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An $\mathcal{H}_2$-Type Error Bound for Balancing-Related Model Order Reduction of Linear Systems with Lévy Noise

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Abstract

To solve a stochastic linear evolution equation numerically, finite dimensional approximations are commonly used. For a good approximation, one might end up with a sequence of ordinary stochastic linear equations of high order. To reduce the high dimension for practical computations, model order reduction is frequently used. Balanced truncation (BT) is a well-known technique from deterministic control theory and it was already extended for controlled linear systems with Lévy noise. Recently, a new ansatz was investigated which provides an alternative way to generalize BT for stochastic systems. There, the question of the existence of an $\mathcal{H}_2$-error bound was asked which we answer in this paper.

1 Introduction

Model order reduction (MOR) is of major importance in the field of deterministic control theory. It is used to save computational time by replacing large scale systems by systems of low order in which the main information of the original system should be captured. Such kind of high dimensional problems occur for example after the spatial discretization of a PDE which can be used to model chemical, physical or biological phenomena. A particular ansatz to obtain a reduced order model is to balance a system such that the dominant reachable and observable states are the same. Afterwards, the difficult to observe and difficult to reach states are neglected. One way to do that is to use balanced truncation (BT) which is introduced by Moore [11] and a thorough treatment of the topic can be found in Antoulas [1] or Obinata, Anderson [12].

Since many phenomena in computational sciences and engineering contain uncertainties, it is natural to extend PDE models by adding a noise term. This leads to stochastic PDEs (SPDEs) which are studied in Da Prato, Zabczyk [6] and in Prévôt and Röckner [14] for the Wiener case. Peszat, Zabczyk consider more general equations with Lévy noise in [13], where the solutions may have jumps. To solve SPDEs numerically, one can reduce them to large scale ordinary SDEs by using the Galerkin method. For that reason, generalizing MOR techniques to stochastic systems can be motivated. The mentioned Galerkin approximation is for example investigated in Grecksch, Kloeden [8], Hausenblas [9], Jentzen, Kloeden [10] and Redmann, Benner [15].

To reduce large scale SDEs, balancing related methods are generalized. BT is considered for SDEs with Wiener noise in Benner, Damm [2] and for systems with Lévy noise it is done by Benner, Redmann in [5]. Benner and Redmann provide an $\mathcal{H}_2$-type error bound and the preservation of mean square asymptotic stability is shown in Benner et al. [3]. In Benner et al. [4] and Damm, Benner [7] an example is presented which clarifies that the $\mathcal{H}_\infty$-error bound from the deterministic case does not hold for stochastic systems. Recently, a new ansatz to extend BT to SDEs is considered by Benner et al. [4] or Damm, Benner [7] in which a new reachability Gramian is used. This Gramian does not allow an energy interpretation, but the advantage of the new ansatz is the existence of an $\mathcal{H}_\infty$-error bound and the preservation of mean square asymptotic stability. The remaining part to prove an $\mathcal{H}_2$-error bound, we present in this paper.

In this paper, we focus on BT for SDEs with Lévy noise. We start with giving an overview about the two ways to generalize the deterministic framework and state the most important results that are already proven. In Section 2, we briefly discuss the procedure and emphasize results on error bounds and the stability analysis of the methods. In Section 3, we contribute an $\mathcal{H}_2$-type error bound for the new ansatz in [4] and [7] to close the gap in the error bound analysis.

2 Balancing of Stochastic Systems with Lévy Noise

Let A, $N_k \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. We consider the following equations:

$$\begin{aligned} dX(t) &= [AX(t) + Bu(t)]dt + \sum_{k=1}^{q} N_k X(t-) dM_k(t), \quad t \geq 0, \quad X(0) = x_0, \qquad (1)\\ Y(t) &= CX(t), \end{aligned}$$

where $M_1, \dots, M_q$ are scalar uncorrelated and square integrable Lévy processes with mean zero defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.[1] In addition, we assume M_k $(k = 1, \dots, q)$ to be $(\mathcal{F}_t)_{t\geq 0}$-adapted and the increments $M_k(t+h) - M_k(t)$ to be independent of $\mathcal{F}_t$ for $t, h \geq 0$. With L^2_T we denote the space of all $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic processes v with values in $\mathbb{R}^m$, which are square integrable with respect to $\mathbb{P} \otimes dt$. The norm in L^2_T is given by

$$\|v\|^2_{L^2_T} := \mathbb{E} \int_0^T v^T(t)v(t)dt = \mathbb{E} \int_0^T \|v(t)\|_2^2 \, dt,$$

where we define the processes v_1 and v_2 to be equal in L^2_T if they coincide almost surely with respect to $\mathbb{P} \otimes dt$. For the case $T = \infty$, we denote the space by L_2. Further, we assume the control $u \in L^2_T$ for every $T > 0$. The solution of equation (1) we denote by $X(t, x_0, u)$ and the corresponding output by $Y(t, x_0, u)$. Moreover, we assume mean square asymptotic stability which is

$$\mathbb{E} \|X(t, x_0, 0)\|_2^2 \to 0 \quad \text{for } t \to \infty. \tag{2}$$

Below, we set $q = 1$ and $M := M_1$, $N := N_1$ for simplicity of notation. Any of the following results also holds for general q.

2.1 Type 1 balanced truncation

In type 1 BT the idea is to introduce a generalized fundamental solution to the state equation (1) which is a matrix-valued process $(\Phi(t))_{t\geq 0}$ defined by $X(t, x_0, 0) = \Phi(t)x_0$. This can be used to define Gramians

$$P := \int_0^\infty \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right] ds \text{ and } Q := \int_0^\infty \mathbb{E}\left[\Phi^T(s)C^TC\Phi(s)\right] ds. \tag{3}$$

Following the arguments in Section 3 in [5] we know that the Gramians are solutions of generalized Lyapunov equations:

$$AP + PA^T + NPN^T \cdot c = -BB^T \text{ and } A^TQ + QA + N^TQN \cdot c = -C^TC, \tag{4}$$

where $c := \mathbb{E}\left[M(1)^2\right]$. Below, we suppose to have a completely observable and reachable system (1) in terms of the concepts used in [2] or [5] which implies P, $Q > 0$. By Section 3 in [5], we have the following result:

Proposition 2.1. *(i) The minimal energy to steer the average state to $x \in \mathbb{R}^n$ is bounded from below as follows:*

$$x^T P^{-1} x \leq \inf_{\substack{u \in L^2_T, T>0, \\ \mathbb{E}[X(T,0,u)]=x}} \|u\|^2_{L^2_T}.$$

(ii) The energy that is caused by the observation of an initial state $x_0 \in \mathbb{R}^n$ is

$$\|Y(\cdot, x_0, 0)\|^2_{L^2} = x_0^T Q x_0.$$

Due to the energy interpretation in Proposition 2.1, we consider the state x to be difficult to reach if the expression $x^T P^{-1} x$ is large and we call it difficult to observe if the term $x^T Q x$ is small. In order to ensure that the sets of difficult to observe and difficult to reach states coincide, we balance the system as follows.
We apply a state space transformation, which does not change the output, by using an invertible

[1] We assume that $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and that $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets.

matrix T:

$$(A, B, C, N) \mapsto \left(TAT^{-1}, TB, CT^{-1}, TNT^{-1}\right)$$

which leads to transformed Gramians

$$(P, Q) \mapsto \left(TPT^T, T^{-T}QT^{-1}\right).$$

We choose $T = \Sigma^{\frac{1}{2}}K^TU^{-1}$, where $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n) > 0$, U comes from the Cholesky decomposition of $P = UU^T$ and K is an orthogonal matrix corresponding to the spectral decomposition $U^TQU = K\Sigma^2K^T$. This yields

$$TPT^T = T^{-T}QT^{-1} = \text{diag}(\sigma_1, \ldots, \sigma_n).$$

We partition as follows:

$$T = \begin{bmatrix} W^T \\ T_2^T \end{bmatrix} \text{ and } T^{-1} = \begin{bmatrix} V & T_1 \end{bmatrix},$$

where $W^T \in \mathbb{R}^{r\times n}, V \in \mathbb{R}^{n\times r}$ and r represents the reduced order model (ROM) state space dimension. Then, the ROM coefficients, obtained by truncation are

$$(A_{11}, B_1, C_1, N_{11}) = \left(W^TAV, W^TB, CV, W^TNV\right).$$

Type 1 balanced truncation preserves mean square asymptotic stability as shown in Theorem 2.3 in [3].

Theorem 2.2. *Let $\sigma_r \neq \sigma_{r+1}$, then the ROM*

$$dX_R(t) = A_{11}X_R(t)dt + N_{11}X_R(t-)dM(t), \quad t \geq 0, \quad X_R(0) = x_{R,0}$$

is mean square asymptotically stable if

$$dX(t) = AX(t)dt + NX(t-)dM(t), \quad t \geq 0, \quad X(0) = x_0$$

is mean square asymptotically stable.

The result in Theorem 2.2 is vital for the existence of the ROM reachability Gramian $P_R := \int_0^\infty \mathbb{E}\left[\Phi_R(s)B_1B_1^T\Phi_R^T(s)\right] ds$ which occurs in the $\mathcal{H}_2$-type error bound below. Here, Φ_R denotes the fundamental solution of the ROM. The matrix P_R fulfills

$$A_{11}P_R + P_RA_{11}^T + N_{11}P_RN_{11}^T \cdot c = -B_1B_1^T.$$

The following result is proven in Theorem 4.5 and Proposition 4.6 in [5]. For simplicity we assume to already have a balanced realization (A, B, C, N) with partitions

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \text{ and } N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{5}$$

for the next theorem.

Theorem 2.3. *Let system (1) with the coefficients (A, B, C, N) be balanced, i.e. $P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n)$, and P_M the solution to*

$$AP_M + P_MA_{11}^T + NP_MN_{11}^T \cdot c = -BB_1^T,$$

then, if $x_0 = 0$ *and* $x_{R,0} = 0$*, we have*

$$\sup_{t\in[0,T]} \mathbb{E} \left\| Y(t) - Y_R(t) \right\|_2 \leq \epsilon \left\| u \right\|_{L^2_T},$$

where Y_R *is the output of the ROM and*

$$\epsilon^2 = \operatorname{tr}(\Sigma_2(B_2 B_2^T + 2P_{M,2}A_{21}^T)) + \operatorname{tr}(\Sigma_2(2N_{22}P_{M,2}N_{21}^T + 2N_{21}P_{M,1}N_{21}^T - N_{21}P_R N_{21}^T))c,$$

with $\Sigma_2 = \operatorname{diag}(\sigma_{r+1}, \ldots, \sigma_n)$*,* $P_{M,1}$ *representing the first* r *and* $P_{M,2}$ *being the last* $n - r$ *rows of the matrix* P_M*.*

If $N = 0$, then the error bound in Theorem 2.3 coincides with the $\mathcal{H}_2$-error bound proven in [1].

For an $\mathcal{H}_\infty$-type error bound we desire to find an arbitrary constant $a > 0$ such that

$$\left\| Y - Y_R \right\|_{L^2(\mathbb{R}^p)} \leq a \operatorname{tr}(\Sigma_2) \left\| u \right\|_{L^2(\mathbb{R}^m)}$$

holds for controls $u \in L^2(\mathbb{R}^m)$. For the deterministic case ($N = 0$), this constant is $a = 2$, see [1]. Example I.3 in [4] or Example II.2 in [7], respectively, show that such a number $a > 0$ does not exist for general matrices N.

2.2 Type 2 balanced truncation

Type 2 BT for stochastic systems is introduced in [7] and there is a more detailed paper [4] with improved results on that ansatz.
The idea is to replace the reachability Gramian P defined by the fundamental solution of the system in (3) by a matrix $\tilde{P}$ which solves

$$A^T \tilde{P}^{-1} + \tilde{P}^{-1} A + N^T \tilde{P}^{-1} N \cdot c = -\tilde{P}^{-1} B B^T \tilde{P}^{-1}. \tag{6}$$

This new choice has the disadvantage that $\tilde{P}$ does not allow an energy interpretation like in Proposition 2.1 (i).
If we set $N = 0$ in the first equation of (4) and in (6), we obtain the same Lyapunov equation that is used for deterministic BT. Hence, both types of introducing BT for SDEs are possible generalizations. Unfortunately, there exist no criteria for the existence of a positive definite solution to (6). There are matrices A, B and N such that the reachability Gramian P of type 1 BT is positive definite, but there is no positive definite solution to (6), see Example II.5 in [4]. For that reason, we turn to a more general matrix *inequality*

$$A^T \tilde{P}^{-1} + \tilde{P}^{-1} A + N^T \tilde{P}^{-1} N \cdot c \leq -\tilde{P}^{-1} B B^T \tilde{P}^{-1}. \tag{7}$$

For this inequality we have an existence and uniqueness result from [4].

Lemma 2.4. *Suppose that (2) holds, then there is a unique solution* $\tilde{P} > 0$ *to inequality (7).*

Balancing for the new method means that we diagonalize the positive definite solutions to the second equation in (4) and inequality (7) simultaneously.
Analogous to the type 1 approach, there is an invertible transformation matrix $\tilde{T}$, so that

$$(A, B, C, N) \mapsto \left(\tilde{T} A \tilde{T}^{-1}, \tilde{T} B, C \tilde{T}^{-1}, \tilde{T} N \tilde{T}^{-1} \right)$$

leads to a system with transformed Gramians

$$\tilde{T} \tilde{P} \tilde{T}^T = \tilde{T}^{-T} Q \tilde{T}^{-1} = \tilde{\Sigma} = \operatorname{diag}(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n) > 0.$$

With the partition

$$\tilde{T} = \begin{bmatrix} \tilde{W}^T \\ \tilde{T}_2^T \end{bmatrix} \text{ and } \tilde{T}^{-1} = \begin{bmatrix} \tilde{V} & \tilde{T}_1 \end{bmatrix},$$

where $\tilde{W}^T \in \mathbb{R}^{r\times n}, \tilde{V} \in \mathbb{R}^{n\times r}$, we obtain the ROM coefficients

$$\left(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1, \tilde{N}_{11}\right) = \left(\tilde{W}^T A\tilde{V}, \tilde{W}^T B, C\tilde{V}, \tilde{W}^T N\tilde{V}\right).$$

Type 2 balanced truncation preserves mean square asymptotic stability as shown in Theorem II.2 in [4].

Theorem 2.5. *Let $\tilde{\sigma}_r \neq \tilde{\sigma}_{r+1}$, then the ROM*

$$d\tilde{X}_R(t) = \tilde{A}_{11}\tilde{X}_R(t)dt + \tilde{N}_{11}\tilde{X}_R(t-)dM(t), \quad t \geq 0, \quad \tilde{X}_R(0) = \tilde{x}_{R,0}$$

is mean square asymptotically stable if

$$dX(t) = AX(t)dt + NX(t-)dM(t), \quad t \geq 0, \quad X(0) = x_0$$

is mean square asymptotically stable.

The advantage of type 2 BT is the existence of an $\mathcal{H}_\infty$-type error bound which is in contrast to the type 1 method. Below, a result from [4] and [7], respectively is stated.

Theorem 2.6. *If $x_0 = 0$ and $\tilde{x}_{R,0} = 0$, then for all $T > 0$, we have*

$$\left\| Y - \tilde{Y}_R \right\|_{L^2_T(\mathbb{R}^p)} \leq 2(\tilde{\sigma}_{r+1} + \ldots + \tilde{\sigma}_\nu) \left\| u \right\|_{L^2_T(\mathbb{R}^m)},$$

where $\tilde{Y}_R$ is the output of the type 2 approach and $\tilde{\sigma}_{r+1}, \ldots, \tilde{\sigma}_\nu$ are the distinct diagonal entries of $\Sigma_2 = \operatorname{diag}(\tilde{\sigma}_{r+1}, \ldots, \tilde{\sigma}_n) = \operatorname{diag}(\tilde{\sigma}_{r+1}I, \ldots, \tilde{\sigma}_\nu I)$.

As mentioned in [4] the existence of an $\mathcal{H}_2$-type error bound is an open question which we answer in the next section.

3 $\mathcal{H}_2$-type Error Bound for Type 2 Balanced Truncation

For simplicity of notation, we assume to have balanced realization of system (1) in terms of the type 2 approach. This balanced realization we denote by $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{N})$ in order to distinguish between the coefficients of the type 1 and the type 2 ansatz. Since we are in a balanced situation, $\tilde{P} = Q = \tilde{\Sigma}$ such that

$$\tilde{A}^T\tilde{\Sigma} + \tilde{\Sigma}\tilde{A} + \tilde{N}^T\tilde{\Sigma}\tilde{N} \cdot c = -\tilde{C}^T\tilde{C}, \tag{8}$$

$$\tilde{A}^T\tilde{\Sigma}^{-1} + \tilde{\Sigma}^{-1}\tilde{A} + \tilde{N}^T\tilde{\Sigma}^{-1}\tilde{N} \cdot c \leq -\tilde{\Sigma}^{-1}\tilde{B}\tilde{B}^T\tilde{\Sigma}^{-1}, \tag{9}$$

where $c := \mathbb{E}\left[M^2(1)\right]$. Below, we use the following suitable partitions

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \ \tilde{C} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}, \ \tilde{N} = \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix} \text{ and } \tilde{\Sigma} = \begin{bmatrix} \tilde{\Sigma}_1 & \\ & \tilde{\Sigma}_2 \end{bmatrix}.$$

By assuming $x_0 = 0$ and $\tilde{x}_{R,0} = 0$ we obtain representations for the outputs

$$Y(t) = \tilde{C}X(t) = \tilde{C}\int_0^t \Phi(t,s)\tilde{B}u(s)ds \text{ and } \tilde{Y}_R(t) = \tilde{C}_1\tilde{X}_R(t) = \tilde{C}_1\int_0^t \tilde{\Phi}_R(t,s)\tilde{B}_1u(s)ds,$$

where $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$, $t \geq s \geq 0$ and $\tilde{\Phi}_R$ is the fundamental matrix of the reduced order system. These representations are proven in Proposition 3.4 in [5]. Some easy rearrangements yield a first error estimate

$$\begin{aligned}
\mathbb{E}\left\|Y(t)-\tilde{Y}_R(t)\right\|_2 &= \mathbb{E}\left\|\tilde{C}\int_0^t \Phi(t,s)\tilde{B}u(s)ds - \tilde{C}_1\int_0^t \tilde{\Phi}_R(t,s)\tilde{B}_1 u(s)ds\right\|_2 \\
&\leq \mathbb{E}\int_0^t \left\|\left(\tilde{C}\Phi(t,s)\tilde{B} - \tilde{C}_1\tilde{\Phi}_R(t,s)\tilde{B}_1\right)u(s)\right\|_2 ds \\
&\leq \mathbb{E}\int_0^t \left\|\tilde{C}\Phi(t,s)\tilde{B} - \tilde{C}_1\tilde{\Phi}_R(t,s)\tilde{B}_1\right\|_F \|u(s)\|_2\, ds.
\end{aligned}$$

By the Cauchy inequality it holds

$$\mathbb{E}\left\|Y(t)-\tilde{Y}_R(t)\right\|_2 \leq \left(\mathbb{E}\int_0^t \left\|\tilde{C}\Phi(t,s)\tilde{B} - \tilde{C}_1\tilde{\Phi}_R(t,s)\tilde{B}_1\right\|_F^2 ds\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^t \|u(s)\|_2^2\, ds\right)^{\frac{1}{2}}.$$

Following the arguments in Section 4 of [5], we obtain

$$\begin{aligned}
\mathbb{E}\int_0^t \left\|\tilde{C}\Phi(t,s)\tilde{B} - \tilde{C}_1\tilde{\Phi}_R(t,s)\tilde{B}_1\right\|_F^2 ds &= \mathbb{E}\int_0^t \left\|\tilde{C}\Phi(s)\tilde{B} - \tilde{C}_1\tilde{\Phi}_R(s)\tilde{B}_1\right\|_F^2 ds \\
&\leq \mathbb{E}\int_0^\infty \left\|\tilde{C}\Phi(s)\tilde{B} - \tilde{C}_1\tilde{\Phi}_R(s)\tilde{B}_1\right\|_F^2 ds \\
&= \operatorname{tr}\left(\tilde{C}P\tilde{C}^T\right) + \operatorname{tr}\left(\tilde{C}_1\tilde{P}_R\tilde{C}_1^T\right) - 2\ \operatorname{tr}\left(\tilde{C}\tilde{P}_M\tilde{C}_1^T\right),
\end{aligned}$$

where the matrices P, $\tilde{P}_R$ and $\tilde{P}_M$ exist by assumption (2) and Theorem 2.5. They are the unique solutions to

$$\tilde{A}P + P\tilde{A}^T + \tilde{N}P\tilde{N}^T \cdot c = -\tilde{B}\tilde{B}^T \tag{10}$$

$$\tilde{A}_{11}\tilde{P}_R + \tilde{P}_R\tilde{A}_{11}^T + \tilde{N}_{11}\tilde{P}_R\tilde{N}_{11}^T \cdot c = -\tilde{B}_1\tilde{B}_1^T \text{ and} \tag{11}$$

$$\tilde{A}\tilde{P}_M + \tilde{P}_M\tilde{A}_{11}^T + \tilde{N}\tilde{P}_M\tilde{N}_{11}^T \cdot c = -\tilde{B}\tilde{B}_1^T. \tag{12}$$

Thus, we obtain an error bound

$$\sup_{t\in[0,T]} \mathbb{E}\left\|Y(t)-\tilde{Y}_R(t)\right\|_2 \leq \left(\operatorname{tr}\left(\tilde{C}P\tilde{C}^T\right) + \operatorname{tr}\left(\tilde{C}_1\tilde{P}_R\tilde{C}_1^T\right) - 2\ \operatorname{tr}\left(\tilde{C}\tilde{P}_M\tilde{C}_1^T\right)\right)^{\frac{1}{2}} \|u\|_{L^2_T},$$

which we specify in the next theorem.

Theorem 3.1. *Let the realization $(\tilde{A},\tilde{B},\tilde{C},\tilde{N})$ be balanced in terms of the type 2 approach, then*

$$\begin{aligned}
&\operatorname{tr}\left(\tilde{C}P\tilde{C}^T + \tilde{C}_1\tilde{P}_R\tilde{C}_1^T - 2\tilde{C}\tilde{P}_M\tilde{C}_1^T\right) \\
&= \operatorname{tr}(\tilde{\Sigma}_2(\tilde{B}_2\tilde{B}_2^T + 2\tilde{P}_{M,2}\tilde{A}_{21}^T + 2\tilde{N}_{22}\tilde{P}_{M,2}\tilde{N}_{21}^T \cdot c + 2\tilde{N}_{21}\tilde{P}_{M,1}\tilde{N}_{21}^T \cdot c - \tilde{N}_{21}\tilde{P}_R\tilde{N}_{21}^T \cdot c)),
\end{aligned}$$

$\tilde{P}_{M,1}$ are the first r and $\tilde{P}_{M,2}$ the last $n-r$ rows of $\tilde{P}_M$.

Proof. By selecting the left and right upper block of (8), we have

$$\tilde{A}_{11}^T\tilde{\Sigma}_1 + \tilde{\Sigma}_1\tilde{A}_{11} + \tilde{N}_{11}^T\tilde{\Sigma}_1\tilde{N}_{11} \cdot c + \tilde{N}_{21}^T\tilde{\Sigma}_2\tilde{N}_{21} \cdot c = -\tilde{C}_1^T\tilde{C}_1 \tag{13}$$

$$\tilde{A}_{21}^T\tilde{\Sigma}_2 + \tilde{\Sigma}_1\tilde{A}_{12} + \tilde{N}_{11}^T\tilde{\Sigma}_1\tilde{N}_{12} \cdot c + \tilde{N}_{21}^T\tilde{\Sigma}_2\tilde{N}_{22} \cdot c = -\tilde{C}_1^T\tilde{C}_2. \tag{14}$$

We introduce the reduced order system observability Gramian which exist by Theorem 2.5

$$\tilde{A}_{11}^T\tilde{Q}_R + \tilde{Q}_R\tilde{A}_{11} + \tilde{N}_{11}^T\tilde{Q}_R\tilde{N}_{11}\cdot c = -\tilde{C}_1^T\tilde{C}_1. \tag{15}$$

We define

$$\tilde{\epsilon} := \sqrt{\operatorname{tr}(\tilde{C}P\tilde{C}^T) + \operatorname{tr}(\tilde{C}_1\tilde{P}_R\tilde{C}_1^T) - 2\operatorname{tr}(\tilde{C}\tilde{P}_M\tilde{C}_1^T)}.$$

Due to the duality between the equations (8) and (10), the identity $\operatorname{tr}(\tilde{C}P\tilde{C}^T) = \operatorname{tr}(\tilde{B}^T\tilde{\Sigma}\tilde{B})$ holds such that

$$\tilde{\epsilon}^2 = \operatorname{tr}(\tilde{B}_1^T\tilde{\Sigma}_1\tilde{B}_1) + \operatorname{tr}(\tilde{B}_2^T\tilde{\Sigma}_2\tilde{B}_2) + \operatorname{tr}(\tilde{C}_1\tilde{P}_R\tilde{C}_1^T) - 2\operatorname{tr}(\tilde{C}_1\tilde{P}_{M,1}\tilde{C}_1^T) - 2\operatorname{tr}(\tilde{C}_2\tilde{P}_{M,2}\tilde{C}_1^T), \tag{16}$$

where we use the partition $\tilde{P}_M = \begin{bmatrix}\tilde{P}_{M,1}\\ \tilde{P}_{M,2}\end{bmatrix}$. We insert equation (14) which yields

$$\begin{aligned} -\operatorname{tr}(\tilde{C}_2\tilde{P}_{M,2}\tilde{C}_1^T) &= -\operatorname{tr}(\tilde{P}_{M,2}\tilde{C}_1^T\tilde{C}_2) = \operatorname{tr}(\tilde{P}_{M,2}(\tilde{A}_{21}^T\tilde{\Sigma}_2 + \tilde{\Sigma}_1\tilde{A}_{12} + \tilde{N}_{11}^T\tilde{\Sigma}_1\tilde{N}_{12}\cdot c + \tilde{N}_{21}^T\tilde{\Sigma}_2\tilde{N}_{22}\cdot c)) \\ &= \operatorname{tr}(\tilde{\Sigma}_2(\tilde{P}_{M,2}\tilde{A}_{21}^T + \tilde{N}_{22}\tilde{P}_{M,2}\tilde{N}_{21}^T\cdot c)) + \operatorname{tr}(\tilde{\Sigma}_1(\tilde{A}_{12}\tilde{P}_{M,2} + \tilde{N}_{12}\tilde{P}_{M,2}\tilde{N}_{11}^T\cdot c)). \end{aligned}$$

By the upper block of equation (12)

$$\tilde{A}_{11}\tilde{P}_{M,1} + \tilde{A}_{12}\tilde{P}_{M,2} + \tilde{P}_{M,1}\tilde{A}_{11}^T + \tilde{N}_{11}\tilde{P}_{M,1}\tilde{N}_{11}^T\cdot c + \tilde{N}_{12}\tilde{P}_{M,2}\tilde{N}_{11}^T\cdot c = -\tilde{B}_1^T\tilde{B}_1,$$

we have

$$\begin{aligned} -\operatorname{tr}(\tilde{C}_2\tilde{P}_{M,2}\tilde{C}_1^T) = &-\operatorname{tr}(\tilde{\Sigma}_1(\tilde{B}_1\tilde{B}_1^T + \tilde{A}_{11}\tilde{P}_{M,1} + \tilde{P}_{M,1}\tilde{A}_{11}^T + \tilde{N}_{11}\tilde{P}_{M,1}\tilde{N}_{11}^T\cdot c)) \\ &+ \operatorname{tr}(\tilde{\Sigma}_2(\tilde{P}_{M,2}\tilde{A}_{21}^T + \tilde{N}_{22}\tilde{P}_{M,2}\tilde{N}_{21}^T\cdot c)). \end{aligned}$$

Using equation (13), we obtain

$$\begin{aligned} \operatorname{tr}(\tilde{\Sigma}_1(\tilde{A}_{11}\tilde{P}_{M,1} + \tilde{P}_{M,1}\tilde{A}_{11}^T + \tilde{N}_{11}\tilde{P}_{M,1}\tilde{N}_{11}^T\cdot c)) &= \operatorname{tr}(\tilde{P}_{M,1}(\tilde{\Sigma}_1\tilde{A}_{11} + \tilde{A}_{11}^T\tilde{\Sigma}_1 + \tilde{N}_{11}^T\tilde{\Sigma}_1\tilde{N}_{11}\cdot c)) \\ &= -\operatorname{tr}(\tilde{P}_{M,1}(\tilde{N}_{21}^T\tilde{\Sigma}_2\tilde{N}_{21}\cdot c + \tilde{C}_1^T\tilde{C}_1)), \end{aligned}$$

so that

$$\begin{aligned} -\operatorname{tr}(\tilde{C}_2\tilde{P}_{M,2}\tilde{C}_1^T) = &\operatorname{tr}(\tilde{\Sigma}_2(\tilde{P}_{M,2}\tilde{A}_{21}^T + \tilde{N}_{22}\tilde{P}_{M,2}\tilde{N}_{21}^T\cdot c + \tilde{N}_{21}\tilde{P}_{M,1}\tilde{N}_{21}^T)) \\ &- \operatorname{tr}(\tilde{B}_1^T\tilde{\Sigma}_1\tilde{B}_1) + \operatorname{tr}(\tilde{C}_1\tilde{P}_{M,1}\tilde{C}_1^T). \end{aligned}$$

Inserting this result into equation (16) provides

$$\begin{aligned} \tilde{\epsilon}^2 = &\operatorname{tr}(\tilde{\Sigma}_2(\tilde{B}_2\tilde{B}_2^T + 2\tilde{P}_{M,2}\tilde{A}_{21}^T + 2\tilde{N}_{22}\tilde{P}_{M,2}\tilde{N}_{21}^T\cdot c + 2\tilde{N}_{21}\tilde{P}_{M,1}\tilde{N}_{21}^T\cdot c)) \\ &+ \operatorname{tr}(\tilde{C}_1\tilde{P}_R\tilde{C}_1^T) - \operatorname{tr}(\tilde{B}_1^T\tilde{\Sigma}_1\tilde{B}_1). \end{aligned}$$

With the equations (15) and (11) one obtains $\operatorname{tr}(\tilde{C}_1\tilde{P}_R\tilde{C}_1^T) = \operatorname{tr}(\tilde{B}_1^T\tilde{Q}_R\tilde{B}_1)$. So, we have

$$\operatorname{tr}(\tilde{C}_1\tilde{P}_R\tilde{C}_1^T) - \operatorname{tr}(\tilde{B}_1^T\tilde{\Sigma}_1\tilde{B}_1) = \operatorname{tr}(\tilde{B}_1\tilde{B}_1^T(\tilde{Q}_R - \tilde{\Sigma}_1)).$$

Equation (11) again yields

$$\begin{aligned} \operatorname{tr}(\tilde{C}_1\tilde{P}_R\tilde{C}_1^T) - \operatorname{tr}(\tilde{B}_1^T\tilde{\Sigma}_1\tilde{B}_1) &= -\operatorname{tr}((\tilde{A}_{11}\tilde{P}_R + \tilde{P}_R\tilde{A}_{11}^T + \tilde{N}_{11}\tilde{P}_R\tilde{N}_{11}^T\cdot c)(\tilde{Q}_R - \tilde{\Sigma}_1)) \\ &= -\operatorname{tr}(\tilde{P}_R((\tilde{Q}_R - \tilde{\Sigma}_1)\tilde{A}_{11} + \tilde{A}_{11}^T(\tilde{Q}_R - \tilde{\Sigma}_1)\tilde{N}_{11}^T(\tilde{Q}_R - \tilde{\Sigma}_1)\tilde{N}_{11}\cdot c)). \end{aligned}$$

Below, we subtract equation (13) from equation (15) and obtain

$$\operatorname{tr}(\tilde{C}_1\tilde{P}_R\tilde{C}_1^T) - \operatorname{tr}(\tilde{B}_1^T\tilde{\Sigma}_1\tilde{B}_1) = -\operatorname{tr}(\tilde{P}_R\tilde{N}_{21}^T\tilde{\Sigma}_2\tilde{N}_{21}\cdot c).$$

Summarizing the result, we have

$$\tilde{\epsilon}^2 = \operatorname{tr}(\tilde{\Sigma}_2(\tilde{B}_2\tilde{B}_2^T + 2\tilde{P}_{M,2}\tilde{A}_{21}^T + 2\tilde{N}_{22}\tilde{P}_{M,2}\tilde{N}_{21}^T\cdot c + 2\tilde{N}_{21}\tilde{P}_{M,1}\tilde{N}_{21}^T\cdot c - \tilde{N}_{21}\tilde{P}_R\tilde{N}_{21}^T\cdot c)).$$

□

4 Conclusion

In this paper, we described two ways to generalize BT for linear controlled SDEs with Lévy noise, the type 1 and the type 2 ansatz. We discussed the procedures to obtain the ROMs and summarized all known facts in that field including an $\mathcal{H}_2$-type error bound, a stability result for type 1 BT and an $\mathcal{H}_\infty$-type error bound, a stability result for type 2 BT. As our contribution we proved an $\mathcal{H}_2$-type error bound for the type 2 ansatz.

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