, \;\; \tilde{A}_k = S_{k,r}^T A_k T_{k,r}, \;\; \tilde{B}_k = S_{k,r}^T B_k, \;\; \tilde{C}_k = C_k T_{k,r}, \tag{37}$$

Let $\tilde{\mathcal{H}}(z)$ be the transfer functions of the reduced-order lifted system formed from the reduced-order matrices in (37). Then we have the following $\mathbb{H}_\infty$-norm error bound

$$\begin{aligned} \|\mathcal{H} - \tilde{\mathcal{H}}\|_{\mathbb{H}_\infty} &= \sup_{\omega \in [0, 2\pi]} \|\mathcal{H}(e^{i\omega}) - \tilde{\mathcal{H}}(e^{i\omega})\|_2 \\ &\leq 2\,\mathrm{trace}\,(\mathrm{diag}(\Sigma_{0,2}, \ldots, \Sigma_{K-1,2})), \end{aligned} \tag{38}$$

where $\mathcal{H}(z)$ is the transfer function of the lifted system defined in (4), $\|.\|_2$ denotes the spectral matrix norm, and $\Sigma_{k,2}$, $k = 0, 1, \ldots, K-1$, contains the truncated causal Hankel singular values. This error bound can be obtained similarly to the standard state space case [35].

8 Results

Example 1. *We consider first an artificial periodic discrete-time descriptor system from [2, Example 1] which is reformulated from its original model in [6, Example 1]. In this reformulation, the periodic descriptor system has $\mu_k = n_k = 404$, $m_k = 2$ and $p_k = 3$ for the periodicity $K = 10$, i.e., $k = 0, 1, \ldots, 9$. The set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is periodic stable with $n_k^f = 400$ and $n_k^\infty = 4$ for $k = 0, 1, \ldots, 9$. The original lifted system has order $n = 4040$. The finite eigenvalues of the lifted system are shown in Fig. 1 which illustrates the stability of the original descriptor system.*

We solve the causal and noncausal lifted projected Lyapunov equations using Algorithm 2 and Algorithm 4, respectively. Since, Algorithm 2 is the cyclic reformulation of Algorithm 1, we compute the normalized residual norms at each Smith iteration step for the

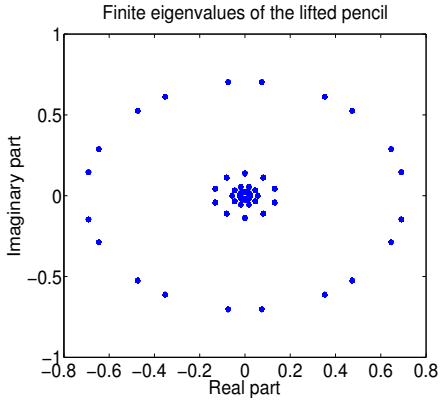


Figure 1: Finite eigenvalues of the pencil $\lambda\mathcal{E} - \mathcal{A}$.

reachability and observability type of the causal lifted Lyapunov equations using Algorithm 1. To illustrate the efficiencies of the proposed algorithms, we compare the results with the corresponding ADI (Alternating Direction Implicit) computation of [2]. Fig. 2 shows the decay of the residual norms computed at each Smith iteration step.

This iteration is stopped as soon as the normalized Lyapunov residual reaches the tolerance tol $= 10^{-10}$*. In Fig. 3(a), we present the approximate causal Hankel singular values* $\sigma_{k,j}$ *computed from the singular value decompositions of the matrices* $L_k^T E_{k-1} R_k$*, where* R_k *and* L_k *are the low-rank Cholesky factors of the causal reachability and observability Gramians, respectively.*

We approximate system (1) by a reduced-order model obtained by truncating the states corresponding to the small causal Hankel singular values satisfying $\sigma_{k,j} < 10^{-4}$*. The largest 80 causal Hankel singular values of the original lifted system, and the approximate 71 causal Hankel singular values for the reduced-order lifted system are shown in Fig. 3(b).*

For different subsystems, the numbers of the computed non-zero noncausal Hankel singular values are identical and given by $r_k^\infty = 2$ *for* $k = 0, 1, \ldots, 9$*. The computed reduced-order model has subsystems of orders* $(9, 9, 9, 9, 9, 9, 9, 9, 10, 9)$*. Note that stability is preserved in the reduced-order system.*

Fig. 4(a) shows the norms of the frequency responses $\mathcal{H}(e^{i\omega})$ *and* $\tilde{\mathcal{H}}(e^{i\omega})$ *of the original and reduced-order lifted systems for a frequency range* $[0, 2\pi]$*. We observe a good match of the system norms. Finally, in Fig. 4(b), we display the absolute error* $\|\mathcal{H}(e^{i\omega}) - \tilde{\mathcal{H}}(e^{i\omega})\|_\infty$ *and the error bound given in (38).*

Example 2. *As a second model problem we consider here an artificial continuous-time model from Section 4.3 of [32], where a spring-damper model is considered as an artificial model of piezo-mechanical systems. We consider* $n = 500, l = 100, nin = 2, nout = 3$ *for our model problem, and hence the dimension of the continuous-time model is* $2n + l = 1100$*. The formulated continuous-time model is converted to a discrete-time model by*

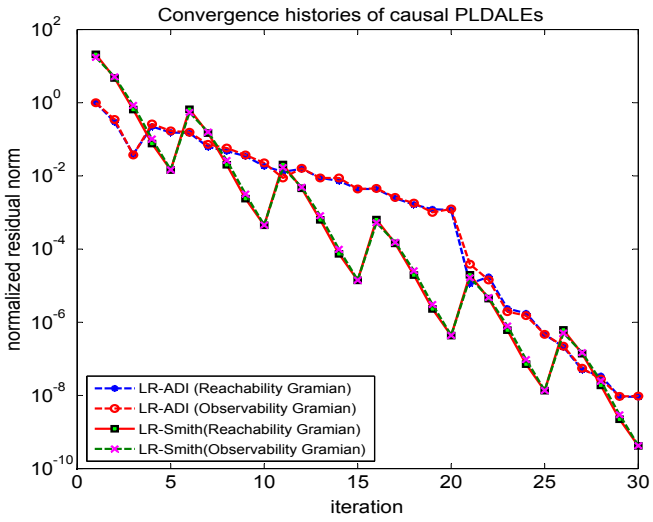
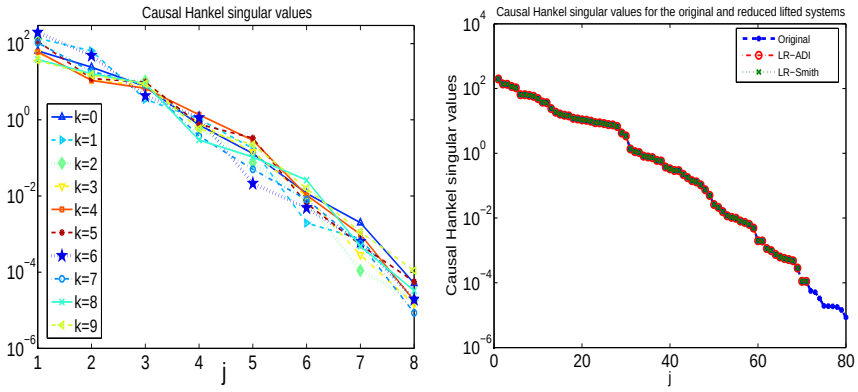
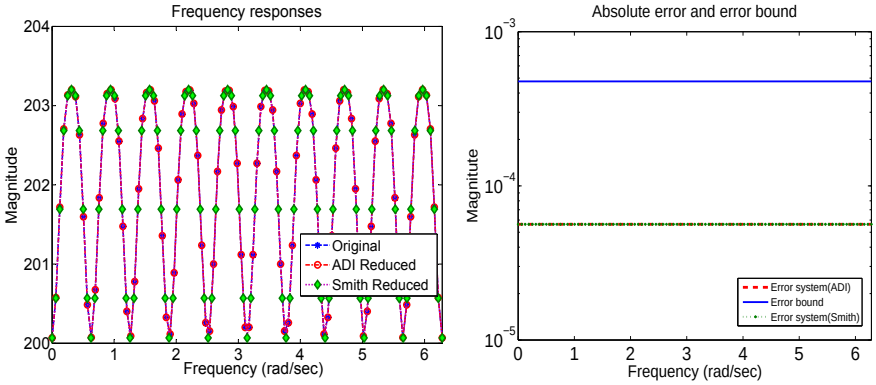


Figure 2: Normalized residual norms for the causal reachability and observability lifted projected Lyapunov equations.



(a) (b)

Figure 3: (a) Causal Hankel singular values for subsystems; (b) Causal Hankel singular values for original and reduced-order lifted systems.



(a) (b)

Figure 4: (a) The frequency responses of the original and the reduced-order lifted systems; (b) absolute error and error bound.

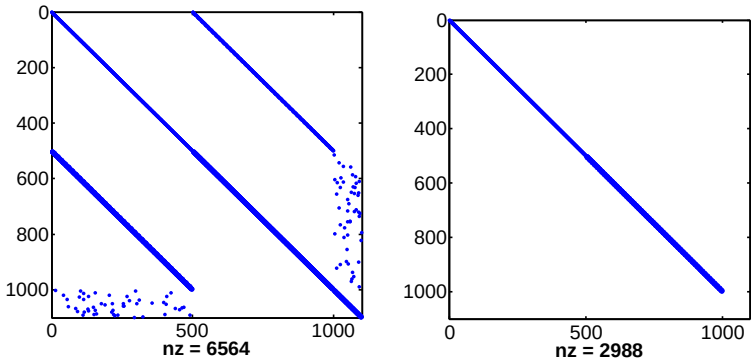


Figure 5: Sparsity patterns of A_0 (left) and E_0 (right).

an Euler discretization method [25]. We then change the damping matrix periodically by introducing some periodic coefficients inside it. As a result, the model is time-varying and periodic. The details of this periodic model formulation are given in the appendix.

For the resulting periodic model, we have $n_k = 1100$, $m_k = 2$, $p_k = 3$, *and a period* $K = 10$. *The periodic matrix pairs* $\{(E_k, A_k)\}_{k=0}^{K-1}$ *are periodic stable with* $n_k^f = 1000$ *and* $n_k^\infty = 100$ *for every* $k = 0, 1, \ldots, 9$. *The resulting periodic system is of index 1, and the original lifted system has order* $\bar{\mathbf{n}} = 11000$. *The sparsity pattern of the periodic pair at* $k = 0$ *is shown in Fig. 5.*

In Fig. 6(a), we present the largest 260 causal Hankel singular values computed by the proposed Smith method in Algorithm 2. We approximate system (1) to the tolerance 10^{-4} *and truncate the states corresponding to the smallest* 200 *causal Hankel singular values. The system has* 20 *noncausal Hankel singular values which are positive, but very small. The values of these noncausal Hankel singular values lie in the range of* $[10^{-13},\ 10^{-15}]$,

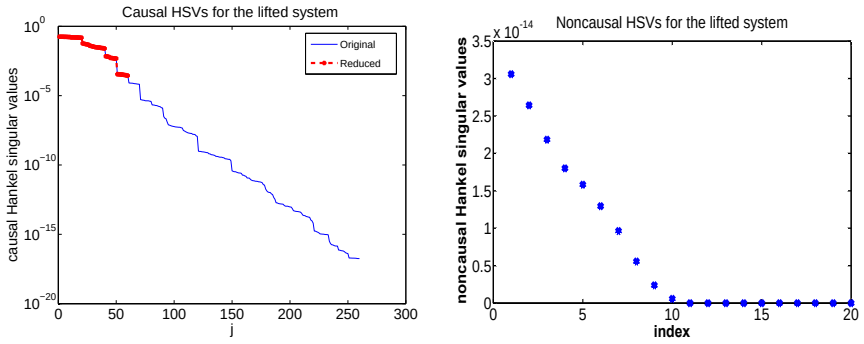


Figure 6: (a) Causal Hankel singular values for original and reduced-order lifted systems. (b) Noncausal Hankel singular values for original and reduced-order lifted systems.

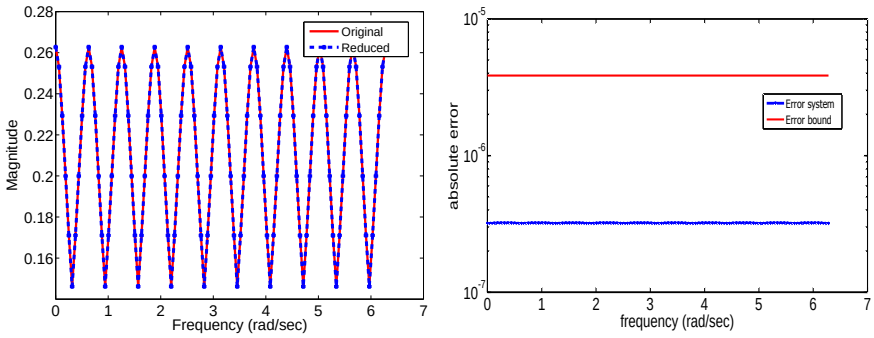


Figure 7: The frequency responses and error bounds of the original and the reduced-order lifted systems.

and they are shown in Fig. 6(b). It is to be mentioned that our model problem is of index 1, and hence we need only one iteration to compute the noncausal Cholesky factor for noncausal PLDALEs using Equation (34).

The computed reduced-order model has subsystems of order $\mathbf{r} = (9, 8, 8, 7, 8, 9, 7, 8, 8, 8)$, *and* $\bar{\mathbf{r}} = 80$. *In Fig. 7(a), we show the norms of the frequency responses* $\mathcal{H}$ *and* $\tilde{\mathcal{H}}$ *for a frequency range* $[0, 2\pi]$. *The absolute error* $\|\mathcal{H} - \tilde{\mathcal{H}}\|_\infty$ *and the error bound are also displayed in Fig. 7(b). One can see that the error bound is tight in this example.*

9 Conclusions

We discussed the structure preserving Smith iterations to compute the low-rank factors for the solutions of large sparse projected periodic discrete-time algebraic Lyapunov equations exploiting the generalized inverses of the periodic matrix pairs associated with the periodic descriptor system. These low-rank factors are used in a balanced

truncation model reduction approach to find a reduced-order model for periodic discrete-time descriptor systems. The proposed model reduction method delivers a reduced-order model that preserves the regularity and stability properties of the original system.

An important advantage of our computational approach is that one can directly compute the generalized inverses of the periodic descriptor system, without explicitly manipulating the lifted representations. Beside this, the proposed Smith iterations preserve the cyclic block diagonal structures at all iteration steps which is the main challenging task in periodic iterative computations.

The major drawback of our proposed method is that it computes the periodic projectors explicitly for the computations of periodic generalized inverses which is numerically expensive and not a wise approach for higher index systems. Hence we restrict the proposed model reduction approach for systems of index-1.

Appendix

The MATLAB codes of the periodic model formulation used in the second model problem. The following codes will construct the periodic matrices E_k, A_k, B_k, and C_k for $k = 0, 1, \ldots, 9$. Note that E_k are constant matrices in the following construction.

```
-----------------------------------------------------
This is an index 1 problem
-----------------------------------------------------
n=500;
l=100;
nin=3;  % number of output
nout=2; % number of output
K=10;   %period
den=0.001;
I=speye(n);
M=.5*I+spdiags(-0.2*ones(n,1),2,n,n)+spdiags(-0.2*ones(n,1),-2,n,n)+...
  spdiags(0.2*ones(n,1),4,n,n)+spdiags(0.2*ones(n,1),-4,n,n);

K_uu=spdiags(5*ones(n,1),0,n,n)+spdiags(-1*ones(n,1),2,n,n)+...
     spdiags(-1*ones(n,1),-2,n,n)+ spdiags(2*ones(n,1),4,n,n)+...
     spdiags(2*ones(n,1),-4,n,n);

D=cell(1,K);
for i=1:K
    mu(i)=0.05+.01*i;
    nu(i)=.8+.01*i;
    D{i}=mu(i)*M+nu(i)*K_uu;
end
```

```
K_pp=spdiags(-5*ones(l,1),0,l,l)+spdiags(ones(l,1),2,l,l)+...
     spdiags(ones(l,1),-2,l,l)+spdiags(-2*ones(n,1),4,l,l)+...
     spdiags(-2*ones(n,1),-4,l,l);
K_up=sprand(n,l,den);

           % Transformation to first order system %

E1=[I spalloc(n,n,0);spalloc(n,n,0) M];
J1=cell(1,K);
for i=1:K
    J1{i}=[spalloc(n,n,0) I;-K_uu -D{i}];
end

J2=[spalloc(n,l,0);-K_up];
J3=[-K_up' spalloc(l,n,0) ];
J4=-K_pp;
clear M K_uu D K_pp K_up
B11=[spalloc(n,nin,0);spdiags(ones(n,1),0,n,nin)];
B2=spdiags(zeros(l,1),0,l,nin);
C11=[spdiags(ones(n,1),0,nout,n)  spalloc(nout,n,0)];
C22=spdiags(zeros(l,1),0,nout,l);

E= [E1 spalloc(size(J2,1),size(J2,2),0);
   spalloc(size(J3,1),size(J3,2),0)
   spalloc(size(J4,1),size(J4,2),0)];  % note E{i}=E; for i=0,1,...,9.

      % Set the periodic matrices %

Ad=cell(1,K);B=cell(1,K); C=cell(1,K);

for i=1:K
    Ad{i}=[J1{i} J2;J3 J4];
    B1{i}=B11*cos(i);
    B{i}=[B1{i}; B2];
    C1{i}=C11*sin(i);
    C{i}=[C1{i} C22];
end

A=cell(1,K);
for i=1:K
    A{i}=(0.6*E-.015*Ad{i});
end
```

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